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Preface

What does it mean to *classify* the equivalence classes of some equivalence relation?

We have some tangible space whose points are very definite. Let X be the space. Maybe it is the reals. Or the complex numbers. Or perhaps all groups with underlying set $\mathbb{N}$. In any case, a concrete object.

In addition there is an equivalence relation E and we consider the quotient X/E. When should we judge this quotient object consisting of all equivalence classes to be comprehensible? When should we allow that the set of all equivalence classes, $\{[x]_E : x \in X\}$, is classifiable?

There is no absolute agreement on what constitutes a good classification theorem. It is necessarily a vague concept. But even granting its vagueness, there are probably some attempts one could dismiss out of hand.

For instance, assigning the equivalence class of x as a complete invariant for each point in the space is hardly satisfactory, since it provides us with no progress at all. The complete invariants should be objects we feel to be reasonably well understood – or at least, less mysterious than the equivalence classes with which we began.

Similarly we do have some standards concerning how the invariants can be produced. Simply appealing to the axiom of choice to well order the equivalence classes of E, and then using some corresponding well order of $\mathbb{R}$ to assign real numbers as complete invariants does not constitute a satisfactory system of classification, even though the complete invariants in this case are indeed well understood. The system of classification should assign invariants to the points in the space X based on their intrinsic properties; even though the properties we may use could be highly subtle or extremely complex, we would have much greater respect for a system of classification that is fantastically difficult than one that just pulls down the axiom of choice and then goes home to bed.

In general terms a complete classification of E should consist of a *reasonably intrinsic* or *definable* function

$$\theta : X \to I$$

from X to a *reasonably well understood* collection of invariants I so that for all x and y in X

$$xEy \Leftrightarrow \theta(x) = \theta(y).$$

That much is a platitude. Beyond this there is great disagreement.

EXAMPLE 0.1. **Low dimensional topology** For compact orientable surfaces a complete invariant can be obtained by simply counting the number of handles, and moreover, this invariant can be produced in a *recursive* or *computable* fashion from a finite triangularization of the manifold; for non-orientable surfaces the structure theorem is more complicated, but again the invariant can be taken to be a finite object – indeed it can be represented by a natural number – and may be computed from a triangularization using a purely finite search. ([**12**], [**46**].) In higher dimensions it is known from [**64**] that the isomorphism relation is not computable in the sense of Turing machines, and this is *sometimes* taken by topologists as indicating that there is no satisfactory system of complete invariants for higher dimension manifolds.

This is an extremely restrictive definition. Here a classification is a computable function

$$\theta : X \to \mathbb{N}$$

where X is a collection of finite triangularizations of manifolds so that $x, y \in X$ code homeomorphic manifolds if and only if $\theta(x) = \theta(y)$. This is the most stringent notion of classification: The invariants are finite, the process by which we assign them finitary.

EXAMPLE 0.2. **Linear algebra** A rather different example is suggested by linear algebra. We may reasonably regard two unitary operators over a finite dimensional (complex) Hilbert space as somehow equivalent if there is a third that conjugates them – so in this sense we obtain an orbit equivalence relation on the space of all unitary operators on $(\mathbb{C}^n, \cdot)$. A complete invariant for an operator is given by its finite set of eigenvalues, considered up to multiplicity, and it is always possible to encode finitely many complex numbers by a single real; thus we can assign in a Borel fashion to each element of U_n, the group of unitary operators over $(\mathbb{C}^n, \cdot)$, a real number as a complete invariant.

Equivalence relations which in a Borel fashion allow real numbers as complete invariants are known as *smooth* or *tame*; thus E_G on X is smooth if there is a Borel

$$\theta : X \to \mathbb{R}$$

that reduces E_G to equality on $\mathbb{R}$, in the sense

$$xE_G y \Leftrightarrow \theta(x) = \theta(y).$$

[**30**], like [**14**] that it followed, takes *classifiable* to mean smooth.

EXAMPLE 0.3. **Ergodic theory** Consider the classification problem for measure preserving transformations of the unit interval. It is natural to say that $\pi_1, \pi_2 : [0,1] \to [0,1]$ are *equivalent* or *isomorphic* if there is some measure preserving bijection $\sigma : [0,1] \to [0,1]$ with

$$\sigma \circ \pi_1 \circ \sigma^{-1} = \pi_2 \text{ a.e.}$$

This equivalence relation arises from a group action – the action of this group on itself by conjugation. In two special cases there are classification theorems.

Ornstein's classification of Bernoulli shifts in [**71**] provides a proof that isomorphism on this class of measure preserving transformations is smooth. One can assign to each Bernoulli shift its *entropy* – a real number that completely classifies a Bernoulli shift up to isomorphism. This is one of the most celebrated theorems of ergodic theory, but it is not true that the *only* notion of classification in ergodic theory is that of being smooth or reducible to the equality relation on $\mathbb{R}$. A rather more generous notion of classification is suggested by a classic paper of Halmos and von Neumann.

In [**29**] Halmos and von Neumann show that for *discrete spectrum* measure preserving transformations we can assign a countable collection $\{c_i(\pi) : i \in \mathbb{N}\}$

of complex numbers that completely describe the equivalence class of π. This assignment is indeed natural, since it arises from taking the eigenvalues of the form $\lambda \in \mathbb{C}$ for which there is some non-zero $f \in L^2([0,1])$ with $f \circ \pi = \lambda f$ a.e.

The notion of classification here cannot be reduced to that of 0.2. For instance [**20**] shows that there is no reasonable method for representing *countably infinite* sets of real or complex numbers by single points in $\mathbb{R}$; indeed without appeal to the axiom of choice we may find it impossible to produce *any* injection from $\mathcal{P}_{\aleph_0}(\mathbb{R})$ (the set of all countable collections of reals) to $\mathbb{R}$. The conjugacy relation on discrete spectrum measure preserving transformations is non-smooth, and yet the perspective of say [**82**] would uphold this as a complete classification for discrete spectrum measure preserving transformations.

Example 0.4. **Locally compact group actions** In [**49**] Kechris proves that the orbit equivalence relations induced by locally compact Polish groups are all reducible to countable equivalence relations. If we have locally compact G acting continuously on Polish X, with orbit equivalence relation E_G, then we can find a Borel equivalence relation F all of whose equivalence classes are countable such that we may assign to each point $x \in X$ some corresponding $\theta(x)$ so that

$$x_1 E_G x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

This result can be viewed as a classification theorem for orbit equivalence relations induced by locally compact group actions. We may assign to each $x \in X$ the countable set $\{y : yF\theta(x)\}$ to obtain an invariant similar in structure to the Halmos-von Neumann spectral invariants.

Example 0.5. **Point set topology** The Cantor-Bendixson derivation as described in [**52**] can be used to provide a classification for countable compact metric spaces. At the first stage we remove the isolated points. At the next we remove the isolated points from the remaining space, and so on, through however many countable ordinals as are needed. This analysis provides a complete invariant of the space consisting of two parts: The ordinal length of this process, along with the number of points left standing before the termination at the final stage. Two countable compact metric spaces will be homeomorphic if and only if their Cantor-Bendixson derivations require the same ordinal number of steps and at the penultimate moment they share the same finite number of points remaining.

Example 0.6. **Abelian group theory** In [**21**] ordinals also enter stage in the famed *Ulm invariants* from abelian group theory. In essence the Ulm invariants are bounded subsets of $\aleph_1$, the first uncountable ordinal, that completely describe the isomorphism type of a countable *torsion* abelian group.

Example 0.7. **Topological dynamics** The authors of [**24**] classify so called *minimal Cantor systems* up to *strong orbit equivalence* by assigning countable ordered abelian groups. Two continuous

$$\varphi_1 : X_1 \to X_1,$$
$$\varphi_2 : X_2 \to X_2$$

which are *minimal* in the sense of having no non-trivial closed invariant sets and are *Cantor* in the sense of X_1, X_2 being compact, uncountable, and zero-dimensional metric spaces, are said to be *strong orbit equivalent* if there is a homeomorphism $F : X_1 \to X_2$ which respects the orbits structure set wise and with the resulting conjugation suffering at most a single discontinuity – so that if $m : X_1 \to \mathbb{Z}$, $n : X_2 \to \mathbb{Z}$ are defined by

$$F \circ \varphi_1(x) = \varphi_2^{n(x)}(F(x)),$$
$$F \circ \varphi_1^{m(x)}(x) = \varphi_2(F(x)),$$

then n, m are continuous on $X \setminus \{x_0\}$ for some $x_0 \in X$.

The countable group associated to (φ, X) itself arises as a homomorphic image of $C(X, \mathbb{Z})$, the space of all continuous maps from X to $(\mathbb{Z}, +)$ with the product topology. As important as the details of the construction may be it is also remarkable that there is *any* reasonable way to assign a countable structure as a complete invariant.

EXAMPLE 0.8. **Stone spaces** Perhaps this preceding example is reminiscent of the duality theorem of [**75**] for compact separable zero-dimensional Hausdorff spaces. To each such space we can assign a countable Boolean algebra with two spaces homeomorphic if and only if there exists an algebraic isomorphism between the Boolean algebras. Similarly Pontryagin duality, as it is found in [**31**], allows us to completely classify compact abelian metric groups by their countable discrete dual groups.

While it is true that complete invariants are provided this is not to say that these dualities are only or even primarily theorems of classification.

EXAMPLE 0.9. **Hom**$^+([0,1])$ Let $\mathrm{Hom}^+([0,1])$ be the group of all orientation preserving ($\pi(0) = 0$, $\pi(1) = 1$) homeomorphisms of the unit interval. The natural equivalence relation is that of conjugation: Two homeomorphisms, π_1, π_2 are equivalent if a homeomorphic "relabeling" of the underlying space transforms one to the other, so that there is some $\sigma \in \mathrm{Hom}^+([0,1])$ with

$$\pi_1 = \sigma^{-1} \circ \pi_2 \circ \sigma.$$

Parallel to 0.3, the equivalence relation arises by the self-action of $\mathrm{Hom}^+([0,1])$ through conjugation.

It is sometimes felt that homeomorphisms of the unit interval are completely understood since we may represent each transformation *symbolically* by indicating the maximal regions on which we have either $\pi(x) > x$, $\pi(x) = x$, or $\pi(x) < x$. This can be made more precise by providing a classification of elements of $\mathrm{Hom}^+([0,1])$ by *countable models*. We naturally assign to each $\pi \in \mathrm{Hom}^+([0,1])$ a countable model $\mathcal{M}(\pi)$ such that for all $\pi_1, \pi_2 \in \mathrm{Hom}^+([0,1])$

$$\exists \sigma \in \mathrm{Hom}^+([0,1])(\sigma \circ \pi_1 \circ \sigma^{-1} = \pi_2) \Leftrightarrow \mathcal{M}(\pi_1) \cong \mathcal{M}(\pi_2).$$

The model $\mathcal{M}(\pi)$ consists of the maximal open intervals on which π displays one of the three possible behaviors indicated above. The language of $\mathcal{M}(\pi)$ encodes the linear ordering between these intervals and indicates which of the three possibilities hold. We will have an ordering $\leq$, and predicates P_-, P_+, and $P_=$. For $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$ maximal open intervals on which the behavior of π is unvarying, we have:

$$I_1 \leq^{\mathcal{M}(\pi)} I_2 \Leftrightarrow a_1 \leq a_2;$$
$$P_-^{\mathcal{M}(\pi)}(I_1) \Leftrightarrow \forall x \in I_1(\pi(x) < x);$$
$$P_+^{\mathcal{M}(\pi)}(I_1) \Leftrightarrow \forall x \in I_1(\pi(x) > x);$$
$$P_=^{\mathcal{M}(\pi)}(I_1) \Leftrightarrow \forall x \in I_1(\pi(x) = x).$$

The outcome is similar for the homeomorphism group of Cantor space – $(\{0,1\})^{\mathbb{N}}$ – in the product topology. Since the homeomorphism group of the Cantor space is isomorphic to a closed subgroup of the infinite symmetric group it follows from [**4**] (see 2.39 below) that we may classify these homeomorphisms by countable models.

This is more than enough examples to be impressed by the diversity. But despite the variation, there are some common themes.

In the above we have natural numbers, real numbers, countable sets of complex numbers, countable ordinals, countable sets of countable ordinals, and various kinds of countable structures considered up to isomorphism being used as complete invariants. The connection between these examples is that in *every one* we may take a countable structure as a complete invariant; as in §2.3 below, we may code complex numbers, countable sets of complex numbers, countable ordinals, and countable sets of countable ordinals by appropriately chosen models. This suggests a notion of classification found at the opposing end of the spectrum to that of 0.1 and which is extreme in its generosity.

