CHAPTER 1

Introduction

The goal of this work is to develop, in a systematic way and in a full natural generality, the foundations of a theory of functions of free[1] noncommuting variables. This theory offers a unified treatment for many free noncommutative objects appearing in various branches of mathematics.

Analytic functions of d noncommuting variables originate in the pioneering work of J. L. Taylor on noncommutative spectral theory [**99, 100**]. The underlying idea is that a function of d noncommuting variables is a function on d-tuples of square matrices of all sizes that respects simultaneous intertwinings (or equivalently — as we will show — direct sums and simultaneous similarities). Taylor showed that such functions admit a good differential (more precisely, difference-differential) calculus, all the way to the noncommutative counterpart of the classical (Brook) Taylor formula. Of course a d-tuple of matrices (say over $\mathbb{C}$) is the same thing as a matrix over $\mathbb{C}^d$, so we can view a noncommutative function as defined on square matrices of all sizes over a given vector space. This puts noncommutative function theory in the framework of operator spaces [**37, 78, 79**]. Also, noncommutative functions equipped with the difference-differential operator form an infinitesimal bialgebra [**90, 3**].[2] The theory has been pushed forward by Voiculescu [**105, 106, 107**], with an eye towards applications in free probability [**102, 103, 104, 108**]. We mention also the work of Hadwin [**46**] and Hadwin–Kaonga–Mathes [**47**], of Popescu [**85, 86, 88, 89**], of Helton–Klep–McCullough [**55, 51, 52, 53**], and of Muhly–Solel [**69, 71**]. The (already nontrivial) case of functions of a single noncommutative variable[3] was considered by Schanuel [**93**] (see also Schanuel–Zame [**94**]) and by Niemiec [**76**].

In a purely algebraic setting, polynomials and rational functions in d noncommuting indeterminates and their evaluations on d-tuples of matrices of an arbitrary fixed size (over a commutative ring $\mathcal{R}$) are central objects in the theory of polynomial and rational identities; see, e.g., [**91, 41**]. A deep and detailed study of the ring of noncommutative polynomials and the skew field of noncommutative rational

[1] We consider only the case of free noncommuting variables, namely a free algebra or more generally the tensor algebra of a module; we will therefore say simply "noncommutative" instead of "free noncommutative".

[2] More precisely, to use our terminology, we have to consider noncommutative functions with values in the noncommutative space over an algebra with a directional noncommutative difference-differential operator as a comultiplication, and it is a topological version of the bialgebra concept where the range of the comultiplication is a completed tensor product. See Section 2.3.4 for the Leibnitz rule, and Section 3.4 for the coassociativity of the comultiplication. We will not pursue the infinitesimal bialgrebra viewpoint explicitly.

[3] See Remark 5.11.

functions has been pursued in the work of P. M. Cohn [**29, 31**]. The noncommutative difference-differential operator in the setting of noncommutative polynomials is well known as the universal derivation on the free algebra, see [**67, 25, 33**].

In systems and control, noncommutative rational functions and formal power series appear naturally as recognizable formal power series of the theory of automata and formal languages [**66, 95, 96, 38, 39, 40, 26**] and as transfer functions of multidimensional systems with evolution along the free monoid [**20, 15, 17, 16, 8, 18**]. In particular, transfer functions of conservative noncommutative multidimensional systems are characterized as formal power series whose values on a certain class of d-tuples of operators are contractive (matrix evaluations actually suffice — see [**9**]). Such classes of formal power series appear as the noncommutative generalization of the classical Schur class of contractive analytic functions on the unit disc [**2, 48, 14**] in the operator model theory for row contractions and more general noncommuting operator tuples [**81, 82, 83, 84, 87**] and in the representation theory of the Cuntz algebra [**27, 32**]; see also the generalized Hardy algebras associated to a W^*-correspondence [**68, 70, 12**].

Coming from a different direction, it turns out that most optimization problems appearing in systems and control are dimension-independent, i.e., the natural variables are matrices, and the problem involves rational expressions in these matrix variables which have therefore the same form independent of matrix sizes; see [**50**]. This leads to exploring such techniques as Linear Matrix Inequalities (LMIs) — see, e.g., [**74, 73, 98**] — in the context of noncommutative convexity and noncommutative real semialgebraic geometry, where one considers polynomials and rational functions in d noncommuting indeterminates evaluated on d-tuples of matrices over $\mathbb{R}$ [**49, 56, 59, 58, 54, 57**].

A key feature of noncommutative functions that we establish in this work is very strong analyticity under very mild assumptions. In an algebraic setting, this means that a noncommutative function which is polynomial in matrix entries when it is evaluated on $n \times n$ matrices, $n = 1, 2, \ldots$, of bounded degree, is a noncommutative polynomial. In an analytic setting, local boundedness implies the existence of a convergent noncommutative power series expansion.

Difference-differential calculus for noncommutative rational functions, its relation to matrix evaluations, and applications were considered in [**64, 65**].

Recent papers [**80, 21, 22**] used the results of the present work on noncommutative function theory to study noncommutative infinite divisibility and limit theorems in operator-valued free probability. For instance, in the scalar-valued case a measure is free infinitely divisible if and only if its so-called R-transform has positive imaginary part on the complex upper half-plane ([**23**], see also [**75**] — this is the free analogue of the classical Levi–Hinçin Theorem). One of the main results of [**80**] is a similar statement in the operator-valued case except that the R-transform of an operator-valued distribution is a noncommutative function.

In a recent paper [**1**], a general fixed point theorem for noncommutative functions has been proved, and, in particular, the corresponding variation of the Banach contraction mapping theorem has been obtained. This result was applied then to prove the existence and uniqueness theorem for ODEs in noncommutative spaces. In addition, a noncommutative version of the principle of nested closed sets has been established.

The recent papers [**5, 6, 7**] made further progress in noncommutative function theory. Specifically, [**5**] established an analogue of the realization theorem of [**10**] and [**13**] for noncommutative functions on a domain defined by a matrix noncommutative polynomial (the result was originally established for special cases in [**17**] in the framework of noncommutative power series, see Section 1.3.5 below for a further discussion). [**5**] used this realization result to establish noncommutative analogues of the Oka–Weil approximation theorem and of the Carleson corona theorem. [**6**] applied these ideas to a noncommutative version of the Nevanlinna–Pick interpolation problem, and [**7**] gave an application to symmetric functions of two noncommuting variables.

We proceed now to give some motivating examples of noncommutative functions followed by their definition; we then discuss the difference-differential calculus and present some of the main results of the theory, and we finish the introduction with a detailed overview. We postpone a review of and a comparison to some of the earlier work on the subject to the short chapter at the end of the book.

1.1. Noncommutative (nc) functions: examples and genesis

Let $\mathcal{R}$ be a unital commutative ring, and let $\mathcal{R}\langle x_1, \ldots, x_d\rangle$ be the ring of nc polynomials (the free associative algebra) over $\mathcal{R}$. Here $x_1, \ldots, x_d$ are nc indeterminates, and $f \in \mathcal{R}\langle x_1, \ldots, x_d\rangle$ is of the form

$$f = \sum_{w \in \boldsymbol{\mathcal{G}}_d} f_w x^w, \tag{1.1}$$

where $\boldsymbol{\mathcal{G}}_d$ denotes the free monoid on d generators (letters) $g_1, \ldots, g_d$ with identity $\emptyset$ (the empty word), $f_w \in \mathcal{R}$, x^w are nc monomials in $x_1, \ldots, x_d$ ($x^w = x_{j_1} \cdots x_{j_m}$ for $w = g_{j_1} \cdots g_{j_m} \in \boldsymbol{\mathcal{G}}_d$ and $x^\emptyset = 1$), and the sum is finite. f can be evaluated in an obvious way on d-tuples of square matrices of all sizes over $\mathcal{R}$: for $X = (X_1, \ldots, X_d) \in (\mathcal{R}^{n\times n})^d$,

$$f(X) = \sum_{w \in \boldsymbol{\mathcal{G}}_d} f_w X^w = \sum_{w \in \boldsymbol{\mathcal{G}}_d} X^w f_w \in \mathcal{R}^{n\times n}. \tag{1.2}$$

We can also consider nc formal power series or nc rational functions. The ring $\mathcal{R}\langle\langle x_1, \ldots, x_d\rangle\rangle$ of nc formal power series over $\mathcal{R}$ is the (formal) completion of the ring of nc polynomials; $f \in \mathcal{R}\langle\langle x_1, \ldots, x_d\rangle\rangle$ is of the same form as in (1.1), except that the sum is in general infinite. There are two settings in which we can define the evaluation of f on d-tuples of square matrices:

- Assume that $X = (X_1, \ldots, X_d) \in (\mathcal{R}^{n\times n})^d$ is a jointly nilpotent d-tuple, i.e., $X^w = 0$ for all $w \in \boldsymbol{\mathcal{G}}_d$ with $|w| \geq k$ for some k, where $|w|$ denotes the length of the word w; when $\mathcal{R} = \mathbb{K}$ is a field, this simply means that X is jointly similar to a d-tuple of strictly upper triangular matrices. Then we can define $f(X)$ as in (1.2), since the sum is actually finite.
- Assume that $\mathcal{R} = \mathbb{K}$ is the field of real or complex numbers and that f has a positive nc multiradius of convergence, i.e., there exists a d-tuple $\rho = (\rho_1, \ldots, \rho_d)$ of strictly positive numbers such that

$$\limsup_{k\to\infty} \sqrt[k]{\sum_{|w|=k} |f_w| \rho^w} \leq 1.$$

Then we can define $f(X)$ as in (1.2), where the infinite series converges absolutely and uniformly on any *nc polydisc*

$$\coprod_{n=1}^{\infty}\left\{X\in\left(\mathbb{K}^{n\times n}\right)^{d}:\|X_j\|<r_j,\ j=1,\ldots,d\right\}$$

of multiradius $r=(r_1,\ldots,r_d)$ with $r_j<\rho_j$, $j=1,\ldots,d$.

The skew field of nc rational functions over a field $\mathbb{K}$ is the universal skew field of fractions of the ring of nc polynomials over $\mathbb{K}$. This involves some nontrivial details since unlike the commutative case, a nc rational function does not admit a canonical coprime fraction representation; see [**11, 24, 28, 30**] for some of the original constructions, and [**91**, Chapter 8] and [**29, 31**] for good expositions and background. The following quick description follows [**64, 65**], to which we refer for both details and further references. We first define (scalar) nc rational expressions by starting with nc polynomials and then applying successive arithmetic operations — addition, multiplication, and inversion. A nc rational expression r can be evaluated on a d-tuple X of $n\times n$ matrices in its *domain of regularity*, $\operatorname{dom} r$, which is defined as the set of all d-tuples of square matrices of all sizes such that all the inverses involved in the calculation of $r(X)$ exist. (We assume that $\operatorname{dom} r\neq\emptyset$, in other words, when forming nc rational expressions we never invert an expression that is nowhere invertible.) Two nc rational expressions r_1 and r_2 are called *equivalent* if $\operatorname{dom} r_1\cap\operatorname{dom} r_2\neq\emptyset$ and $r_1(Z)=r_2(Z)$ for all d-tuples $Z\in\operatorname{dom} r_1\cap\operatorname{dom} r_2$. We define a *nc rational function* $\mathfrak{r}$ to be an equivalence class of nc rational expressions; notice that it has a well-defined evaluation on $\bigcup_{r\in\mathfrak{r}}\operatorname{dom} r$ (in fact, on a somewhat larger set called the extended domain of regularity of $\mathfrak{r}$).

We notice that in all these cases the evaluation of a formal algebraic object f (a nc polynomial, formal power series, or rational function) on d-tuples of matrices possesses two key properties.

- *f respects direct sums*: $f(X\oplus Y)=f(X)\oplus f(Y)$, where

$$X\oplus Y=(X_1\oplus Y_1,\ldots,X_d\oplus Y_d)=\left(\begin{bmatrix}X_1&0\\0&Y_1\end{bmatrix},\ldots,\begin{bmatrix}X_d&0\\0&Y_d\end{bmatrix}\right)$$

  (we assume here that $X=(X_1,\ldots,X_d)$, $Y=(Y_1,\ldots,Y_d)$ are such that $f(X)$, $f(Y)$ are both defined).
- *f respects simultaneous similarities*: $f(TXT^{-1})=Tf(X)T^{-1}$, where

$$TXT^{-1}=(TX_1T^{-1},\ldots,TX_dT^{-1})$$

  (we assume here that $X=(X_1,\ldots,X_d)$ and T are such that $f(X)$ and $f(TXT^{-1})$ are both defined).

More generally, one can consider a $p\times q$ matrix nc polynomial f with coefficients $f_w\in\mathbb{K}^{p\times q}$ and with evaluation

$$f(X)=\sum_{w\in\boldsymbol{\mathcal{G}}_d}X^w\otimes f_w\in\mathbb{K}^{n\times n}\otimes\mathbb{K}^{p\times q}\cong\mathbb{K}^{np\times nq}\tag{1.3}$$

for $X\in\left(\mathbb{K}^{n\times n}\right)^d$, and similarly for nc formal power series with matrix coefficients and matrix nc rational functions.[4] It is still true that the evaluation of f on matrices respects direct sums and simultaneous similarities (where for similarities we replace

[4]Notice that, unlike in [**64**] and [**65**], we write the coefficients f_w on the right.

$Tf(X)T^{-1}$ by $(T \otimes I_p)f(X)(T \otimes I_q)^{-1})$. In the case $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, one can also consider operator nc polynomials and formal power series.

Quasideterminants [**44, 45, 42**] and *nc symmetric functions* [**43**] are important examples of nc rational functions, whereas *formal Baker–Campbell–Hausdorff series* [**36**] are an important example of nc formal power series. Let us also mention here *nc continued fractions* [**109**].

1.2. NC sets, nc functions, and nc difference-differential calculus

Both for the sake of potential applications and for the sake of developing the theory in its natural generality, it turns out that the proper setting for the theory of nc functions is that of matrices of all sizes over a given vector space or a given module. In the special case when the module is $\mathcal{R}^d$, $n \times n$ matrices over $\mathcal{R}^d$ can be identified with d-tuples of $n \times n$ matrices over $\mathcal{R}$, and we recover nc functions of d variables, key examples of which appeared in Section 1.1.