QUESTION 0.10. Let E be an equivalence relation on a space X. When can we assign countable *models* or *structures* considered up to isomorphism as complete invariants?

Recall that HC is the collection of all hereditarily countable sets, and may be defined as the smallest collection of sets containing the natural numbers and closed under the operation of taking a countable subset. These therefore include all countable subsets of $\aleph_1$, all countable sets of subsets of $\mathbb{N}$, and – appropriately understood – all real numbers, all countable sets of real numbers, and so on.

In virtue of the Scott analysis of [**59**] we may equivalently ask:

QUESTION 0.11. For which equivalence relations can we assign elements of HC as complete invariants?

[**4**] allows one more reworking of the question:

QUESTION 0.12. Let E be an equivalence relation on a space X. When can we find a Polish space Y on which the infinite symmetric group S_∞ acts continuously and a reasonable function $\theta : X \to Y$ so that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \exists g \in S_\infty (g \cdot \theta(x_1) = \theta(x_2))?$$

Of course we clearly need to have an assumption that the function θ or the assignments of models or HC sets be *reasonable*. On the whole I will take *reasonable* to mean Borel in an appropriate Borel structure, the technicalities of which are addressed in §2.1, §2.2, §2.3, and §3.1.[1] But I should stress that relatively little change occurs if we extend to much broader classes of functions and far more generous methods of reduction. A point made in the course of §6.2 and §9.1-2 is that if there is any remotely definable assignment of countable models or HC sets in the context of Polish group actions, then we may find a reduction that is at worst only slightly more complicated than Borel.

This monograph can be viewed as part of a broad project to understand *effective* cardinalities in the sense raised by Luzin in [**62**], in the sense which reappears

[1]In the context of Borel reduction, 0.10 and 0.12 are known to be equivalent problems from 2.7.3 of [**4**]. Provided we are willing to countenance reductions somewhat more complicated than Borel the Scott analysis of [**59**] shows both equivalent to 0.11. §6.1 and §9.1 return to these points.

briefly in the opening parts of [**11**], but which finds its most forthright statement in modern works of descriptive set theory such as [**60**] and [**50**]. Here the concept is to calculate cardinalities using only functions that lay some claim on being *reasonable* or *definable*. Formally $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ both have cardinality $2^{\aleph_0}$; a well ordering of $\mathbb{R}$ will enable us to find a bijection. *Effectively* $\mathbb{R}$ is smaller than $\mathbb{R}/\mathbb{Q}$, since we may find reasonable injections of $\mathbb{R}$ into $\mathbb{R}/\mathbb{Q}$ (2.59, 2.63 below), but not the converse (3.8).

Papers such as [**14**], [**63**], and [**25**] have previously addressed the question of which naturally occuring objects have effective cardinality no greater than that of $\mathbb{R}$. In a great many specific instances the answer has been determined, and one finds in [**30**] and [**14**] a kind of theory, recounted in §3.1 and §7.1, regarding when a reduction exists and why in certain cases it cannot. In turn this monograph tries to understand which objects have effective cardinality below HC and develop a parallel theory of why some do not.

We will only be concerned with the case that E arises from a Polish group action. Admittedly this may seem very restrictive, and it would certainly be desirable to have an analysis for all Borel or even $\underset{\sim}{\Sigma}^1_1$ equivalence relations. On the other hand most naturally occurring examples can be subsumed under an appropriately chosen Polish group action, and the impression left by chapter 8 is that the exceptions are somehow pathological. This is the point of question 10.9 near the end of the book, and even if that conjecture should fail it seems plausible that a similar outlook is justified.

§3 isolates a dynamical property for analyzing which Polish group actions allow reduction to countable models:

DEFINITION 0.13. Let G be a topological group acting on a space X. The action is said to be *turbulent* if:

(i) every orbit is dense;

(ii) every orbit is meager;

(iii) for all $x, y \in X$, $U \subset X$, $V \subset G$ open with $x \in U$, $1 \in V$, there exists $y_0 \in [y]_G =_{df} G \cdot y$ and $(g_i)_{i \in \mathbb{N}} \subset V$, $(x_i)_{i\in\mathbb{N}} \subset U$ with

$$x_0 = x,$$

$$x_{i+1} = g_i \cdot x_i,$$

and for some subsequence $(x_{n(i)})_{i\in\mathbb{N}} \subset (x_i)_{i\in\mathbb{N}}$

$$x_{n(i)} \to y_0.$$

Turbulence is a sufficient condition for the orbit equivalence relation of a Polish group to refuse classification by countable structures; further: for a turbulent orbit equivalence relation any function assigning countable models up to isomorphism as invariants must be constant on a comeager set. (3.18, 3.19)

A transfinite analysis shows turbulence to be necessary for non-classification:

THEOREM 0.14. *Let G be a Polish group acting continuously on a Polish space X. Exactly one of the following holds:*

1. *the orbit equivalence relation E^X_G is reasonably reducible to isomorphism on countable models;*
2. *there is a turbulent Polish G-space Y and a continuous G-embedding from Y to X.*

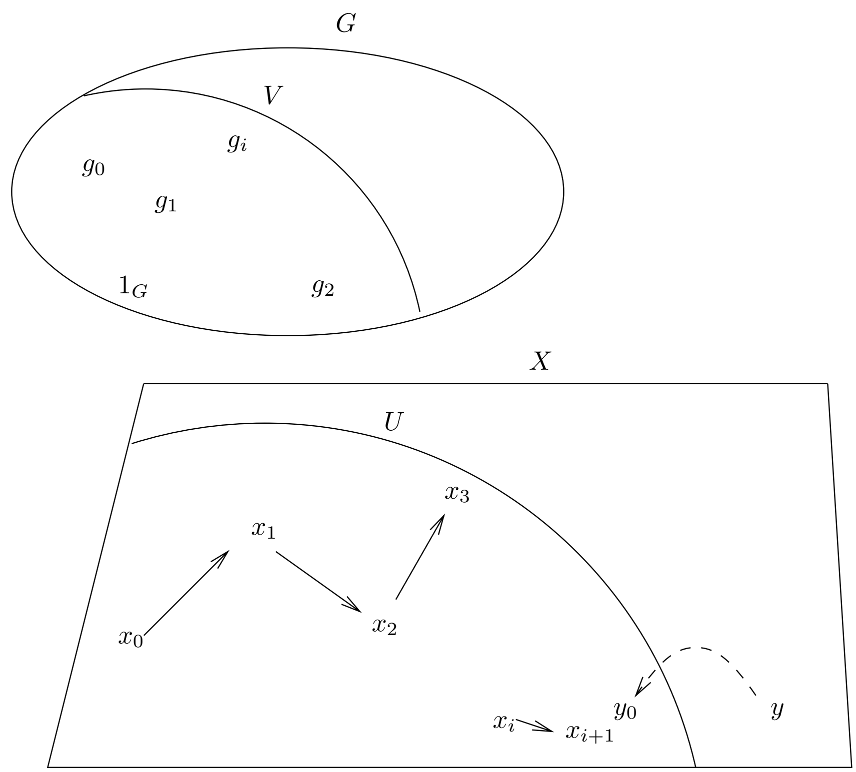

FIGURE 0.1. Turbulence

Here the notion of reduction is somewhat more complicated than Borel. In special cases, such as for G abelian or invariantly metrizable, one can obtain a reduction in 1. that is not only Borel but admits a Borel inverse up to orbit equivalence (6.40 and 6.30). The proof that 2. implies the negation of 1. is given in §3.2; the converse requires a long argument, finally concluding in chapter 9, and uses an elaboration on the Scott analysis presented in §6.2 that may have independent interest.

The form of 0.14 is intended to make it into a tool that can be easily applied in concrete cases.

CHAPTER 1

An outline

This chapter discusses a few specific classification problems, but leaves until later the precise definitions.

1.1. Some specific classification problems

In light of 0.2, we may very well ask regarding the difficulty of classifying unitary operators on *infinite dimensional* Hilbert space. Let

$$l^2 = \{(x_i)_{i\in\mathbb{N}} | x_i \in \mathbb{C}, \sum |x_i|^2 < \infty\}$$

with the usual inner product and let U_∞ be the set

$$\{T : l^2 \to l^2 | T \text{ linear}, \forall \vec{x}, \vec{y} \in l^2(\langle \vec{x}, \vec{y}\rangle = \langle T(\vec{x}), T(\vec{y})\rangle)\}.$$

We can set

$$T_1 \approx T_2 \Leftrightarrow \exists S \in U_\infty(ST_1S^{-1} = T_2)$$

and obtain an infinite dimensional analog of the classification problem for unitary matrices. It was known from [**10**] that $\approx$ is non-smooth, in the sense of not allowing points in $\mathbb{R}$ or any other Polish space as complete invariants.

Building on earlier results of Kechris' the classifiability of $\approx$ by countable structures has been refuted in the following dramatic form:

THEOREM 1.1. *(Kechris, Sofronidis) The self action of U_∞ by conjugation is turbulent (on an invariant dense G_δ).*

While a satisfactory system of complete invariants have been established for the class of discrete spectrum measure preserving transformations of the unit interval and Bernoulli shifts, there is no known classification theorem for arbitrary measure preserving transformations. Certainly we might ask for classification by countable models or even by countable sets of reals.

It can be shown that we may reduce a turbulent orbit equivalence and conclude:

THEOREM 1.2. *(See* [**34**]*) The isomorphism relation on measure preserving transformations does not allow classification by countable models.*

The classification theorem for locally compact groups from 0.4 hints at the existence of classifiability for broader classes of Polish groups actions, in particular those with a kinship to locally compact. Even though an infinite product group such as $\mathbb{R}^{\mathbb{N}}$ would seem to bear little topological or algebraic kinship to S_∞:

THEOREM 1.3. *(§7.4) Let $\prod_{\mathbb{N}} H_n$ be the product of countably many locally compact Polish groups and suppose it acts continuously on a Polish space. Then the resulting orbit equivalence relation allows classification by countable models.*

Given the existence of a classification for orientation preserving homeomorphisms of the unit interval (0.9 and §4.2), we might hope for a similar classification of the homeomorphisms of the unit square. Here there is a distinction between dimensions 1 and 2.

THEOREM 1.4. *(§4.3) Let $H = Hom([0,1]^2)$ act upon itself by conjugation. Then the resulting equivalence relation does not admit classification by countable models.*

After showing that the equivalence relation of conformal equivalence for Riemann surfaces is not reducible to the equality relation on any reasonably concrete space, the authors of [**5**] ask whether a classification may be produced using *countable unordered sets* of reals as complete invariants. [**40**] uses the uniformization theorem to reduce this classification problem to the orbit equivalence relation of a locally compact group, and then appeals to Kechris' theorem as mentioned at 0.4 to obtain an affirmative answer.

No such result can be obtained for general complex manifolds:

THEOREM 1.5. *(Hjorth, Kechris; see* [**40**]*) Biholomorphism of 2-dimensional complex manifolds reduces a turbulent orbit equivalence relation; thus it does not admit classification by countable models, and in particular there is no reasonable method of assigning countable sets of reals as complete invariants.*

Finally some words about the study of infinite dimensional group representations, since this is the area in which the smooth/non-smooth division received perhaps its first clear formulation. Given a countable group G we can consider the space of all representations on Hilbert space – that is to say, all homomorphisms of G into the infinite dimensional unitary group U_∞. The natural notion of classification is up to pointwise conjugation. Given $\rho_1, \rho_2 : G \to U_\infty$ let us write $\rho_1 \sim_r \rho_2$ if there is some $T \in U_\infty$ such that at each $g \in G$

$$T \circ \rho_1(g) \circ T^{-1} = \rho_2(g).$$

Already the Kechris-Sofronidis result shows that the classification problem for $G = \mathbb{Z}$ does not allow reduction to countable models, but one might argue that here we should restrict attention to the irreducible transformations. Building on [**79**]:

THEOREM 1.6. *(*[**35**]*) The irreducible representations considered up to $\sim_r$ admit classification by countable models if and only if G is abelian-by-finite.*

In §5 we prove this just for the specific case of G being the free group on infinitely many generators.

1.2. The form of this book (and one time paper)

This began as some seminars given at Caltech. The paper grew longer and longer, and at some point there entered the optimistic idea that by adding just a couple more pages it could be made totally self contained.

The desire to avoid any black boxes and reach a completely general audience has determined the choice of material. I have not tried to cover everything, but concentrated only on developing what seemed like the main ideas. In the exercises of §6.1 a set theorist may recognize that some calculations can be given swifter proofs using the relatively sophisticated techniques of [**52**] and [**67**]. §3.2 could be completed more quickly using "forcing", but I have instead taken every step by hand.