Let $\mathcal{M}$ be a module over a unital commutative ring $\mathcal{R}$; we call

$$\mathcal{M}_{\rm nc} = \coprod_{n=1}^{\infty} \mathcal{M}^{n \times n}$$

the *nc space over* $\mathcal{M}$. A subset $\Omega \subseteq \mathcal{M}_{\rm nc}$ is called a *nc set* if it is closed under direct sums, i.e., we have

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$$

for all $n, m \in \mathbb{N}$ and all $X \in \Omega_n$, $Y \in \Omega_m$, where we denote $\Omega_n = \Omega \cap \mathcal{M}^{n \times n}$. NC sets are the only reasonable domains for nc functions, but additional conditions on the domain are needed for the development of the nc difference-differential calculus. Essentially, we need the domain to be closed under formation of upper-triangular block matrices with an arbitrary upper corner block, but this is too stringent a requirement (e.g., this is false for nc polydiscs or nc balls — see Section 1.2.3 below). The proper notion turns out to be as follows: a nc set $\Omega \subseteq \mathcal{M}_{\rm nc}$ is called *right admissible*[5] if for all $X \in \Omega_n$, $Y \in \Omega_m$ and all $Z \in \mathcal{M}^{n \times m}$ there exists an invertible $r \in \mathcal{R}$ such that

$$\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Our primary examples of right admissible nc sets are as follows:

1.2.1. The set $\Omega = \operatorname{Nilp}(\mathcal{M})$ of nilpotent matrices over a module $\mathcal{M}$. Here $X \in \mathcal{M}^{n \times n}$ is called nilpotent if $X^{\odot k} = 0$ for some k, where $X^{\odot k}$ denotes the power of X as a matrix over the tensor algebra

$$\mathbf{T}(\mathcal{M}) = \bigoplus_{j=0}^{\infty} \mathcal{M}^{\otimes j}$$

of $\mathcal{M}$ (this is often called the "faux" product in operator space theory, when $\mathcal{M} = \mathcal{V}$ is an operator space); in the case where $\mathcal{R} = \mathbb{K}$ is a field, this means that there exists an invertible $T \in \mathbb{K}^{n \times n}$ such that TXT^{-1} is strictly upper triangular.

[5] "Upper admissible" could have been a more appropriate terminology, however we stick with "right admissible" since it is related to the right difference-differential operator, see below. A similar comment applies to the analogous notion of "left admissible" which could have been called "lower admissible".

1.2.2. Assume that $\mathcal{V}$ is a Banach space (so $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$) and that so are the spaces $\mathcal{V}^{n\times n}$, $n = 2, 3, \ldots$. We usually need the topologies on $\mathcal{V}^{n\times n}$ to be compatible in the following sense: we require that the corresponding system of matrix norms $\|\cdot\|_n$ is *admissible*, i.e., for every $n, m \in \mathbb{N}$ there exist $C_1(n,m)$, $C_1'(n,m) > 0$ such that for all $X \in \mathcal{V}^{n\times n}$ and $Y \in \mathcal{V}^{m\times m}$,

$$C_1(n,m)^{-1} \max\{\|X\|_n, \|Y\|_m\} \le \|X \oplus Y\|_{n+m} \le C_1'(n,m) \max\{\|X\|_n, \|Y\|_m\}, \tag{1.4}$$

and for every $n \in \mathbb{N}$ there exists $C_2(n) > 0$ such that for all $X \in \mathcal{V}^{n\times n}$ and $S, T \in \mathbb{K}^{n\times n}$,

$$\|SXT\|_n \le C_2(n)\|S\|\,\|X\|_n\|T\|, \tag{1.5}$$

where $\|\cdot\|$ denotes the operator norm of $\mathbb{K}^{n\times n}$ with respect to the standard Euclidean norm of $\mathbb{K}^n$. If $\Omega \subseteq \mathcal{V}_{\rm nc}$ is open in the sense that $\Omega_n \subseteq \mathcal{V}^{n\times n}$ is open for all n, then Ω is right admissible.

1.2.3. Assume that $\mathcal{V}$ is an *operator space*, see, e.g., [**37, 78, 79, 92**]. Recall that this means (by Ruan's Theorem) that there exists a system of norms $\|\cdot\|_n$ on $\mathcal{V}^{n\times n}$, $n = 1, 2, \ldots$, satisfying

$$\|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\} \quad \text{for all } X \in \mathcal{V}^{n\times n},\ Y \in \mathcal{V}^{m\times m}, \tag{1.6}$$

and

$$\|TXS\|_n \le \|T\|\|X\|_n\|S\| \quad \text{for all } X \in \mathcal{V}^{n\times n},\ T, S \in \mathbb{C}^{n\times n}. \tag{1.7}$$

Clearly, this system of norms is admissible, with the constants in (1.4) and (1.5) satisfying

$$C_1(n,m) = C_1'(n,m) = C_2(n) = 1, \quad n, m \in \mathbb{N}.$$

For $Y \in \mathcal{V}^{s\times s}$ and $r > 0$, define a *nc ball centered at Y of radius r* as

$$B_{\rm nc}(Y,r) = \coprod_{m=1}^{\infty} B\Big(\bigoplus_{\alpha=1}^{m} Y, r\Big) = \coprod_{m=1}^{\infty} \Big\{X \in \mathcal{V}^{sm\times sm} \colon \Big\|X - \bigoplus_{\alpha=1}^{m} Y\Big\|_{sm} < r\Big\}.$$

NC balls form a basis for a topology on $\mathcal{V}_{\rm nc}$, that we call the *uniformly-open topology*, which is weaker than the disjoint union topology of Section 1.2.2. In particular, every uniformly-open nc set is right admissible.

Let $\mathcal{M}$ and $\mathcal{N}$ be modules over a unital commutative ring $\mathcal{R}$, and let $\Omega \subseteq \mathcal{M}_{\rm nc}$ be a nc set. A function $f \colon \Omega \to \mathcal{N}_{\rm nc}$ with $f(\Omega_n) \subseteq \mathcal{N}^{n\times n}$ is called a *nc function* if:

- f *respects direct sums*: $f(X \oplus Y) = f(X) \oplus f(Y)$ for all $X \in \Omega_n$, $Y \in \Omega_m$.
- f *respects similarities*: $f(TXT^{-1}) = Tf(X)T^{-1}$ for all $X \in \Omega_n$ and invertible $T \in \mathcal{R}^{n\times n}$ such that $TXT^{-1} \in \Omega_n$.

It turns out that these two conditions are equivalent to a single one: f *respects intertwinings*, namely if $XS = SY$ then $f(X)S = Sf(Y)$, where $X \in \Omega_n$, $Y \in \Omega_m$, and $S \in \mathcal{R}^{n\times m}$. This condition originates in the pioneering work of Taylor [**100**]. We denote the module of nc functions on Ω with values in $\mathcal{N}_{\rm nc}$ by $\mathcal{T}(\Omega; \mathcal{N}_{\rm nc})$.

The main idea behind the nc difference-differential calculus is to evaluate a nc function on block upper triangular matrices. Let f be a nc function on a right admissible nc set $\Omega \subseteq \mathcal{M}_{\rm nc}$ with values in $\mathcal{N}_{\rm nc}$. Let $X \in \Omega_n$, $Y \in \Omega_m$, and

$Z \in \mathcal{M}^{n\times m}$, and let $r \in \mathcal{R}$ be invertible and such that $\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$. Then it turns out that

$$f\left(\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} f(X) & \Delta_R f(X,Y)(rZ) \\ 0 & f(Y) \end{bmatrix},$$

where the mapping $Z \mapsto \Delta_R f(X,Y)(Z)$ from $\mathcal{M}^{n\times m}$ to $\mathcal{N}^{n\times m}$ is $\mathcal{R}$-linear.

We will call $\Delta = \Delta_R$ the *right nc difference-differential operator.* (The left nc difference-differential operator Δ_L can be defined analogously via evaluations on block lower triangular matrices.) Its main property is that for all $n, m \in \mathbb{N}$, $X \in \Omega_n$, $Y \in \Omega_m$, and $S \in \mathcal{R}^{m\times n}$ one has

$$Sf(X) - f(Y)S = \Delta f(Y,X)(SX - YS). \tag{1.8}$$

In particular, when $n = m$ and $S = I_n$, we have the following formula of finite differences:

$$f(X) - f(Y) = \Delta f(Y,X)(X - Y) \quad (= \Delta f(X,Y)(X - Y)). \tag{1.9}$$

Thus, the linear mapping $\Delta f(Y,Y)(\cdot)$ plays the role of a nc differential.

Let $\mathcal{R} = \mathbb{K}$ be a field of real or complex numbers. Setting $X = Y + tZ$ (with $t \in \mathbb{K}$), we obtain from (1.9) that

$$f(Y + tZ) - f(Y) = t\Delta f(Y, Y + tZ)(Z).$$

Under appropriate continuity conditions, it follows that $\Delta f(Y,Y)(Z)$ is the directional derivative of f at Y in the direction Z.

In the case of $\mathcal{M} = \mathcal{R}^d$, (1.9) turns into

$$f(X) - f(Y) = \sum_{i=1}^{d} \Delta_i f(Y,X)(X_i - Y_i), \quad X, Y \in \Omega_n, \tag{1.10}$$

where $\Delta_i f(Y,X)(C) = \Delta f(Y,X)(0,\ldots,0,C,0,\ldots,0)$, and $C \in \mathcal{R}^{n\times n}$ is at the i-th position. The linear mapping $\Delta_i f(Y,Y)(\cdot)$ plays the role of an i-th partial nc differential at the point Y.

For $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{N}_0$, $\mathcal{N}_1$ modules over a unital commutative ring $\mathcal{R}$, and $\Omega^{(0)} \subseteq \mathcal{M}_{0,\mathrm{nc}}$, $\Omega^{(1)} \subseteq \mathcal{M}_{1,\mathrm{nc}}$ nc sets, we define a *nc function of order 1* to be a function f on $\Omega^{(0)} \times \Omega^{(1)}$ so that for $X^0 \in \Omega^{(0)}_{n_0}$ and $X^1 \in \Omega^{(1)}_{n_1}$, $f(X^0, X^1)\colon {\mathcal{N}_1}^{n_0\times n_1} \to {\mathcal{N}_0}^{n_0\times n_1}$ is a linear mapping, and so that f respects, in a natural way, direct sums and similarities in each argument. A typical nc function of order 1 is $f(X,Y)(Z) = f_0(X)(Zf_1(Y))$, where $f_0 \in \mathcal{T}(\Omega^{(0)}; \mathcal{N}_{0,\mathrm{nc}})$ and $f_1 \in \mathcal{T}(\Omega^{(1)}; \mathcal{N}^*_{1,\mathrm{nc}})$. We denote the class of nc functions of order 1 by $\mathcal{T}^1(\Omega^{(0)}, \Omega^{(1)}; \mathcal{N}_{0,\mathrm{nc}}, \mathcal{N}_{1,\mathrm{nc}})$.

It turns out that for $f \in \mathcal{T}(\Omega; \mathcal{N}_{\mathrm{nc}})$ one has $\Delta f \in \mathcal{T}^1(\Omega, \Omega; \mathcal{N}_{\mathrm{nc}}, \mathcal{M}_{\mathrm{nc}})$. More generally, one can define *nc functions of order k*,

$$\mathcal{T}^k(\Omega^{(0)}, \ldots, \Omega^{(k)}; \mathcal{N}_{0,\mathrm{nc}}, \ldots, \mathcal{N}_{k,\mathrm{nc}}),$$

where $\Omega^{(0)} \subseteq \mathcal{M}_{0,\mathrm{nc}}$, $\ldots$, $\Omega^{(k)} \subseteq \mathcal{M}_{k,\mathrm{nc}}$, to be functions of $k+1$ arguments in $\Omega^{(0)}_{n_0}, \ldots, \Omega^{(k)}_{n_k}$ whose values are k-linear mappings

$${\mathcal{N}_1}^{n_0\times n_1} \times \cdots \times {\mathcal{N}_k}^{n_{k-1}\times n_k} \longrightarrow {\mathcal{N}_0}^{n_0\times n_k},$$

and that respect, in a natural way, direct sums and similarities in each argument. There is a difference-differential operator $\Delta\colon \mathcal{T}^k \to \mathcal{T}^{k+1}$, so that by iteration we obtain for $f \in \mathcal{T}(\Omega; \mathcal{N}_{\mathrm{nc}})$ that $\Delta^\ell f \in \mathcal{T}^\ell(\Omega, \ldots, \Omega; \mathcal{M}_{\mathrm{nc}}, \mathcal{N}_{\mathrm{nc}}, \ldots, \mathcal{N}_{\mathrm{nc}})$. (We view usual nc functions as nc functions of order 0: $\mathcal{T}(\Omega; \mathcal{N}_{\mathrm{nc}}) = \mathcal{T}^0(\Omega; \mathcal{N}_{\mathrm{nc}})$.)

Rather than using iterations, we can also calculate $\Delta^\ell f(X^0,\dots,X^\ell)(Z^1,\dots,Z^\ell)$ directly, by evaluating f on a block matrix having $X^0,\dots,X^\ell$ on the main diagonal, $Z^1,\dots,Z^\ell$ just above the main diagonal, and all the other block entries zero, and taking the $(1,\ell+1)$-th block entry.

The difference-differential operators Δ (and Δ_i) are obviously linear. They also satisfy a version of the chain rule (for a composition of nc functions), and — in the case when a product operation is defined — a version of the Leibnitz rule.