Chapter 2 gives some background in classical descriptive set theory and equivalence relations. 3 provides the sufficient condition for non-classification:

THEOREM 1.7. *(See 3.19) A turbulent orbit equivalence relation does not allow classification by countable models.*

and 4 and 5 applies this to homeomorphism groups and the unitary group of Hilbert space. Returning to the theory at 6 it is shown that turbulence is essential to non-classification for a broad class of groups:

THEOREM 1.8. *Let G be a GE group (for instance abelian) and X a Polish G-space. Then the following are equivalent:*

1. *the orbit equivalence relation admits classification by countable models;*
2. *there is no turbulent Polish G-space Y and continuous G-embedding from Y to X.*

In 7 a complete and self contained account is given of the GE groups and the Glimm-Effros dichotomies, and more generally the smooth\non-smooth dichotomy. There is a new result to the effect that countable products of locally compact groups admit a Glimm-Effros dichotomy. This in turn is used in the proof 1.3 at 7.43.

8 passingly covers Borel groups.

9 presents the general classifiability versus turbulence analysis, and is unlike the other chapters in the sense of being written purely with the expert in mind. I regret that it seemed impossible to give a self contained exposition of these proofs.

10 lists some open problems.

There is an appendix discussing ordinals, another clarifying notation, especially conventions that are idiosyncratic among set theorists, and there is an index.

Although the direction of this book is towards explicating a notion of classification there are also comments along the way regarding the broader area of Polish group actions. General results are not presented axiomatically in layered fashion, but are mostly developed at the point of use. Even as late as §7.1 basic facts appear for the first time.

The logical order of the chapters is that 2 precedes 3, and from there one can read any of chapters 4 through 8 just assuming those two. Chapter 9 however requires the reader to be familiar with most of the book.

For many purposes, shorter routes may be possible. Simply reading chapter §2.1, §2.2, and then up through the first half of §2.4, and §3.1, and §7.1-3 and §7.5 covers most of what is known about the Glimm-Effros dichotomy for Polish group actions. In a 10 week course at UCLA we went up through to 2.51, then §3.1, then lemma 3.14, which constitutes the essence of why turbulent equivalence relations are complicated, a few examples, and then finally we discussed §6.1.

There are many exercises, some of which are difficult and some of which are even theorems I wanted to mention briefly without taking a great deal of time to prove.

1.3. Acknowledgments

The work here was partially supported by NSF grant DMS 96-22977 and a generous grant from the Alfred P. Sloan Foundation.

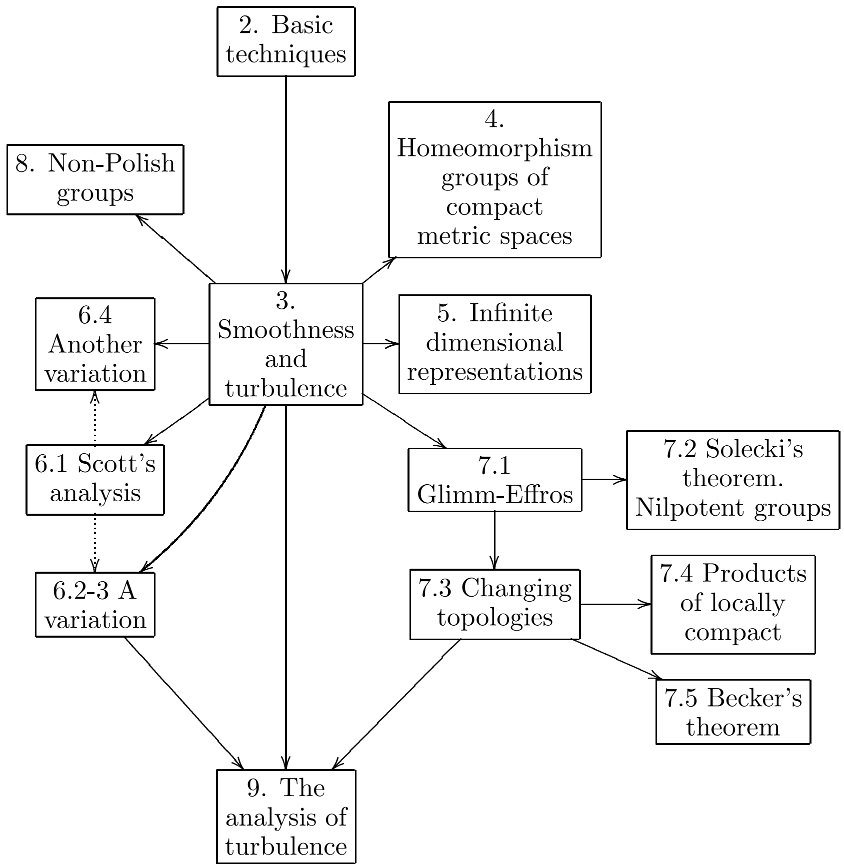

FIGURE 1.1. The order of dependence

I am indebted to Harvey Friedman for showing me the unpublished proof alluded to in [**20**].

I must twice thank Alexander Kechris.

Once for his general works on Polish groups, and many conversations, freely given, that served to introduce me to this area.

And once more for some very specific discussions on the notion of turbulence. During the Caltech seminars the definitions were the focus of his unrelenting criticism; this in turn molded the order of presentation at §3.2, and the decisions of where to make a theorem from a definition and where to do the reverse.

I am further grateful to Alexander Andretta, Lyman Chaffee, David Fremlin, Su Gao, and Nathan Schroeder for reading through various parts of earlier drafts and making invaluable comments. Undoubtedly obscurities and oversights remain, but they would number far greater without the suggestions I received. The research underpinning chapter 4 was especially influenced by a number of conversations with Su.

CHAPTER 3

Turbulence

Here we discuss various sufficient conditions for non-classifiability. §3.1 begins with smoothness, proving that *properly generically ergodic* group actions induce non-smooth orbit equivalence relations. This is the model for §3.2, which introduces the more elaborate notion of *turbulence* and in turn establishes radical non-classifiability results for actions in this class. §3.3 presents some examples and applications. §3.4 consists in historical remarks.

3.1. Generic ergodicity

DEFINITION 3.1. Let G be Polish group and X a Polish G-space. The action is said to be *properly generically ergodic* if:

(i) every orbit is dense; and

(ii) every orbit is meager.

By analogy we may say that the orbit equivalence relation is *properly generically ergodic* if it arises, or can be made to arise, from such an action.

THEOREM 3.2. *(classical) Suppose G is a Polish group, X is a Polish G-space, and every orbit in X is dense. Suppose $\theta : X \to \mathbb{R}$ is Baire measurable and G-invariant (in the sense that $\forall x \in X g \in G(\theta(x) = \theta(g \cdot x))$). Then there is a comeager set $C \subset X$ on which θ is constant.*

PROOF. Let C be a comeager set on which θ is continuous. Let $x \in X$ be such that $\forall^* g \in G(g \cdot x \in C)$; x exists by 2.47. Then $[x]_G \cap C$ is dense in C, so by continuity θ is constant on C. □

Thus we obtain a sufficient condition for an orbit equivalence relation to be non-smooth.

COROLLARY 3.3. *Suppose G is a Polish group and X is a properly generically ergodic Polish G-space. Then $X/G \not\leq_B \mathrm{id}(\mathbb{R})$.*

In the proof of 3.2 we did not need every orbit dense; only a comeager set of dense orbits. But if one orbit is dense then generically so are all.

LEMMA 3.4. *Suppose G is a Polish group and X is a Polish G-space. Suppose some orbit is dense. Then the set of dense orbit is an invariant dense G_δ.*

PROOF. Let $\{U_n : n \in \mathbb{N}\}$ enumerate the non-empty basic open sets in X. Then for each l the set of $x \in X$ such that

$$[x]_G \cap U_l \neq \emptyset$$

is invariant and open. Thus the set of dense orbits is an invariant G_δ.

Therefore as soon as we know this set to be non-empty we are done. □

The next statement foreshadows lemmas from §3.2.

Corollary 3.5. *Suppose G is a Polish group and X is a Polish G-space. Suppose some orbit is dense and every orbit is meager.*

Then there is an invariant dense G_δ set $X_0 \subset X$ such that X_0 is a properly generically ergodic Polish G-space. In particular $E_G^X \not\leq_B \mathrm{id}(\mathbb{R})$.

Proof. Let X_0 be the set of points with dense orbits, and note the notions of categoricity coincide between X and X_0 since X_0 is comeager by 3.4. □

By the same proof we obtain $E_G^X \not\leq_B \mathrm{id}(Y)$ for any Polish space Y.

Definition 3.6. Let E and F be equivalence relations on X and Y. E is said to be *generically F-ergodic* if whenever $\theta : X \to Y$ is Baire measurable with $\forall x_1, x_2 \in X$,

$$x_1 E x_2 \Rightarrow \theta(x_1) F \theta(x_2),$$

then we must have some comeager set $C \subset X$ such that for all $x_1, x_2 \in C$

$$\theta(x_1) F \theta(x_2).$$

So 3.2 states that any properly generically ergodic orbit equivalence relation is $\mathrm{id}(\mathbb{R})$-ergodic, and in particular, since every orbit is meager, the equivalence relation is non-smooth.

Exercise 3.7. Let G be a countable infinite group. Show that $2^G/G$ (in the shift action from 2.24) is not smooth. (Hint: The main issue is the existence of some dense orbit. But given finite $F, F' \subset G$ and basic open sets

$$U = \{f \in 2^G : \forall g \in F(f(g) = i_g)\},$$
$$U' = \{f \in 2^G : \forall g \in F'(f(g) = j_g)\},$$

where each $i_g, j_g \in \{0, 1\}$, we may obtain $h \in G$ such that $h \cdot F \cap F' = \emptyset$ and hence $h \cdot U \cap U' \neq \emptyset$. This point granted, a diagonalization argument enables us to produce a sequence of non-empty basic open sets $(U_i)_{i \in \mathbb{N}}$, each

$$\overline{U_{i+1}} \subset U_i,$$

and

$$d(U_i) < 2^{-i},$$

with respect to a complete metric d on 2^G, so that for any basic open set W we have some i and $h \in G$ with $h \cdot U_i \subset W$. For $\{f\} = \bigcap U_i$ we have $[f]_G$ dense.)

Exercise 3.8. Show: $\mathrm{id}(\mathbb{R}) <_B E_0$; $\mathrm{id}(\mathbb{R}) <_B \mathbb{R}/\mathbb{Q}$; $\mathrm{id}(\mathbb{R}) <_B 2^{\mathbb{Z}}/\mathbb{Z}$. (3.3 is applicable to E_0, since this equivalence relation arises from an action of the infinite rank $\mathbb{Z}_2$-module on $2^{\mathbb{N}}$; compare for instance 7.9.)

We will see in §7 that if X is a properly generically ergodic Polish G-space then E_0 is continuously reducible to E_G^X.

Exercise 3.9. Let G be a Polish group and let X be a Polish G-space. Show that if E_G^X is generically $\mathrm{id}(\mathbb{R})$-ergodic, then

$$\forall^* x \in X([x]_G \text{ is dense}).$$

Exercise 3.10. The isomorphism relation on rank one torsion free abelian groups is not smooth. Before proving or even precisely stating this it is necessary to work out some definitions:

(i) (The set of models displaying group structures on $\mathbb{N}$ forms a Polish S_∞-space.) Let $R_\bullet(\cdot, \cdot, \cdot)$ be a three placed relation and let $R_{-1}(\cdot, \cdot)$ be a two placed relation and let $\mathcal{L}$ be the language generated by these two. Let Mod(Grp) $\subset$Mod($\mathcal{L}$) be the set of $\mathcal{M}$ for which:

(a) for all $a, b \in \mathbb{N}$ there exists $c, d \in \mathbb{N}$ so that

$$\mathcal{M} \models R_\bullet(a, b, c) \wedge R_{-1}(a, d)$$

(b) for all $a, b, c, c', d, d' \in \mathbb{N}$

$$\mathcal{M} \models (R_\bullet(a, b, c) \wedge R_\bullet(a, b, c')) \Rightarrow c = c'$$

$$\mathcal{M} \models (R_{-1}(a, d) \wedge R_{-1}(a, d')) \Rightarrow d = d';$$

and if we therefore define $(\bullet(\cdot, \cdot))^{\mathcal{M}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $((\cdot)^{-1})^{\mathcal{M}} : \mathbb{N} \to \mathbb{N}$ accordingly, so that for all a, b

$$\mathcal{M} \models R_\bullet(a, b, (\bullet(a, b))^{\mathcal{M}})$$

$$\mathcal{M} \models R_{-1}(a, ((a)^{-1})^{\mathcal{M}})$$

then

(c) $(\bullet(\cdot, \cdot))^{\mathcal{M}}$ defines a group operation on $\mathbb{N}$ with inverse $((\cdot)^{-1})^{\mathcal{M}}$ (so that for all a, b, c, d, e, f, g associativity amounts to

$$\mathcal{M} \models (R_\bullet(a, b, d) \wedge R_\bullet(d, c, e) \wedge R_\bullet(b, c, f) \wedge R_\bullet(a, f, g)) \Rightarrow g = e$$

while the defining condition on the inverse for a group comes out as for all a, b, c, d

$$\mathcal{M} \models (R_{-1}(a, b) \wedge R_\bullet(a, b, c)) \Rightarrow (R_\bullet(c, d, d) \wedge R_\bullet(d, c, d)),$$

$$\mathcal{M} \models (R_{-1}(a, d) \Rightarrow R_{-1}(d, a));$$

Show that Mod(Grp) is an invariant G_δ subset of Mod($\mathcal{L}$) and thereby forms a Polish S_∞-space.