Using these higher order difference-differential operators, iterating the first order finite difference formula (1.9), and using the fact that direct sums are respected, leads to the following nc counterpart of the classical Taylor formula (see, e.g., [**97**] for the commutative multivariable version), that we call the *Taylor–Taylor formula* (or simply the *TT formula*), in honor of Brook Taylor and of Joseph L. Taylor. Let $f\colon \Omega\to\mathcal{N}_{\rm nc}$ be a nc function on a right admissible nc set $\Omega\subseteq\mathcal{M}_{\rm nc}$, let $s\in\mathbb{N}$, and let $Y\in\Omega_s$; then for all $m\in\mathbb{N}$, $X\in\Omega_{ms}$, and $N=0,1,\dots$ one has

$$f(X)=\sum_{\ell=0}^{N}\Big(X-\bigoplus_{\alpha=1}^{m}Y\Big)^{\odot_s\,\ell}\Delta^\ell f(\underbrace{Y,\dots,Y}_{\ell+1\ \text{times}}) + \Big(X-\bigoplus_{\alpha=1}^{m}Y\Big)^{\odot_s\,N+1}\Delta^{N+1}f(\underbrace{Y,\dots,Y}_{N+1\ \text{times}},X). \tag{1.11}$$

Here $(X-\bigoplus_{\alpha=1}^m Y)^{\odot_s\,\ell}$ denotes the ℓ-th power of $X-\bigoplus_{\alpha=1}^m Y$ viewed as a $m\times m$ matrix over the tensor algebra $\mathbf{T}(\mathcal{M}^{s\times s})$; this power is a $m\times m$ matrix over $(\mathcal{M}^{s\times s})^{\otimes\ell}$, to which the mapping $\Delta^\ell f(Y,\dots,Y)$ viewed as a linear mapping on $(\mathcal{M}^{s\times s})^{\otimes\ell}$ is applied entrywise, yielding a $m\times m$ matrix over $\mathcal{N}^{s\times s}$, i.e., an element of $\mathcal{N}^{ms\times ms}$. The remainder term has a similar meaning.

The TT formula (1.11) makes it natural to consider the infinite TT series of f around $Y\in\Omega_s$,

$$\sum_{\ell=0}^{\infty}\Big(X-\bigoplus_{\alpha=1}^{m}Y\Big)^{\odot_s\,\ell}\Delta^\ell f(Y,\dots,Y). \tag{1.12}$$

In the case $\mathcal{M}=\mathcal{R}^d$ and $Y=\mu\in\mathbb{K}^d$ is a scalar point (so $s=1$), (1.12) can be rewritten as an ordinary nc power series using the d-tuple of partial nc difference-differential operators $\Delta=(\Delta_1,\dots,\Delta_d)$ (with an obvious abuse of notation),

$$\sum_{w\in\boldsymbol{\mathcal{G}}_d}(X-\mu I_m)^w\Delta^{w^\top}f(\mu,\dots,\mu), \tag{1.13}$$

where $\Delta^{w^\top}=\Delta_{i_\ell}\cdots\Delta_{i_1}$ for a word $w=g_{i_1}\cdots g_{i_\ell}\in\boldsymbol{\mathcal{G}}_d$. There is a version of (1.13) for a general matrix center Y:

$$\sum_{w\in\boldsymbol{\mathcal{G}}_d}\Big(X-\bigoplus_{\alpha=1}^{m}Y\Big)^{\odot_s w}\Delta^{w^\top}f(Y,\dots,Y). \tag{1.14}$$

For $f\in\mathcal{T}(\Omega;\mathcal{N}_{\rm nc})$ and $Y\in\Omega_s$, the sequence of ℓ-linear mappings $f_\ell:=\Delta^\ell f(Y,\dots,Y)\colon\ (\mathcal{M}^{s\times s})^\ell\to\mathcal{N}^{s\times s}$, $\ell=0,1,\dots$, satisfies the following conditions:

$$Sf_0-f_0S=f_1(SY-YS), \tag{1.15}$$

and for $\ell = 1, \ldots$,

$$Sf_\ell(Z^1, \ldots, Z^\ell) - f_\ell(SZ^1, Z^2, \ldots, Z^\ell) = f_{\ell+1}(SY - YS, Z^1, \ldots, Z^\ell), \tag{1.16}$$

$$\begin{aligned} f_\ell(Z^1, \ldots, Z^{j-1}, Z^j S, Z^{j+1}, \ldots, Z^\ell) - f_\ell(Z^1, \ldots, Z^j, SZ^{j+1}, Z^{j+2}, \ldots, Z^\ell) \\ = f_{\ell+1}(Z^1, \ldots, Z^j, SY - YS, Z^{j+1}, \ldots, Z^\ell), \end{aligned} \tag{1.17}$$

$$f_\ell(Z^1, \ldots, Z^{\ell-1}, Z^\ell S) - f_\ell(Z^1, \ldots, Z^\ell)S = f_{\ell+1}(Z^1, \ldots, Z^\ell, SY - YS), \tag{1.18}$$

for every $S \in \mathcal{R}^{s\times s}$. (Notice, that in the case $s = 1$, conditions (1.15)–(1.18) are trivial.) Conversely, given a sequence of ℓ-linear mappings $f_\ell \colon \left(\mathcal{M}^{s\times s}\right)^\ell \to \mathcal{N}^{s\times s}$, $\ell = 0, 1, \ldots$, satisfying conditions (1.15)–(1.18), the sum of the series

$$f(X) = \sum_{\ell=0}^\infty \Big(X - \bigoplus_{\alpha=1}^m Y\Big)^{\odot_s \ell} f_\ell, \tag{1.19}$$

whenever it makes sense, defines a nc function. Similar direct and inverse statement can also be formulated in the case where $\mathcal{M} = \mathcal{R}^d$ for the series along $\boldsymbol{\mathcal{G}}_d$.

1.3. Applications of the Taylor–Taylor formula

The Taylor–Taylor formula is the main tool for the study of local behavior of nc functions. Here are some sample results.

1.3.1. Any nc function on $\mathrm{Nilp}_d(\mathcal{R}) := \mathrm{Nilp}(\mathcal{R}^d)$ (the set of jointly nilpotent d-tuples of matrices over a unital commutative ring $\mathcal{R}$) is given by a nc power series

$$f(X) = \sum_{w \in \boldsymbol{\mathcal{G}}_d} X^w \Delta^{w^\top} f(0, \ldots, 0),$$

where the sum is finite. More generally, let $Y \in \left(\mathcal{R}^{s\times s}\right)^d$. Denote by $\mathrm{Nilp}_d(\mathcal{R}, Y)$ the set of d-tuples of $sm \times sm$ matrices over $\mathcal{R}$, $m = 1, 2, \ldots$, such that $X - \bigoplus_{\alpha=1}^m Y \in \mathrm{Nilp}_d(\mathcal{R})$. Then any nc function on $\mathrm{Nilp}_d(\mathcal{R}, Y)$ can be written as in (1.14) where, again, the sum is finite.

1.3.2. Let $\mathbb{K}$ be an infinite field, and let f be a nc function on $(\mathbb{K}^d)_{\mathrm{nc}}$ such that for every n each matrix entry of $f(X_1, \ldots, X_d)$ is a polynomial in matrix entries of $X_1, \ldots, X_d$ of a uniformly (in n) bounded degree. Then f is a nc polynomial.

The condition of uniform boundedness of the degrees is necessary. However, we can get a sharp result without this condition as well: namely, f belongs to the completion of the ring of nc polynomials with respect to the decreasing sequence of ideals $\{\boldsymbol{\mathcal{I}}_n\}_{n=1}^\infty$, where $\boldsymbol{\mathcal{I}}_n$ denotes the ideal of identities [**91**] for $n \times n$ matrices (over $\mathbb{K}$ and in d variables).