(ii) (The torsion free abelian rank 1 groups form a Polish S_∞-space.) Let TFA_1 consist of all $\mathcal{M} \in$ Mod(Grp) such that:

(d) (abelian) for all $a, b, d, d' \in \mathbb{N}$

$$\mathcal{M} \models (R_\bullet(a, b, d) \wedge R_\bullet(b, a, d')) \Rightarrow d = d';$$

(e) (torsion free) for all $a = a_0, a_1, ...a_n \in \mathcal{M}$ such that at each $i < n$

$$\mathcal{M} \models R_\bullet(a, a_i, a_{i+1})$$

$$\mathcal{M} \models R_\bullet(a_n, a_n, a_n)$$

we have

$$\mathcal{M} \models R_\bullet(a, a, a);$$

(f) (rank one) for all $a, b \in \mathbb{N}$ there are $a_1 = a, a_2, ..., a_n$ and $b_1 = b, b_2, ..., b_k$ so that at each $i < n$

$$\mathcal{M} \models R_\bullet(a, a_i, a_{i+1})$$

and for $i < k$

$$\mathcal{M} \models R_\bullet(b, b_i, b_{i+1})$$

and

$$a_n = b_k$$

(in other words $n \cdot a = k \cdot b$).

Show that TFA_1 is an invariant G_δ subset of Mod(Grp).

(iii) Let $(\mathbb{Q}^+, \cdot)$ be the multiplicative group of the positive rationals. Let $2^{\mathbb{Q}}$ be the set of $f : \mathbb{Q} \to \{0, 1\}$ with the usual product topology, so that basic open sets have the form

$$\{f \in 2^{\mathbb{Q}} : f(q_1) = i_1, ..., f(q_n) = i_n\}.$$

Let X be the space of $f \in 2^{\mathbb{Q}}$ for which the set A_f of $q \in \mathbb{Q}$ with $f(q) = 1$ forms a subgroup of $(\mathbb{Q}, +)$. Let $(\mathbb{Q}^+, \cdot)$ act on X by

$$\hat{q} \cdot f(q) = f(q\hat{q}).$$

(iv) Show that X is a Polish $(\mathbb{Q}^+, \cdot)$-space.

(v) Use 3.5 to show that $E^X_{(\mathbb{Q}^+,\cdot)}$ is non-smooth.

(vi) Show that $X/(\mathbb{Q}^+, \cdot) \leq_B \mathrm{TFA}_1/S_\infty$ and hence conclude that isomorphism relation on elements of $\mathrm{Mod}(\mathcal{L})$ corresponding to torsion free rank 1 abelian groups is non-smooth.

(vii) On the other hand, show that the action of S_∞ on TFA_1 is not properly generically ergodic since there *is* a dense G_δ orbit.

It is implicit in [**1**] that we conversely obtain $\mathrm{TFA}_1/S_\infty \leq_B E_0$. Since E_0 can be thought of as a kind of Borel instantiation of the subsets of $\mathbb{N}$ considered up to finite difference, this is tantamount to classifying rank one torsion free abelian groups by elements of $\mathcal{P}(\mathbb{N})/\mathrm{Finite}$. The method of proof for this is to take a rank one torsion free abelian group G with underlying set $\mathbb{N}$ and assign $k_G \in \mathbb{N}$ to be the first non-zero element of G. One then defines a function $\theta(G)$ in $2^{\mathbb{N}\times\mathbb{N}}$, where for p_n the nth prime we let $(\theta(G))(m, n) = 1$ if and only if $(p_n)^m$ divides k_G. With some effort one can show that for any two G_1 and G_2 we have $\theta(G_1)E_0(2^{\mathbb{N}\times\mathbb{N}})\theta(G_2)$ if and only if $G_1 \cong G_2$.

Exercise 3.11. The action of S_∞ on itself by conjugation is smooth. (The number of n-cycles for $n \in \{1, 2, 3, ..., \infty\}$ is a complete invariant.)

The next exercise may seem artificial. The point is that this is exactly kind of reduction one obtains by the Ulm invariants (see [**48**]) for countable abelian p-groups. The definition itself is technical, and does not play an important part until chapter 9.

Exercise 3.12. Recall that ω_1 is the first uncountable ordinal, so that $(\omega_1, <)$ is an uncountable well ordered set all of whose proper initial segments are countable. Let $2^{<\omega_1}$ be the set of all bounded subsets of ω_1 – that is to say the $A \subset \omega_1$ for which there is some $\alpha \in \omega_1$ such that for all $\beta \in A$ we have $\beta < \alpha$.

Let us say that a function $\theta : X \to 2^{<\omega_1}$ from a standard Borel space space to the bounded subsets of ω_1 is *UBMC* (short for *universally Baire measurable in the codes*) if we have the following:

> whenever $g : Y \to X$ is a Borel function from a Polish space there will be some $\alpha \in \omega_1$ and comeager $C \subset Y$ such that:
>
> (i) for all $y \in C$
>
> $$\theta(g(y)) \subset \{\beta \in \omega_1 : \beta < \alpha\};$$
>
> (ii) if $\pi : \mathbb{N} \to \{\beta \in \omega_1 : \beta < \alpha\}$ is a bijection, then the function
>
> $$C \to 2^{\mathbb{N}}$$
>
> $$y \mapsto \{n \in \mathbb{N} : \pi(n) \in \theta(g(y))\}$$

is Borel. (The issue of this function being Borel does not depend on the choice of π.)

Let us say that an equivalence relation E on a standard Borel space X is *UBMC reducible to* $2^{<\omega_1}$ if there a UBMC function $\theta : X \to 2^{<\omega_1}$ such that for all $x_0, x_1 \in X$

$$x_0 E x_1 \Leftrightarrow \theta(x_0) = \theta(x_1).$$

Show that if $E_0 \leq_B E$ then E is not UBMC reducible to $2^{<\omega_1}$.

One can define a similar notion for functions from a space to the hereditarily countable sets. Thus we can say

$$\theta : X \to \mathrm{HC}$$

is UBMC if whenever $g : Y \to X$ is Borel then there is a comeager set $C \subset Y$ and Borel

$$y \mapsto \mathcal{M}_y$$
$$C \to \mathrm{Mod}(\mathcal{L}),$$

where $\mathcal{L}$ is the language generated by a single binary relation E, such that for all $y \in C$

$$\mathcal{M}_y \cong (\{\theta(g(y))\} \cup \mathrm{TC}(\theta(g(y))), \in);$$

in other words, on the comeager set C we have a Borel manner of supplying representatives of the hereditarily countable set $\theta(g(y))$.

3.2. The definition of turbulence

Definition 3.13. Let G be a Polish group acting continuously on a Polish space X. The action is said to be *turbulent* if:

(i) every orbit is dense;

(ii) every orbit is meager;

(iii) for all $x, y \in X$, $U \subset X$, $V \subset G$ open with $x \in U$, $1 \in V$, there exists $y_0 \in [y]_G$ and $(g_i)_{i\in\mathbb{N}} \subset V$, $(x_i)_{i\in\mathbb{N}} \subset U$ with

$$x_0 = x,$$

$$x_{i+1} = g_i \cdot x_i,$$

and for some subsequence $(x_{n(i)})_{i\in\mathbb{N}} \subset (x_i)_{i\in\mathbb{N}}$

$$x_{n(i)} \to y_0.$$

Part (i) in this definition is implied by (iii), but left in place to indicate the resemblance with 3.2. (i) and (ii) are already familiar from §3.1. Rather it is part (iii) which desires some comment.

Roughly speaking we might view condition (i) as saying that every orbit acts *chaotically* – a point in the space may begin at some specific place, but if we wait long enough the group action has the ability to rub it up arbitrarily close to any other orbit. Condition (iii) on the other hand states that not only do the orbits behave chaotically, but even when we try to restrict them and localize their behavior, they still have the capacity to approach representatives of any other orbit.

Some analogy might be provided by the old standby of a billiard ball bouncing back and forth on a perfectly smooth and frictionless table with perfectly elastic boundaries. Thinking of the position the ball as a function of time we may naturally associate this with an $\mathbb{R}$-action. Given reasonably generic starting conditions, we might expect that the ball will on its travels eventually pass through every part of the surface.

Now suppose we cordon off one section of the table by an array of one way doors. The ball may exit the cordoned section whenever moved to do so, but having done so it can never return. It we the examine the *local orbit* of the billiard ball inside this section we no longer expect the orbit to fill out the space. It should depart to never return after some finite time, and its entire local orbit consists of just finitely many finite line segments, and thus, since this is a compact 1-dimensional set, we may find a subregion of the cordoned section on which it has not had the chance to encroach.

Indeed it is true that the associated action of $\mathbb{R}$ is *not* turbulent, and appropriately modified the above considerations can be turned into a proof that there are no turbulent $\mathbb{R}$-actions. The examples of turbulent actions in §3.3 all arise from infinite dimensional groups, and from 3.14 and 0.4, or the more general construction of §7.4 below, we can conclude that locally compact groups never give rise to turbulent actions. The notion of turbulence does not appear in the dynamics of locally compact groups, but rather is motivated by the general theory of equivalence relations and the much nastier groups which operate in this context.

We will prove that if E_G arises from a turbulent action and E_{S_∞} from a continuous S_∞ action, then E_G is generically E_{S_∞}-ergodic. Before then there are some preliminaries, the first of which states that turbulent orbit equivalence relations are generically ergodic with respect to the equivalence relation on codes for countable subsets of $2^{\mathbb{N}}$. In other words, any Baire measurable in the codes function from a turbulent Polish G-space to the countable subsets of $2^{\mathbb{N}}$ is constant on a comeager set.

LEMMA 3.14. *Let G be a Polish group acting continuously on a Polish space X with 3.13(iii): Namely: for all $x, y \in X$, $U \subset X$, $V \subset G$ open with $x \in U$, $1 \in V$, there exists $y_0 \in [y]_G$ and $(g_i)_{i\in\mathbb{N}} \subset V$, $(x_i)_{i\in\mathbb{N}} \subset U$ with*

$$x_0 = x,$$

$$x_{i+1} = g_i \cdot x_i,$$

and for some subsequence $(x_{n(i)})_{i\in\mathbb{N}} \subset (x_i)_{i\in\mathbb{N}}$

$$x_{n(i)} \to y_0.$$

Then E_G, the orbit equivalence relation of G on X, is generically F_2-ergodic.

Prelude to the proof. As a preliminary to proving this lemma, let us first discuss the proof of the much easier result that if $\theta : X \to B_2$ and various other relevant functions are all *continuous* with

$$\forall x \in X \forall g \in G(\theta(x) F_2 \theta(g \cdot x))$$

then $x \mapsto [\theta(x)]_{F_2}$ is constant on X. Recall that $B_2 = \{z \in 2^{\mathbb{N}\times\mathbb{N}} : \forall k \neq l(z(k,\cdot) \neq z(l,\cdot)\}$. We have already seen at 2.30 that $F_2 \leq_B F_2|_{B_2}$, and it is somewhat more natural to work on the set B_2 where the equivalence relation F_2 is actually induced by an S_∞-action. We fix $x, y \in X$ and $k \in \mathbb{N}$ and attempt to show how one might set about proving $\theta(x)(k,\cdot) \in \{\theta(y)(n,\cdot) : n \in \mathbb{N}\}$. Note that for any z in the space we may cover G by countably many sets $\{A^z_l : l \in \mathbb{N}\}$ such that on each A^z_l we have for all $g \in A^z_l$

$$\theta(g \cdot z)(l, \cdot) = \theta(z)(k, \cdot).$$

This is just at once true by the invariance assumption on θ.