1.3.3. Let $\mathbb{K} = \mathbb{C}$, let $\mathcal{V}$ and $\mathcal{W}$ be Banach spaces equipped with admissible systems of matrix norms over $\mathcal{V}$ and over $\mathcal{W}$ — see Section 1.2.2, and let $\Omega \subseteq \mathcal{V}_{\mathrm{nc}}$ be an open nc set. If a nc function f on Ω is locally bounded, then it is analytic, i.e., $f|_{\Omega_n}$ is an analytic function on $\Omega_n \subseteq \mathcal{V}^{n\times n}$ with values in $\mathcal{W}^{n\times n}$ (see, e.g., [**61, 72**] for the general theory of analytic functions between Banach spaces), and the TT series (1.12) converges to f locally uniformly. More precisely, if f is bounded on $B(\bigoplus_{\alpha=1}^m Y, \delta)$, then the series (1.12) converges absolutely and uniformly on $B(\bigoplus_{\alpha=1}^m Y, r)$ for all $r < \delta$. In the case $\mathcal{V} = \mathbb{C}^d$ and $\mathcal{W} = \mathbb{C}$, analyticity simply means that for every n each matrix entry of $f(X_1, \ldots, X_d)$ is an analytic function of the matrix entries of $X_1, \dots, X_d$; in this case it is enough to require that for every n, $f|_{\Omega_n}$ is locally bounded on slices, that is, for any fixed d-tuples of matrices $X = (X_1, \dots, X_d) \in \Omega_s$ and $Z = (Z_1, \dots, Z_d) \in (\mathbb{C}^{s\times s})^d$, $f(X + tZ)$ is bounded for all $t\colon |t| < \epsilon$, with some $\epsilon > 0$. In fact, if a nc function $f\colon \Omega \to \mathcal{W}_{\rm nc}$ (for a general vector space $\mathcal{V}$ and a Banach space $\mathcal{W}$ as before) is locally bounded on slices, then f is Gâteaux differentiable and its TT series converges.

1.3.4. Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces, and let $\Omega \subseteq \mathcal{V}_{\rm nc}$ be a uniformly-open nc set (see Section 1.2.3). If a nc function f on Ω is locally bounded in the uniformly-open topology, then the TT series (1.12) converges to f locally uniformly in that topology. More precisely, if f is bounded on $B_{\rm nc}(Y, \delta)$, then the series converges absolutely and uniformly on $B_{\rm nc}(Y, r)$ for all $r < \delta$. f is called a uniformly analytic nc function.

In the case where $\mathcal{V} = \mathbb{C}^d$ with some operator space structure given by a sequence of norms $\|\cdot\|_n$, it is further true that the series (1.13) — without grouping together terms of the same degree into homogeneous nc polynomials — converges absolutely and uniformly on somewhat smaller sets, namely on every *open nc diamond about* Y,

$$\diamondsuit_{\rm nc}(Y, r) := \coprod_{m=1}^{\infty} \left\{X \in \Omega_{sm}\colon \sum_{j=1}^{d} \|e_j\|_1 \left\|X_j - \bigoplus_{\alpha=1}^{m} Y_j\right\| < r\right\}$$

with $r < \delta$; here $e_1, \dots, e_d$ denote the standard basis for $\mathbb{C}^d$.

1.3.5. TT series expansions allow us to reformulate some results that were originally established for nc power series as results about nc functions. This includes, in particular, realizations of nc power series as transfer functions of noncommutative multidimensional systems (systems with evolution along the free monoid $\boldsymbol{\mathcal{G}}_d$[6]). These systems were first studied in [**20**] in the conservative infinite-dimensional setting, in the context of operator model theory for row contractions due mostly to Popescu [**81, 82, 83, 84**] and of representation theory of the Cuntz algebra [**27, 32**]. A comprehensive study of nc realization theory, in both finite-dimensional and infinite-dimensional setting, appears in [**15, 17, 16**]; these papers give a unified framework of structured nc multidimensional linear systems for different kinds of realization formulae. We also mention the paper [**18**] where an even more general class of nc systems (given though in a frequency domain) was described and the corresponding dilation theory was developed.

We give (a somewhat imprecise version of) the main conservative realization theorem from [**17**] restated as a theorem about nc functions. Let $Q_1, \dots, Q_d$ be $p\times q$ complex matrices with a certain additional structure that comes from a bipartite graph. Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces, and let $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ be the space of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$. The corresponding *nc Schur–Agler class* consists of contraction-valued $f \in \mathcal{T}(B_{\rm nc}(0,1); \mathcal{L}(\mathcal{U}, \mathcal{Y})_{\rm nc})$, where $B_{\rm nc}(0,1) \subseteq (\mathbb{C}^d)_{\rm nc}$, and $\mathbb{C}^d$ is equipped with the operator space structure defined by the system of matrix norms

$$\|X\|_n = \|Q(X)\|_{\mathcal{L}(\mathbb{C}^n\otimes\mathbb{C}^q, \mathbb{C}^n\otimes\mathbb{C}^p)},$$

where $Q(X) := X_1 \otimes Q_1 + \cdots + X_d \otimes Q_d$. Then the following are equivalent:

[6] Referred to often as systems with evolution along the free semigroup.

(1) f is in the nc Schur–Agler class.
(2) There exists an *Agler decomposition*

$$I_{\mathbb{C}^n\otimes\mathcal{Y}} - f(X)f(Y)^* = H(X)(I_{\mathbb{C}^n\otimes\mathbb{C}^p\otimes\mathcal{G}} - (Q(X)\otimes I_{\mathcal{G}})(Q(Y)^*\otimes I_{\mathcal{G}}))H(Y)^*$$

for an auxiliary Hilbert space $\mathcal{G}$ and some $H \in \mathcal{T}(B_{\rm nc}(0,1); \mathcal{L}(\mathbb{C}^p\otimes\mathcal{G},\mathcal{Y})_{\rm nc})$ which is bounded on every nc ball $B_{\rm nc}(0,r)$ of radius $r < 1$.
(3) There exists a conservative realization:

$$f(X) = I_n \otimes D + (I_n \otimes C)(I_{\mathbb{C}^n\otimes\mathbb{C}^p\otimes\mathcal{G}} - Q(X)(I_n\otimes A))^{-1}Q(X)(I_n\otimes B).$$

Here the colligation matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p\otimes\mathcal{G} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q\otimes\mathcal{G} \\ \mathcal{Y} \end{bmatrix}$$

is unitary.

(The statement in [**17**] is considerably more precise in identifying the state spaces $\mathbb{C}^p\otimes\mathcal{G}$ and $\mathbb{C}^q\otimes\mathcal{G}$ of the unitary colligation in terms of the bipartite graph generating the matrices $Q_1,\ldots,Q_d$.) This is the nc version of the commutative multidimensional realization theorem of [**10**] and [**13**] for the linear nc function Q defining the domain of f, which in turn generalizes Agler's seminal result on the polydisc [**4**], see also [**19**]. Such generalized transfer-function realizations occur also in the context of the generalized Hardy algebras of Muhly–Solel (elements of which are viewed as functions on the unit ball of representations of the algebra—see [**68, 70, 12**]) and of the generalized Schur–Agler class associated with an admissible class of test functions (see [**34, 35**]).

We note that the formulation of the realization theorem for the nc Schur–Agler class given above suggests generalizations to an arbitrary finite- or infinite-dimensional operator space (instead of $\mathbb{C}^d$ equipped with the system of norms associated with the linear nc function Q) and to more general nc domains (a nc counterpart of the commutative domains of [**10**] and [**13**], where the defining nc function Q is a nc polynomial). In the finite-dimensional case such a generalization has been carried out recently in [**5**]; we refer to both this paper and [**6, 7**] for applications.

1.4. An overview

In Chapter 2, we introduce nc functions and their difference-differential calculus. In Section 2.1, we define nc spaces, nc sets, and nc functions, and prove that respecting direct sums and similarities is equivalent to respecting intertwinings (Proposition 2.1). In Section 2.2, we introduce the right and left difference-differential operators Δ_R and Δ_L via evaluation of nc functions on block upper triangular matrices and then taking the upper right (left) corner block of the value, and show that $\Delta_R f(X,Y)(Z)$ and $\Delta_L f(X,Y)(Z)$ are linear in Z (Propositions 2.4 and 2.6). We proceed to formulate basic nc calculus rules in Section 2.3 and to establish the first-order finite difference formulae in Section 2.4. Then we prove in Section 2.5 that $\Delta_R f(X,Y)(Z)$ and $\Delta_L f(X,Y)(Z)$ respect direct sums and similarities in X and in Y (in the terminology of Chapter 3, $\Delta_R f$ and $\Delta_L f$ are nc functions of order 1). We conclude Chapter 2 by discussing in Section 2.6 directional and (in the special case $\mathcal{M} = \mathcal{R}^d$) partial nc difference-differential operators.

In Chapter 3, we first introduce higher order nc functions, and then extend the difference-differential operators to these functions, which leads to a higher order calculus for ordinary and higher order nc functions. In Section 3.1, we define nc functions of order k and introduce the classes $\mathcal{T}^k$ of those functions (so that the original nc functions belong to the class $\mathcal{T}^0$). The values of nc functions of order k are k-linear mappings of matrices over modules, which can also be interpreted as elements of tensor products of $k+1$ matrix modules (Remarks 3.4 and 3.5). This allows us to define a natural mapping $\mathcal{T}^k \otimes \mathcal{T}^\ell \to \mathcal{T}^{k+\ell+1}$, $k, \ell = 0, 1, \ldots$, and consequently — a mapping from the tensor product of $k+1$ classes $\mathcal{T}^0$ (perhaps over different operator spaces) to $\mathcal{T}^k$ (Remark 3.6) and interpret a higher order nc function in terms of nc functions of order 0. We then extend in Section 3.2 the right difference-differential operator Δ_R to a mapping of classes $\mathcal{T}^k \to \mathcal{T}^{k+1}$, $k = 0, 1, \ldots$. In this definition, the last argument of $f(X^0, \ldots, X^k)$, where f is a nc function of order k, is taken in the form of a block matrix $X^k = \begin{bmatrix} X^{k\prime} & Z \\ 0 & X^{k\prime\prime} \end{bmatrix}$, and then we take the upper right corner block of the matrix value. The extended operator Δ_R is also linear as a function of Z (Propositions 3.8 and 3.9). Starting from this point, we concentrate on the right version of calculus, since the left version can be developed analogously or simply obtained from the right one by symmetry.

The iterated operator $\Delta_R^\ell \colon \mathcal{T}^k \to \mathcal{T}^{k+\ell}$ can also be computed directly, by evaluating a nc function on block upper bidiagonal matrices and then taking the upper right corner block of its matrix value (Therems 3.11 and 3.12). The definition of the extended operator Δ_R becomes more transparent when we interpret higher order nc functions using tensor products of nc functions of order 0 (Remark 3.17).

We can also take X^j in $f(X^0, \ldots, X^k)$ in the block upper triangular form, which leads to operators ${}_j\Delta_R \colon \mathcal{T}^k \to \mathcal{T}^{k+1}$, $j = 0, 1, \ldots$ (so that $\Delta_R = {}_k\Delta_R$), that also extend the original operator $\Delta_R \colon \mathcal{T}^0 \to \mathcal{T}^1$ (Remark 3.18). In Section 3.3, we establish the first order difference formulae for higher order nc functions (Theorem 3.19) and its generalized version for the case of matrices X and Y where we evaluate the higher order nc function in these formulae, of possibly different size (Theorem 3.20). Similar formulae are valid for operators ${}_j\Delta_R$ (Remark 3.22).

In Section 3.4, we show that for a nc function $f \in \mathcal{T}^k$ which is k times integrable, i.e., can be represented as $\Delta_R^k g$ with $g \in \mathcal{T}^0$, one has ${}_i\Delta_R\, {}_j\Delta_R f = {}_j\Delta_R\, {}_i\Delta_R f$, for all $i, j = 0, \ldots, k$. Finally, we discuss in Section 3.5 higher order directional nc difference-differential operators; in particular, when the underlying module is $\mathcal{R}^d$, we discuss higher order partial difference-differential operators Δ_R^w, $w \in \boldsymbol{\mathcal{G}}_d$.

In Chapter 4, we establish the Taylor–Taylor formula (Theorem 4.1 or more general Theorem 4.2; in the case $\mathcal{M} = \mathcal{R}^d$ — Corollary 4.4 and, in tensor product interpretation of $\Delta_R^{w^\top} f$, Theorem 4.6). We show in Remark 4.3 that the coefficients in the TT formula, the multilinear mappings $f_\ell = \Delta_R f(Y, \ldots, Y)$, $\ell = 0, 1, \ldots$, satisfy conditions (1.15)–(1.18) that we mentioned in Section 1.2. The counterpart of these condition for the sequence $f_w = \Delta_R^{w^\top} f(Y, \ldots, Y)$, $w \in \boldsymbol{\mathcal{G}}_d$, in the case $\mathcal{M} = \mathcal{R}^d$ is established in Remark 4.5.

As a first application of the TT formula, we describe in Chapter 5 nc functions on the set $\operatorname{Nilp}(\mathcal{M})$ (or, more generally, on $\operatorname{Nilp}(\mathcal{M}; Y)$) of nilpotent matrices (or nilpotent matrices about Y) over a module $\mathcal{M}$ — see Section 1.2.1 for the definition. In Section 5.1 we show that every such a nc function is a sum of its TT series (Theorems 5.2, 5.4, 5.6, and 5.8). Then we show that the coefficients of a nc power series representing the nc function are uniquely determined. In fact, this series is unique and is equal to the corresponding TT series (Theorems 5.9 and 5.10).