Continuing with the policy of attempting to see how the proof moves forward under the most favorable circumstances, let us suppose that each such A^z_l is always open. Then for each z there will be some A^z_l which is not empty, and so we may find some $z' \in [z]_G$ and V a basic open neighborhood of the identity such that for all $g \in V$

$$\theta(g \cdot z')(l, \cdot) = \theta(z)(k, \cdot).$$

For each basic open V we may let $B(V, l)$ be the set of z' such that for all $g \in V$

$$\theta(g \cdot z')(l, \cdot) = \theta(z')(l, \cdot).$$

Again, let us just optimistically assume that $B(V, l)$ is always open.

Now going back to our original x and y, we will have some $x' \in [x]_G$ and basic open V containing the identity such that $x' \in B(V, l)$ and for all $g \in V$

$$\theta(g \cdot x')(l, \cdot) = \theta(x')(l, \cdot) = \theta(x)(k, \cdot).$$

Let $U \subset B(V, l)$ be some basic open set containing x'. Using the assumption of turbulence applied to x', y, U and V we may find $y_0 \in [y]_G$ and $h_i \in V$ such that if we let $x_0 = x'$ and $x_{i+1} = h_i \cdot x_i$ then y_0 is an accumulation point of $\{x_i : i \in \mathbb{N}\}$ and each $x_i \in U \subset B(V, l)$.

At this point we can use definition of $B(V, l)$ to note that each $\theta(x_{i+1})(l, \cdot) = \theta(x_i)(l, \cdot)$, and then obtain by induction i that each $\theta(x_i)(l, \cdot) = \theta(x)(k, \cdot)$. Thus by continuity of θ we have

$$\theta(x)(k, \cdot) = \theta(y_0)(l, \cdot),$$

as required.

Of course this is all rather wishful. In general these sets will be far from being open. But we will steadily verify that the sets in question have the Baire property, and so there will be some open set which they equal on a comeager subset of X or G; these repeated verifications make the actual proof seem technical at first sight.

Proof. Choose $\theta : X \to B_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ Borel such that for all $x \in X$ and $g \in G$

$$\{\theta(x)(n, \cdot) : n \in \mathbb{N}\} = \{\theta(g \cdot x)(n, \cdot) : n \in \mathbb{N}\}.$$

Let $G_0 < G$ be a countable dense subgroup, with $(g_i)_{i\in\mathbb{N}}$ some enumeration. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of X and $(V_n)_{n\in\mathbb{N}}$ a neighborhood basis for G at the identity. Choose $C \subset X$ a comeager Borel set on which θ is continuous by 2.49, and let $C_0 = \bigcap_{i\in\mathbb{N}} g_i \cdot C$ so that it is G_0 invariant and then $C_1 = \{x \in X : \forall^* g \in G(g \cdot x \in C_0)\}$. By 2.47 C_1 is comeager; by 2.50 Borel.

Claim(1): For all $x \in C_1$ and $k \in \mathbb{N}$ there exists $W \subset G$ open and non-empty and $l \in \mathbb{N}$ so that for all $h \in W$

$$h \cdot x \in C_0 \Rightarrow \theta(x)(k, \cdot) = \theta(h \cdot x)(l, \cdot).$$

Proof of claim: Note that we may cover $S_\infty \cdot \theta(x) =_{df} [\theta(x)]_{S_\infty}$ by countably many closed sets $(A_j)_{j\in\mathbb{N}}$ so that for each j and $y \in A_j$

$$\theta(x)(k, \cdot) = y(j, \cdot).$$

Let D_x be the preimage of C_0 under $g \mapsto g \cdot x$ and note that $g \mapsto \theta(g \cdot x)$ is continuous on D_x; since the preimage of each A_j is relatively closed and D_x is not meager, there must be some $W \subset G$ non-empty and open so that for some j

$$\forall g \in W \cap D_x(\theta(g \cdot x) \in A_j),$$

and $l =$ this j is as required. ($\square$Claim(1))

Now define $I : \mathbb{N} \times X \to \mathbb{N} \cup \{\infty\}$ so that $I(l, x)$ equals the least n so that for all g_i and $g_{i'}$ in V_n

$$g_i \cdot x, g_{i'} \cdot x \in C_0 \Rightarrow \theta(g_i \cdot x)(l, \cdot) = \theta(g_{i'} \cdot x)(l, \cdot),$$

and equals ∞ if no such n exists. Note that if $x \in C_0$ and $I(l, x) = j$ then *for all* $g \in V_j$

$$g \cdot x \in C_0 \Rightarrow \theta(g \cdot x)(l, \cdot) = \theta(x)(l, \cdot)$$

by the continuity of θ on C_0 and the density of $\{g_i \cdot x : i \in \mathbb{N}\}$ in $G \cdot x \cap C_0$.

Considering $\mathbb{N} \cup \{\infty\}$ as a discrete space, I is transparently Borel, so choose comeager $C_2 \subset C_0 \cap C_1$ on which $I|_{\mathbb{N}\times C_2}$ is continuous, and then take $C_3 = \{x \in X : \forall^* g \in G(g \cdot x \in C_2)\}$. Note $C_3 \subset C_1$. Claim(1) implies that for all $x \in C_3$ and $k \in \mathbb{N}$ there will exist some $\hat{x} \in [x]_G \cap C_2$ and l for which $I(l, \hat{x}) < \infty$ and $\theta(x)(k, \cdot) = \theta(\hat{x})(l, \cdot)$. (We may assume that the $h \cdot x = \hat{x}$ promised by claim(1) is in C_2, since if not then we replace $\hat{x}$ by $\epsilon \cdot \hat{x}$ for some $\epsilon \in G$ which is close to the identity and has $\epsilon \cdot \hat{x} \in C_2$.)

With all this preparation let us finally suppose that $x, y \in C_3$; fixing k it is enough to show that $\theta(x)(k, \cdot)$ appears in $\{\theta(y)(n, \cdot) : n \in \mathbb{N}\}$. By the above reasoning we can assume that $x \in C_2$ and that

$$I(k, x) = m < \infty.$$

Since I is continuous on $\mathbb{N} \times C_2$ we can find some open set $U \subset X$ containing x so that for all $z \in C_2 \cap U$

$$I(k, z) = m.$$

Choose an open neighborhood W of the identity in G so that $W^3 \subset V_m$ and let $V = W \cap W^{-1}$.

Now apply the definition of turbulence to x, y, U, and V to find $y_0 \in [y]_G$ and $\{h_i : i \in \mathbb{N}\} \subset V$ so that for $x_0 = x$ and $x_{i+1} = h_i \cdot x_i$ we have each $x_i \in V$ and that y_0 is an accumulation point of the set $\{x_i : i \in \mathbb{N}\}$.

Claim(2): There are $\{\delta_i : i \in \mathbb{N}\} \subset V$ so that $d(x_i, \delta_i \cdot x_i) < 2^{-i}$, and $\delta_i \cdot x_i \in U$, and $\delta_i \cdot x_i \in C_2$

Proof of claim: For each i let $W_i \subset V$ be an open neighborhood of the identity such that for all $\delta \in W_i$

$$d(x_i, \delta \cdot x_i) < 2^{-i}$$

and

$$\delta \cdot x_i \in U.$$

By definition of C_3, for each x_i there is a comeager subset of $D_i \subset G$ on which $g \cdot x_i \in C_2$. Choosing δ_i inside $D_i \cap W_i$ suffices. ($\Box$Claim(2))

Let $\hat{x_i} = \delta_i \cdot x_i$ for $i > 0$; $\hat{x}_0 = x_0$. Let $\hat{h}_i = \delta_{i+1} h_i \delta_i^{-1}$ for $i > 0$; at $i = 0$ just let $\hat{h}_i = \delta_1 h_0$. Note that y_0 is still an accumulation point of $\{\hat{x}_i : i \in \mathbb{N}\}$.

Since each $\hat{x}_i \in C_2 \cap U$ we have $I(k, \hat{x}_i) = m$ for each i. Thus by $\hat{h}_i \in W^3 \subset V_m$, the definition of I, and induction on i,

$$\theta(\hat{x}_i)(k, \cdot) = \theta(\hat{x}_0)(k, \cdot) =_{df} \theta(x)(k, \cdot).$$

Claim(3): There is $\delta \in V_m$ such that $\delta \cdot y_0 \in C_0$ and at each i

$$\delta \cdot \hat{x}_i \in C_0,$$

$$\theta(\delta \cdot \hat{x}_i)(k, \cdot) = \theta(\hat{x}_i)(k, \cdot).$$

Proof of claim. The set of δ with $\delta \cdot y_0 \in C_0$ is comeager in G and therefore relatively comeager in V_m by $y \in C_1$, and similarly the set for which $\delta \cdot \hat{x}_i \in C_0$ is comeager at each i. However by the assumption that $\hat{x}_i \in U \cap C_2$ we have that the set of $\delta \in V_m$ with $\theta(\delta \cdot \hat{x}_i)(k, \cdot) = \theta(\hat{x}_i)(k, \cdot)$ is relatively comeager. Choosing δ in the intersection of all these comeager sets suffices. ($\Box$Claim(3))

By continuity of θ on C_0,

$$\theta(\delta \cdot y_0)(k, \cdot) = \lim_{i \to \infty} \theta(\delta \cdot \hat{x}_i)(k, \cdot) = \lim_{i \to \infty} \theta(\hat{x}_i)(k, \cdot) = \theta(x)(k, \cdot),$$

as required. $\Box$

Definition 3.15. Let G be a Polish group, X a Polish G-space. For $x \in X$, $U \subset X$, $V \subset G$ both open, with $x \in U$ and $1_G \in V$, we define the *local U-V-orbit of x* to be the set of $\hat{x}$ for which there exists some $k \in \mathbb{N}$ and $x = x_0, x_1, x_2, ..., x_k \in U$ and $g_0, g_1, ..., g_{k-1} \in V$

$$x_k = \hat{x}$$

and for all $i < k$

$$x_{i+1} = g_i \cdot x_i.$$

This local orbit is denoted by $\mathcal{O}(x, U, V)$. We then define $\varphi_0(x, U, V)$ to encode $\overline{\mathcal{O}(x, U, V)}$, the closure of $\mathcal{O}(x, U, V)$, with the specification that if $\{U_l : l \in \mathbb{N}\}$ is a basis of X then

$$\varphi_0(x, U, V) = \{l \in \mathbb{N} : U_l \cap \mathcal{O}(x, U, V) \neq \emptyset\}.$$

Under the natural identification of $2^{\mathbb{N}}$ with the collection of all subsets of $\mathbb{N}$ the object $\varphi_0(x, U, V)$ can be calculated in a Borel fashion from x, in the sense that

$$\varphi_0(\cdot, U, V) : X \to 2^{\mathbb{N}}$$

$$x \mapsto \{l : \exists k \exists (x_i)_{i \leq k} \subset U \exists (g_i)_{i<k} \subset V (x_0 = x, x_k \in U_l, \forall i < k(x_{i+1} = g_i \cdot x_i))\}$$

is a Borel function for each open U and V: For each k let $A_k(U, V, U_l)$ be the set of x for which there exist open $U_0', U_1', ...U_k' = U_l \cap U$, and $g_0, ..., g_{k-1} \in V$ with $x \in U_0'$, $g_i \cdot (U_i') \subset U_{i+1}'$; each $A_k(U, V, U_l)$ is itself open, and $l \in \varphi_0(x, U, V)$ if and only if $x \in A_k(U, V, U_l)$ for some k.

With the definition of 3.15 we can express 3.13(iii) compactly: for all $x \in U \subset X$ and $1_G \in V \subset G$

$$[y]_G \cap \overline{\mathcal{O}(x, U, V)} \neq \emptyset.$$

Lemma 3.16. *Let G be a Polish group and X a Polish G-space, and suppose that E_G^X is generically F_2-ergodic. Then for all $U \subset X$ open, $V \subset G$ open containing the identity there is a comeager $C \subset X$ so that for all $x \in C$ and $x_0 \in [x]_G \cap U$ the set $\overline{\mathcal{O}(x_0, U, V)}$ has non-empty interior.*

Proof. Let $(g_i)_{i \in \mathbb{N}}$ enumerate a countable dense subset of G. Let W be the interior of the complement of U.

First note that for any closed nowhere dense A and $i \in \mathbb{N}$ the set

$$B_i(A) = \{x \in X : g_i \cdot x \in W\} \cup \{x \in X : g_i \cdot x \in U \setminus A\}$$

is open dense, and so

$$B(A) = \bigcap_{i \in \mathbb{N}} B_i(A)$$

is a dense G_δ.

Using the assumptions of the lemma we may go to a comeager set C on which

$$x \mapsto \{\varphi_0(g_i \cdot x, U, V) : g_i \cdot x \in U\} = \{\varphi_0(x_0, U, V) : x_0 \in [x]_G \cap U\}$$

is constant. And now for any $x \in C$ and $A \in \{\overline{\mathcal{O}(g_i \cdot x, U, V)} : g_i \cdot x \in U\}$ it must be the case that A contains an open set, or else we find $y \in C \cap B(A)$ for which A will not be an element of $\{\overline{\mathcal{O}(g_i \cdot y, U, V)} : g_i \cdot y \in U\}$. □

If X is a Polish G-space with (i) and (ii) from the definition of turbulence, and the conclusion of 3.16 holds for *every* $x \in X$ then the action is turbulent.

LEMMA 3.17. *Let G and H be Polish groups and X and Y Polish G and H-spaces respectively. Suppose that*

$$\theta : X \to Y$$

is a Borel function such that for all $x \in X$ and $g \in G$

$$\theta(x) E_H \theta(g \cdot x).$$

Then for any open neighborhood W of the identity in H there is a comeager set of $x \in X$ for which there is a basic open neighborhood V of the identity in G with

$$\forall^* g \in V \exists h \in W(h \cdot \theta(x) = \theta(g \cdot x)).$$

PROOF. First:

Claim: For all $x \in X$, there is a comeager set of $g_0 \in G$ for which there exists some open neighborhood V of the identity on which

$$\forall^* g_1 \in V \exists h \in W(h \cdot \theta(g_0 \cdot x) = \theta(g_1 g_0 \cdot x)).$$

Proof of claim: Fix x and W. Choose a smaller open

$$(W')^2 \subset W$$

$$1_H \in W'$$

$$(W')^{-1} = W'.$$

We may cover H by countably many right translates $\{W' h_i : i \in \mathbb{N}\}$, for appropriately chosen $(h_i)_{i \in \mathbb{N}}$. Therefore we may cover

$$\theta[[x]_G] =_{df} \{\theta(x_0) : x_0 \in [x]_G\}$$

by $\{W' h_i \cdot \theta(x) : i \in \mathbb{N}\}$. Each $W' h_i \cdot \theta(x)$ is $\underset{\sim}{\Sigma}^1_1$, and thus its pullback under the function

$$g \mapsto \theta(g \cdot x)$$

is $\underset{\sim}{\Sigma}^1_1$, and so has the Baire property by 2.53. So find open $(O_i)_{i \in \mathbb{N}}$ with $\bigcup_{\mathbb{N}} O_i$ open dense in G and for all $i \in \mathbb{N}$

$$\forall^* g \in O_i(\theta(g \cdot x) \in W' h_i \cdot \theta(x)).$$

At each i let $C_i \subset O_i$ be this relatively comeager set; $\bigcup_{i \in \mathbb{N}} C_i$ is comeager in G, so consider any i and choose $g_0 \in C_i$. Take V to be a basic open neighborhood of the identity for which we have $V g_0 \subset O_i$. $C_i(g_0)^{-1}$ is still relatively comeager in V, and for all $g_1 \in C_i \cap C_i(g_0)^{-1} \cap V$ we have some $h \in W'$ with

$$\theta(g_0 \cdot x) = h h_i \cdot \theta(x);$$

since $g_1 g_0 \in C_i$ there is some $h' \in W'$ with

$$\theta(g_1 g_0 \cdot x) = h' h_i \cdot \theta(x);$$

and so for $h^* \in W'(W')^{-1} \subset W$ given by $h^* = h' h^{-1}$ we have

$$h^* \cdot \theta(g_0 \cdot x) = \theta(g_1 g_0 \cdot x).$$

Thus for a comeager set of g_1 in V we do indeed have the conclusion of the claim. ($\square$Claim)

The set of x for which the conclusion of the lemma holds is $\underset{\sim}{\Sigma}^1_1$ by 2.55, 2.61(ii) and hence has the Baire property by 2.53 again. If the lemma fails then for all $g_0 \in G$ we would have that on a non-meager set of $x \in X$ it is the case that for all open neighborhoods V of the identity,

$$\exists^* g_1 \in V \forall h \in W (h \cdot \theta(g_0 \cdot x) \neq \theta(g_1 g_0 \cdot x)).$$

By Kuratowski-Ulam as at 2.46, there is consequently a non-meager set of $x \in X$ and a non-meager set of $g_0 \in G$ such that for all open V containing 1_G

$$\exists^* g_1 \in V \forall h \in W (h \cdot \theta(g_0 \cdot x) \neq \theta(g_1 g_0 \cdot x)),$$

with a contradiction to the claim. $\square$

THEOREM 3.18. *Let G be a Polish group, X a Polish G-space, satisfying (iii) from the definition of turbulence: For all $x, y \in X$, all $U \subset X$ open containing x, all $V \subset G$ open containing 1_G,*

$$\overline{\mathcal{O}(x, U, V)} \cap [y]_G \neq \emptyset.$$

Let Y be a Polish S_∞-space.

Then E^X_G is generically $E^Y_{S_\infty}$- ergodic.

PROOF. Let $\theta : X \to Y$ be a Borel map such that $x_1 E^X_G x_2 \Rightarrow \theta(x_1) E^Y_{S_\infty} \theta(x_2)$.

Let $(U_l)_{l \in \mathbb{N}}$ be a basis of X and $(V_l)_{l \in \mathbb{N}}$ a neighborhood basis of G at the identity. Define $I : X \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by setting $I(x, N)$ to be the least l so that

$$\forall^* g \in V_l \exists h \in S_\infty (\forall i \leq N (h(i) = i) \wedge h \cdot \theta(x) = \theta(g \cdot x)),$$

if such l exists ($= \infty$ if not). If $\mathbb{N} \cup \{\infty\}$ is given the discrete topology then the preimage of a basic open set is in the σ-algebra generated by $\underset{\sim}{\Sigma}^1_1$ sets.

Using lemmas 2.49 and 3.17, let C_0 be a comeager set on which I and θ are continuous and that for all $x \in C_0, N \in \mathbb{N}$

$$I(x, N) < \infty.$$

Following 3.16 find $C_1 \subset X$ comeager and invariant so that for all $l, k \in \mathbb{N}$ and $x \in C_1 \cap U_l$ the closure of the local U_l-V_k-orbit,

$$\overline{\mathcal{O}(x, U_l, V_k)},$$

contains an open set. Now let

$$C_2 = \{x \in C_0 : \forall^* g \in G (g \cdot x \in C_0 \cap C_1)\};$$

by 2.47 this is comeager. Fix $x, y \in C_2$. We will see that there exists $g, h \in S_\infty$ with $g \cdot \theta(x) = h \cdot \theta(y)$, and thereby complete the proof.

We choose successively

$$x_0 = x, x_1, x_2, ...,$$
$$y_0 = y, y_1, y_2, ...,$$

in C_2, and

$$g_0 = 1_{S_\infty}, g_1, g_2 ...,$$
$$h_0 = 1_{S_\infty}, h_1, h_2, ...,$$

in S_∞, and

$$W_0 = X, W_1, ..., W'_0, W'_1, ...,$$

basic open in X,

$$V_{n(0)}, V_{n(1)}, ..., V_{k(0)}, V_{k(1)}, ...,$$

basic open neighborhoods of 1_G so that at each i

(ia) $g_i \cdot \theta(x) = \theta(x_i)$;
(ib) $h_i \cdot \theta(y) = \theta(y_i)$;
(iia) $x_{i+1} \in W_{i+1} \cap C_2 \cap \mathcal{O}(x_i, W_i, V_{n(i)})$;
(iib) $y_{i+1} \in W'_{i+1} \cap C_2 \cap \mathcal{O}(y_i, W'_i, V_{k(i)})$;
(iii) $W_i \supset W'_i \supset W_{i+1}$;
(iv) for d a previously fixed complete metric on Y and $i > 0$

$$d(\theta[W_i \cap C_2]) < 2^{-i};$$

(va) for

$$N_{i+1} =_{df} \sup\{g_{i+1}(l), g_{i+1}^{-1}(l) : l \leq i+1\}$$

we have

$$(W_{i+1} \cap C_2) \times \{N_{i+1}\} \subset I^{-1}[\{n(i+1)\}];$$

(vb) for

$$K_i =_{df} \sup\{h_i(l), h_i^{-1}(l) : l \leq i\}$$

we have

$$(W'_i \cap C_2) \times \{K_i\} \subset I^{-1}[\{k(i)\}];$$

(via) for all $l \leq i+1 < j+1$

$$g_{j+1}(l) = g_{i+1}(l)$$
$$g_{j+1}^{-1}(l) = g_{i+1}^{-1}(l);$$

(vib) for all $l \leq i < j$

$$h_j(l) = h_i(l)$$
$$h_j^{-1}(l) = h_i^{-1}(l);$$

(viia) $\mathcal{O}(x_i, W_i, V_{n(i)})$ is dense in W'_i;
(viib) $\mathcal{O}(y_i, W'_i, V_{k(i)})$ is dense in W_{i+1}.

Note that if all this succeeds we may in the end let

$$g_\infty = \lim_{i \in \mathbb{N}} g_i$$

and

$$h_\infty = \lim_{i \in \mathbb{N}} h_i;$$

by (vi) the limits exist. Then

$$g_\infty \cdot \theta(x) = h_\infty \cdot \theta(y)$$

by (ii)-(iv).

We can begin the construction with $x_0 = x$, $W_0 = X$, $V_{n(0)} = G$, $g_0 = 1_{S_\infty}$. Then choose $y_0 = y$, W'_0 containing y with

$$(W'_0 \cap C_2) \times \{0\} \subset I^{-1}[\{k(0)\}],$$

some $k(0) \in \mathbb{N}$, and let $h_0 = 1_{S_\infty}$.

So let us suppose we have built these as above for all $j \leq i$ and concentrate on producing x_{i+1}, g_{i+1}, and W_{i+1}. The further step of producing y_{i+1}, h_{i+1}, and W'_{i+1} is completely symmetrical and can be omitted.

Fix nonempty open $U \subset W'_i$ in which $\mathcal{O}(y_i, W'_i, V_{k(i)})$ is dense and then also demand that $d(\theta[U \cap C_2]) < 2^{-i-1}$. Then choose $x_{i+1} \in \mathcal{O}(x_i, W_i, V_{n(i)}) \cap U$, and let $z_0 = x_i, z_1, ..., z_k \in W_i$ and $g'_0, g'_1, ..., g'_{k-1} \subset V_{n(i)}$ witness this, in the sense that

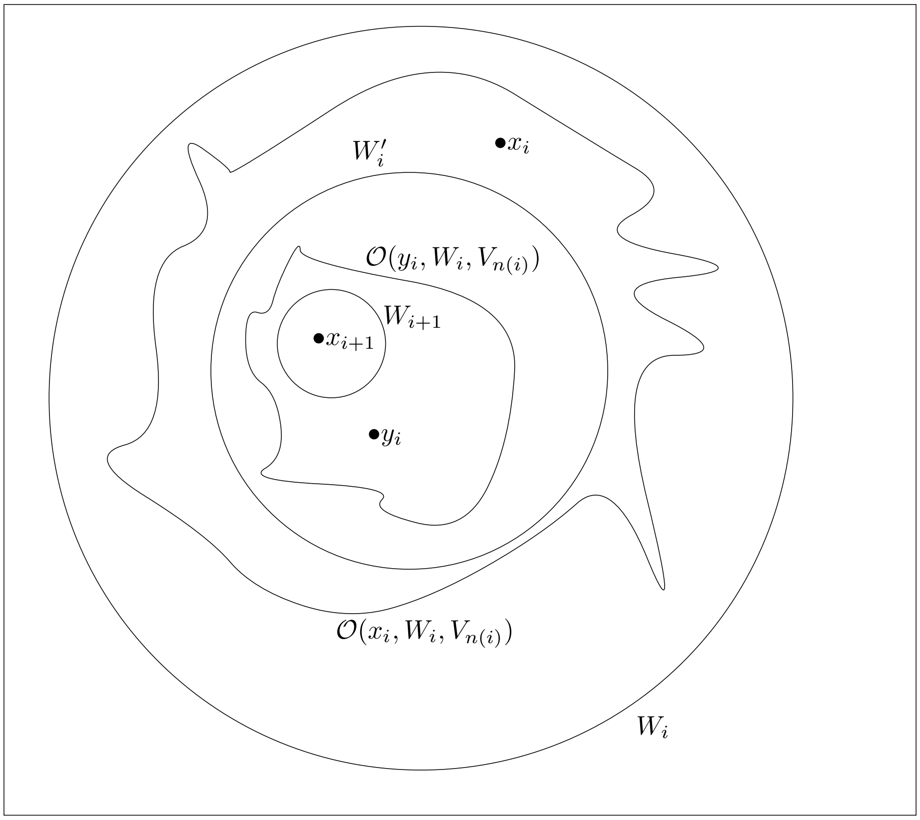

FIGURE 3.1. The local orbits folding in

$z_{i+1} = g_i' \cdot z_i$, $z_0 = x_i, z_k = x_{i+1}$. From now on we may as well assume that $i > 0$, or else (via) places no restriction on our choice of g_1.