In Section 5.2 we show that, conversely, given a sequence of multilinear mappings $f_\ell\colon (\mathcal{M}^{s\times s})^\ell \to \mathcal{N}^{s\times s}$, $\ell = 0, 1, \ldots$, satisfying conditions (1.15)–(1.18), the sum of the nc power series $\sum_{\ell=0}^{\infty}(X - \bigoplus_{\alpha=1}^{m} Y)^{\odot_s \ell} f_\ell$ is a nc function on $\mathrm{Nilp}(\mathcal{M}; Y)$ with values in $\mathcal{N}_{\mathrm{nc}}$ (Theorem 5.13), and a version of this result, Theorem 5.15, for the case $\mathcal{M} = \mathcal{R}^d$ and a sequence of multilinear mappings $f_w\colon (\mathcal{M}^{s\times s})^\ell \to \mathcal{N}^{s\times s}$, $w \in \boldsymbol{\mathcal{G}}_d$, satisfying the conditions analogous to (1.15)–(1.18). These results are algebraic in their nature, since the sum of the corresponding series is finite at every point of $\mathrm{Nilp}(\mathcal{M}; Y)$.

In Chapter 6, we give some other algebraic applications of the TT formula. We prove that a nc function on $(\mathbb{K}^d)_{\mathrm{nc}}$, with $\mathbb{K}$ an infinite field, which is polynomial in matrix entries when evaluated on $n \times n$ matrices, $n = 1, 2, \ldots$, of bounded degree, is necessarily a nc polynomial (Theorem 6.1). The degree boundedness condition is essential and cannot be omitted (Example 6.3). However, if we do not require this condition, then the following is true: an arbitrary nc function on $(\mathbb{K}^d)_{\mathrm{nc}}$, which is polynomial of degree M_n in matrix entries when evaluated on $n \times n$ matrices, is a sum of a nc polynomial and an infinite series of homogeneous polynomials f_j vanishing on all d-tuples of $n \times n$ matrices for $j > M_n$ (Theorem 6.4). We also obtain a generalization of Theorem 6.1 to nc functions on (much more general) nc spaces, which are polynomial on slices (Theorem 6.8).

In Chapter 7 we study analytic nc functions and the convergence of their TT series. We consider three different topologies on a nc space $\mathcal{V}_{\mathrm{nc}}$ (over a complex vector space $\mathcal{V}$), or three types of convergence of nc power series, and, correspondingly, three types of analyticity of nc functions: finitely open topology and analyticity on slices, norm topology and analyticity on $n \times n$ matrices over $\mathcal{V}$ for every $n = 1, 2, \ldots$, and uniformly-open topology and uniform (in n) analyticity. The main feature of analytic nc functions established in Chapter 7 is that local boundedness implies analyticity, in each of the three settings. In the first part of Section 7.1, we introduce the finitely open topology on $\mathcal{V}_{\mathrm{nc}}$ and prove that a nc function which is locally bounded on slices is analytic on slices and discuss the convergence of the corresponding TT series (Theorem 7.2). In the second part of Section 7.1, we show that a nc function which is locally bounded (with respect to the norm topology on $\mathcal{V}^{n\times n}$, for every n) is analytic, with the TT series convergent uniformly and absolutely on certain open complete circular nc sets about the center $Y \in \Omega_s$ (Theorem 7.4) or on open balls centered at Y (Corollary 7.5). We also establish the uniqueness of the convergent nc power series expansions, in both of the above topologies (Theorem 7.9). In Section 7.2, we introduce and study the uniformly-open topology on a nc space $\mathcal{V}_{\mathrm{nc}}$ over an operator space $\mathcal{V}$. In Section 7.3, we show that a uniformly locally bounded nc function is uniformly analytic, and its TT series converges uniformly and absolutely on certain uniformly-open complete circular nc sets about Y (Theorem 7.21) or on certain uniformly-open matrix circular nc sets (Theorem 7.23) or on uniformly-open nc balls (Corollary 7.26). We also prove a version of this result for the case $\mathcal{V} = \mathbb{C}^d$ and the TT series convergent along $\boldsymbol{\mathcal{G}}_d$ (Theorem 7.29 and Corollary 7.31). In Section 7.4 we introduce and study analytic higher order nc functions, also in the three different settings. The main results are analogous to those in Sections 7.1 and 7.3.

In Chapter 8, we study the convergence of nc power series in the three different topologies, as in Chapter 7, to an (analytic in the corresponding sense) nc function. The results are the converse of the results of Sections 7.1 and 7.3. We discuss the convergence of nc power series in the finitely open topology (resp., in the norm topology and in the uniformly-open topology) in Section 8.1 (resp., 8.2 and 8.3). In each topology, we have Cauchy–Hadamard type estimates for radii of convergence and sharp estimates for the size of convergence domains of various shapes. We also characterize the maximal nc sets where the sum of the series is an analytic on slices (resp., analytic and uniformly analytic) nc function and give many examples illustrating the differences between various types of convergence domains.

In Chapter 9, we define and study so-called direct summands extensions of nc sets and nc functions. If a nc set Ω is invariant under similarities, then one can extend Ω to a larger nc set, $\Omega_{\text{d.s.e.}}$, which contains, together with a matrix which is decomposable into a direct sum of matrices, every direct summand of the decomposition. This extension preserves many properties of Ω (Proposition 9.1). We then can extend a nc function f on Ω to a nc function $f_{\text{d.s.e.}}$ on $\Omega_{\text{d.s.e.}}$, which inherits most important properties of f, in particular, analyticity (Proposition 9.2). Similarly, we define and establish the properties of the direct summands extensions of higher order nc functions (Proposition 9.3). We also define and study direct summands extensions of sequences f_ℓ, $\ell = 0, 1, \ldots$ (resp., f_w, $w \in \boldsymbol{\mathcal{G}}_d$) of multilinear mappings satisfying conditions (1.15)–(1.18) (resp., their counterparts in the case $\mathcal{M} = \mathcal{R}^d$) in Proposition 9.4 (resp., in Proposition 9.6).

As we mentioned in Section 1.2, we need a nc set Ω to be right (left) admissible in order to define right (left) difference-differential operators via evaluations of nc functions f on Ω at block upper (lower) triangular matrices. However, for a (say, upper) block triangular matrix $\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}$ we need to scale Z, so that $\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega$. Then we can define $\Delta_R f(X,Y)(rZ)$ and extend this definition to arbitrary Z by linearity of $\Delta_R f(X,Y)(\cdot)$. Although the properties of Δ_R (Δ_L) can be established using these scalings, this creates cumbersome technicalities in the proofs. To bypass those technicalities, we define the so-called similarity invariant envelope $\widetilde{\Omega}$ of Ω, which is the smallest nc set containing Ω and invariant under similarities. Then it turns out that $\widetilde{\Omega}$ is also invariant under formation of block triangular matrices (Proposition A.2). Our next step is to extend a nc function f on Ω to a nc function $\widetilde{f}$ on $\widetilde{\Omega}$. Such an extension is unique (Proposition A.3). A similar extension is constructed for higher order nc functions (Proposition A.5). It turns out that $(\Delta_R \widetilde{f})|_{\Omega\times\Omega} = \Delta_R f$ (Remark 2.5), which allows us to prove various statements about the difference-differential operators throughout the monograph assuming, without loss of generality, that the underlying nc set Ω is similarity invariant. The results on similarity invariant envelopes and the corresponding extensions of nc functions are presented in Appendix A, which is written in collaboration with Shibananda Biswas.

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