Claim(I): Without loss of generality each $z_j \in C_2$.

Proof of claim: Choose V_j' open neighborhoods of the identity of G so that $V_j' = (V_j')^{-1}$, $V_{j+1}' \subset V_j'$, and

$$V_j' g_j' V_j' \subset V_{n(i)}.$$

Since

$$\forall^* g \in G (g \cdot x_i \in C_2)$$

and each $z_j \in [x]_G$ we may find $\epsilon_j \in V_j'$ so that

$$\epsilon_j \cdot z_j \in C_2 \cap W_i.$$

If we now replace

$$z_0, z_1, z_2, ..., z_k$$

by

$$z_0, \epsilon_1 \cdot z_1, \epsilon_2 \cdot z_2, ..., \epsilon_k \cdot z_k,$$

and

$$x_{i+1}$$

by

$$\epsilon_k \cdot x_{i+1},$$

and

$$g_0', g_1', ..., g_{k-1}'$$

by

$$\epsilon_1 g_0', \epsilon_2 g_1'(\epsilon_1)^{-1}, \epsilon_3 g_2'(\epsilon_2)^{-1}, ..., \epsilon_k g_{k-1}'(\epsilon_{k-1})^{-1}$$

then we are done. ($\square$Claim(I))

Thus in particular for $i > 0$ $I(z_j, N_i) = n(i)$ for each j. With slight perturbation of the z_j's we can use the assumption

$$I(z_j, N_i) = n(i)$$

as follows:

Claim(II): Without loss of generality for each j there is $f_j \in S_\infty$ with

$$f_j \cdot \theta(z_j) = \theta(z_{j+1})$$

$$\forall l \leq N_i(f_j(l) = l).$$

Proof. First fix a relatively comeager $B(z_0) \subset V_{n(i)}$ so that $\forall g \in B(z_0)$ we have that there exists $f_0 \in S_\infty$ with

$$f_0 \cdot \theta(z_0) = \theta(g \cdot z_0),$$

$$\forall l \leq N_i(f_0(l) = l).$$

This uses $I(z_0, N_i) = n(i)$. Then by the argument of claim(I) we may replace g_0' by some $\epsilon \cdot g_0' \in B(z_0)$ and then by the definition of $B(z_0)$ find f_0 as desired. Clearly this can be continued inductively so that each $g_j' \in B(z_j)$, for $B(z_j)$ defined by the requirement that it be relatively comeager and so that $\forall g \in B(z_j)$ we have that there exists $f_j \in S_\infty$ with $f_j \cdot \theta(z_j) = \theta(g \cdot z_j)$ and $\forall l \leq N_i(f_j(l) = l)$.
($\square$Claim(II))

Let $f_i^* = f_{k-1} f_{k-2} ... f_0$ and note then $\theta(z_k) =_{df} \theta(x_{i+1}) = f_i^* \cdot \theta(x_i)$. Each f_j leaves $0, 1, ..., N_i$ fixed, and hence so does f_i^*; then we may let $g_{i+1} = f_i^* g_i$, which indeed gives us (va) above for g_{i+1}. Now take basic non-empty open $W_{i+1} \subset U$ containing x_{i+1} and some $n(i+1)$ so that

$$(W_{i+1} \cap C_2) \times \{N_{i+1}\} \subset I^{-1}[\{n(i+1)\}],$$

where as indicated above N_{i+1} is the sup of $\{g_{i+1}(l), g_{i+1}^{-1}(l) : l \leq i\}$. $\square$

Corollary 3.19. *(A) If G is a Polish group and X is a turbulent Polish G-space, then for any Polish S_∞-space Y*

$$E_G^X \not\leq_B E_{S_\infty}^Y.$$

(B) No turbulent orbit equivalence relation admits classification by countable models.

Also following from the proof of 3.18 is that there can be no *Baire measurable* reduction of a turbulent orbit equivalence relation to any $E_{S_\infty}^Y$. Since these arguments were based on Baire category methods it is natural to relax the definitions to the requirement that they hold on a comeager set.

Definition 3.20. Let G be a Polish group and X a Polish G-space. The action is said to be *generically turbulent* if:

(i) $\forall^* x \in X([x]_G$ is dense);

(ii) every orbit is meager;

(iii) $\forall^* x, y \in X$, for all $U \subset X$ and $V \subset G$ basic open with $x \in U$ and V containing the identity, there exists $y_0 \in [y]_G$ with

$$y_0 \in \overline{\mathcal{O}(x, U, V)}.$$

THEOREM 3.21. *Let G be a Polish group, X a Polish G-space. Then the following are equivalent:*

(I) The action is generically turbulent.

(II) The orbit equivalence relation E_G^X is generically F_2-ergodic.

(III) For any countable language $\mathcal{L}$, the orbit equivalence relation is generically $\cong_{\mathrm{Mod}(\mathcal{L})}$-ergodic.

(IV) Every orbit is meager; for a comeager collection of $x \in X$ the orbit $[x]_G$ is dense and for all $U \subset X$, $V \subset G$ basic open with $x \in U$, $1 \in V$, there exists $U_0 \subset U$ non-empty with $\mathcal{O}(x, U, V)$ dense in U_0.

(V) Every orbit is meager, and for a comeager collection of $x \in X$ it is the case that $[x]_G$ is dense and for all open $U \subset X$, $V \subset G$ with $x \in U$, $1 \in V$, there exists $U_0 \subset U$ non-empty with $\mathcal{O}(x, U, V)$ dense in U_0.

(VI) Some orbit is dense, every orbit is meager, and there exists $x \in X$, such that for all $U \subset X$, $V \subset G$ basic open with $x \in U$, $1 \in V$, there exists open $U_0 \subset U$ non-empty with $\mathcal{O}(x, U, V)$ dense in U_0.

(VII) There is an invariant dense G_δ subspace $X_0 \subset X$ with X_0 a turbulent Polish G-space.

PROOF. (I)$\Leftrightarrow$(II)$\Leftrightarrow$(III)$\Leftrightarrow$(IV): (I)$\Rightarrow$(II) follows exactly as in 3.14, and (II)$\Rightarrow$(III) is as in 3.18, while (III)$\Rightarrow$(II) is trivial from 2.41. (II)$\Rightarrow$(IV) is the proof of 3.16. (IV)$\Rightarrow$(I): since if

$$U_0 \subset \overline{\mathcal{O}(x, U, V)}$$

is open and non-empty and if $[y]_G$ is dense, then we may find $y_0 \in [y]_G \cap \overline{\mathcal{O}(x, U, V)}$, as desired.

(IV)$\Leftrightarrow$(V): It is immediate that (V) entails (IV). As for the converse, if x satisfies the statement of (IV), with its limitation to basic open sets, and U and V are as in (V), then we may find basic open $U' \subset U$ and $V' \subset V$ containing x and 1 respectively; noting that $\mathcal{O}(x, U', V') \subset \mathcal{O}(x, U, V)$ we are done by assumption that $\mathcal{O}(x, U', V')$ is somewhere dense.

So we have the equivalence of (I)-(V) and we just need to tie these to (VI) and (VII).

Trivially (IV)$\Rightarrow$(VI).

(VI)$\Rightarrow$(IV): For this purpose let $(g_i)_{i\in\mathbb{N}}$ be a dense countable subset of G, $\{U_l : l \in \mathbb{N}\}$ a countable basis of X, $\{V_l : l \in \mathbb{N}\}$ a countable neighborhood basis at 1_G, and let $x \in X$ witness the conclusion of (VI).

Now for each $g_i \cdot x \in U_l$ and V_k let $W_{l,i,k} \subset U_l$ be the interior of $\overline{\mathcal{O}(g_i \cdot x, U_l, V_k)}$. This set will be non-empty by assumption on $[x]_G$.

Note that $[x]_G$ must be dense, since if we choose some $[z]_G$ which is dense, then we may find $z' \in W_{l,i,k} \cap [z]_G$ – and since z' is a limit point of $[x]_G$ and since $[z']_G = [z]_G$ is dense, $[x]_G$ is dense as well.

Let $B_{l,k}$ be the set of $y \in X$ such that either y is not in U_l or there exists some i with

$$\mathcal{O}(y, U_l, V_k) \cap W_{l,i,k} \neq \emptyset.$$

Note that each $B_{l,k}$ is G_δ by direct calculation, and dense since $[x]_G \subset B_{l,k}$. For $V_m \cap W_{l,i,k} \neq 0$ and $g_i \cdot x \in U_l$, let $C_{l,i,k,m}$ be the set of $y \in W_{l,i,k}$ for which

$$\mathcal{O}(y, U_l, V_k) \cap V_m \neq \emptyset.$$

$C_{l,i,k,m}$ is open and dense in $W_{l,i,k}$ whenever $g_i \cdot x \in U_l$ since

$$W_{l,i,k} \cap \mathcal{O}(g_i \cdot x, U_l, V_k) \subset C_{l,i,k,m}.$$

Hence if we let $D_{l,i,k}$ be the set of $y \in Y$ such that

$$\forall m((y \in W_{l,i,k} \wedge (U_m \cap W_{l,i,k} \neq \emptyset)) \Rightarrow \mathcal{O}(y, U_l, V_k) \cap V_m \neq \emptyset)$$

then $D_{l,i,k}$ is a dense G_δ.

Now for any y with $[y]_G$ dense and in the intersection of all the $g_j \cdot B_{l,k}$ and $g_j \cdot D_{l,i,k}$ we will have that the local orbits of y are all somewhere dense. By 3.4 and the above calculations, the set of such y is comeager.

(VII)$\Rightarrow$(IV): At once since the conclusion of (IV) must hold on X_0 by 3.16, and therefore X since X_0 is comeager in X.

For (IV)$\Rightarrow$(VII): Following the notation of (VI)$\Rightarrow$(IV) let X_0 be the set

$$(\bigcap_{i,l,k} (B_{l,k} \cap D_{l,i,k}))^{*G}.$$

$\square$

EXERCISE 3.22. If X is a turbulent Polish G-space, then there is no $\underset{\sim}{\Sigma}^1_1$ set $A \subset X$ meeting every orbit in a countable but non-empty set. (Use 2.54.)

3.3. Examples

EXAMPLE 3.23. Recalling $c_0 = \{\vec{x} \in \mathbb{R}^{\mathbb{N}} : x_i \to 0\}$ equipped with the topology given by the sup norm, $d(\vec{x}, \vec{y}) = \sup_{i \in \mathbb{N}} |x_i - y_i|$, we may let c_0 act on the ambient space $\mathbb{R}^{\mathbb{N}}$ by translation. The inclusion map $c_0 \hookrightarrow \mathbb{R}^{\mathbb{N}}$ is continuous and so this makes $\mathbb{R}^{\mathbb{N}}$ a Polish c_0-space. This is one of the simplest examples of a turbulent action.

It is clear that every orbit is dense, since c_0 is dense. We obtain every orbit being meager from c_0 being meager, which in turn follows by 2.56(i) and $(-c_0) + c_0 = c_0$ not containing an open set. As for the final condition 3.13(iii) to the effect that every orbit is somehow "locally dense", fix $\vec{x}, \vec{y} \in \mathbb{R}^{\mathbb{N}}$, $U \subset \mathbb{R}^{\mathbb{N}}$ containing $\vec{x}$, $V \subset c_0$ an open neighborhood of the identity in c_0. We may assume that for some $\epsilon > 0, l \in \mathbb{N}$

$$V = \{\vec{z} \in c_0 : \sup_{i \in \mathbb{N}} z_i < \epsilon\}$$

$$U = \{\vec{z} \in \mathbb{R}^{\mathbb{N}} : \forall i < l(|z_i - x_i| < \epsilon)\},$$

and that $\vec{y} \in U$. Fixing some arbitrary open neighborhood U_0 of $\vec{y}$ it suffices to find some $\vec{z} \in \mathcal{O}(x, U, V) \cap U_0$.

So using the density of $[x]_{c_0}$ we may find $\vec{g} \in c_0$ with $\vec{g} \cdot x \in U_0 \cap U$. Let $\delta = \sup_{\mathbb{N}} |g_i|$, and choose $k \in \mathbb{N}$ large enough that $\delta / k < \epsilon$. Let $\vec{h} = \vec{g}/k$, in the sense that $h_i = g_i / k$ at each $i \in \mathbb{N}$. Let $\vec{h}^j = \vec{h}$ for each $j < k$, and

$$\vec{x}_0 = \vec{x},$$

$$\vec{x}^{j+1} = \vec{h}^j \cdot \vec{x}^j.$$

Then $\vec{x}^k \in U_0$ and each $\vec{h}^j \in V$.

A similar result holds for $l^1(\mathbb{R}) = \{\vec{x} \in \mathbb{R}^{\mathbb{N}} : \sum |x_i| < \infty\}$ or $l^1(\mathbb{C}) = \{\vec{x} \in \mathbb{C}^{\mathbb{N}} : \sum |x_i| < \infty\}$. There is a completely general result regarding Polishable subgroups of $\mathbb{R}^{\mathbb{N}}$.

Definition 3.24. Let $G \subset \mathbb{R}^{\mathbb{N}}$ be a Polishable subgroup. G is said to be *super dense* if for all $(x_0, x_1, ...x_n) \in \mathbb{R}^{<\mathbb{N}}$ there is some $\vec{y} \in G$ with $y_i = x_i$ for all $i \leq n$.

Proposition 3.25. *Let $G \subset \mathbb{R}^{\mathbb{N}}$, $G \neq \mathbb{R}^{\mathbb{N}}$ be super dense. Then the action by translation of G on $\mathbb{R}^{\mathbb{N}}$ is turbulent.*

Proof. Let τ be the Polish topology on G. Note that by 2.56

$$\pi_n : G \to \mathbb{R}^n$$

$$\vec{x} \mapsto (x_0, x_1, ..., x_{n-1})$$

is a continuous map; by assumption of super density it is onto and so open.

Now we can simply repeat the proof from 3.23 with suitable changes. The argument that every orbit is dense and meager is as before.

So fix $U \subset \mathbb{R}^{\mathbb{N}}$ open containing $\vec{x}$, and assume that $U = \{\vec{z} \in \mathbb{R}^{\mathbb{N}} : \forall i < l(|z_i - x_i| < \epsilon)\}$. Fix $\vec{y} \in \mathbb{R}^{\mathbb{N}}$, and note that we may find some $\vec{g} \in G$ with $\vec{g} \cdot \vec{y} \in U$ by assumption on G. Let U_0 be an open set containing $g \cdot \vec{y}$ and V an open set containing 1_G. We may assume

$$U_0 = \{\vec{z} \in \mathbb{R}^{\mathbb{N}} : \forall i < n(|z_i - y_i| < \epsilon_0)\}.$$

Then we may find some $(h_0, h_1, ...) \in \mathbb{R}^n$ so that $h_i \cdot x_i = y_i$.

Since π_n is an open map we can find some open $W \subset \mathbb{R}^n$ with $\pi_n[V] = W$. Then for some sufficiently large k we will have $(h_0, h_1, ...h_{n-1})/k \in W$. Choosing $\vec{g} \in V$ with $g_i = h_i/k$ for $i < n$, we may let $\vec{x}^0 = \vec{x}, \vec{x}^{j+1} = \vec{g} \cdot \vec{x}^j$. □

Examples 3.26. Under the natural identification of $2^{\mathbb{N}}$ with $\mathbb{Z}_2^{\mathbb{N}}$ we may view the Cantor space as a compact Polish group with the operation of pointwise addition. A further identification is obtained by associating every $f \in 2^{\mathbb{N}}$ with the set $A_f \in \mathcal{P}(\mathbb{N})$ (the collection of all subsets of $\mathbb{N}$) defined by

$$n \in A_f \Leftrightarrow f(n) = 1.$$

Under this identification the group operation on $2^{\mathbb{N}}$ corresponds in $\mathcal{P}(\mathbb{N})$ to the operation of symmetric difference:– $A \Delta B = A \setminus B \cup B \setminus A$.

A subset of $\mathcal{P}(\mathbb{N})$ is said to be an *ideal* if closed under finite unions and the process of going to an arbitrary subset. An ideal on $\mathcal{P}(\mathbb{N})$ is said to be *Polishable* if it is Polishable as a group with respect to the operation of symmetric difference.

Two examples of Polishable ideals on $\mathcal{P}(\mathbb{N})$ are:

(i) the *density ideal*:

$$I_d = \{A \in \mathcal{P}(\mathbb{N}) : \frac{|A \cap \{0, 1, .., n\}|}{n+1} \to 0\};$$ [1]

(ii) the *summable ideal*:

$$I_2 = \{A \in \mathcal{P}(\mathbb{N}) : \sum_{n \in A} \frac{1}{n+1} < \infty\}.$$

These groups are Polish in the topologies generated by the metrics

$$d_0(A, B) = \sup_{n \in \mathbb{N}} \frac{|A \Delta B \cap \{0, 1, .., n\}|}{n+1}$$

[1] Where $|A \cap \{0, 1, .., n\}|$ denotes the size of the set $A \cap \{0, 1, .., n\}$.

and

$$d_1(A,B) = \sum_{n\in A\Delta B} \frac{1}{n+1}.$$

Both the actions by translation on the surrounding space $2^{\mathbb{N}}$ are turbulent. This also provides a counterpoint to the examples of 3.25, since here $\mathcal{O}(x,U,V)$ depends as much on V as it does on U – for instance in (i) (or (ii)) above, if we let $U = \mathcal{P}(\mathbb{N})$, A be any element in $\mathcal{P}(\mathbb{N})$ and

$$V_k = \{B \in I_2 : \forall n < k(n \notin B)\}$$

then

$$\overline{\mathcal{O}(A,U,V_k)} = \{C \subset \mathbb{N} : \forall n < k(n \in C \Leftrightarrow n \in A)\}.$$

More recently Alexander Kechris has in [**54**] used the notion of turbulence and work of Slawek Solecki from [**77**] to completely characterize for which Borel ideals $I \subset \mathcal{P}(\mathbb{N})$ we have the equivalence relation E_I given by

$$AE_IB \Leftrightarrow A\Delta B \in I$$

admitting classification by countable models.

A rather different kind of example for a group studied in functional analysis:

Theorem 3.27. *(Kechris, Sofronidis) The action of U_∞ upon itself by conjugation is generically turbulent.*

[**10**] shows the conjugation action of U_∞ to be meager and to have a dense orbit, but the condition (iii) from turbulence did not follow from this almost classical argument. Kechris and Sofronidis not only obtained (iii) but determined the orbits $x \in U_\infty$ satisfying condition (V) from 3.21.

We will investigate a sequence of examples similar to this one in chapters 4 and 5.

Example 3.28. Choose a rapidly increasing sequence of primes, $p_0, p_1, \ldots$, so that $p_{k+1} > kp_0p_1\ldots p_k$. Let $G = \mathbb{Z}_{p_0} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\ldots$, so the elements of G will be functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) < p_n$ and addition is mod p_n on the nth coordinate. Then let

$$H = \{f \in G : \sup_{\mathbb{N}} \frac{|f(n)|_n}{p_n}\} < \infty,$$

where $|f(n)|_n$ should be understood as indicating the least $m \geq 0$ for which $f(n) + m = p_n$ or $f(n) - m = 0$. H is a Polish group with respect to this sup norm and the action of H on G is turbulent.

In some respects this example is one of minimal algebraic complexity. Not only is the group H abelian, but also *monothetic* – the group element $\vec{1} \in H$ given by $\vec{1}(n) = \bar{1}$ for all n generates a dense cyclic subgroup of H.

Example 3.29. Finally Kechris has found an application in [**55**] which would initially appear to have no relation to the study of equivalence relations. Here he unearthed a completely new proof of a result of Preiss and Rataj to the effect that there is no analytic maximal set of pairwise orthogonal measures on any uncountable standard Borel space; in fact, this result was not only given a faster proof but considerably strengthened. Turbulence found an entrance by Kechris' embedding of a turbulent orbit equivalence relation into the equivalence relation of absolute continuity of measures, to which he then applied a version of 3.22.

3.4. Historical remarks

3.4.1. Smoothness as a notion of classification. The results of §3.1 are implicit in classical papers. Indeed some version of these results have appeared independently in many different places – for instance [**25**] and [**5**], and in [**47**] one may find a discussion of even earlier proofs that $\mathbb{R}/\mathbb{Q}$ is not smooth.

Smoothness is very commonly taken to be archetypal notion of classifiable. In the writings of some analysts, such as perhaps [**18**], one almost senses a feeling that smoothness might be the only thing that could be *meant* by providing a complete classification of an equivalence relation; that somehow this is the only sensible way one could even think about the problem. The interesting properties are those that can be presented as Borel subsets of a Polish space. A classification theorem for an equivalence relation E on a space X should consist in finding a family of invariant, preferably Borel invariant, subsets of X, and the classification is complete when any two distinct equivalence classes can be separated by a member of our family. In order for the notion of completely classifiable to have any content, and not just apply across the board to every virtually every equivalence relation, we should require that the family of invariant Borel sets be countable. Thus by 2.27(ii) we are inexorably led to the conclusion that an equivalence relation is completely classifiable if and only if it is smooth.

This is also the view point the authors of [**25**] and [**14**] have drawn from [**63**], and despite its plausibility it turns out to be a surprisingly harsh standard. It is a position from which we must conclude that even the rank one torsion free abelian groups are unclassifiable and must reject the K-theoretic classification of [**24**]; any equivalence relation into which we may embed E_0 is by this definition unclassifiable.[2]

On the other hand a completely different culture is ascendant in the branch of mathematical logic known as model theory. For instance one finds implicit in the introductory passages of [**2**] that the classification of countable structures is completely and satisfactorily settled by the Scott analysis, which we will recall at §6.1 below, and that the crucial directions of research are related to which classes of *uncountable, discrete* structures allow some higher level analogue of Scott's original analysis. From this standing it is simply a triviality that all the finite rank, or even infinite rank, torsion free abelian groups can be completely classified.

This book has tried to take from both these points of view. §3.1 recalls a coherent theory of smoothness, and the work of §3.2 came about as an attempt to parallel these classical and folklore results to more permissive notions of classification, such as the assignment of a countable structure considered up to isomorphism as a complete invariant.

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[2]A footnote is irresistible . The perspective of [**21**] is to indeed embrace rank one torsion free abelian groups as classifiable since we may associate to them elements of $\mathcal{P}(\mathbb{N})/\mathrm{Finite}$ as complete invariants – in fact they are considered classifiable precisely because $\cong|_{\mathrm{TFA}_1} \leq_B E_0$. Rather [**21**] seems to take the battle ground to be for rank *two* torsion free abelian groups.

Recently [**36**] showed that for all $n \geq 2$ one has $E_0 <_B \cong|_{\mathrm{TFA}_n}$; the argument used the non-amenability of $\mathrm{SL}_2(\mathbb{Z})$ for rank 2 and Kazhdan's property (T) for $n \geq 3$ to obtain not only $E_0 <_B \cong|_{\mathrm{TFA}_n}$ but also the *non-treeability* of $\cong|_{\mathrm{TFA}_n}$ – which in essence states that $\cong|_{\mathrm{TFA}_n}$ is not $\leq_B$ reducible to any free Borel action of a countable free group. Since then [**56**] obtained the parallel non-treeability result for $n = 2$ through an appeal to deep facts about algebraic groups.

The exact classification difficulty of the $\cong|_{\mathrm{TFA}_n}$'s remains undetermined, but at the time of writing it would appear intertwined with the general ergodic theory of countable group actions.

3.4.2. Alternative proofs. In particular §3.2 and §3.3 show that there are Polish group actions whose orbit equivalence relations are not reducible to orbit equivalence relations induced by S_∞. This question was raised at 3.5.4 and 8.2.4 of [**4**]. As a comment on the viewpoint at the time of posing, almost all the Polish group actions which had been seriously studied were so reducible; most examples of orbit equivalence relations commonly considered bore a resemblance to the logic action of S_∞ on $\mathrm{Mod}(\mathcal{L})$ or the actions of locally compact groups.

During 1996 the perspective began to change. Groups arising from real analysis such as the classical Banach spaces were subjected to increasing scrutiny, and the counterexamples that once seemed exotic became commonplace.

It should not however be suggested that 3.18 was the first demonstration that there are such unclassifiable group actions. Earlier in spring of 1995 the author had a rather different construction, using work of [**77**] to the effect that there are F_σ Polishable ideals whose coset equivalence relation is not reducible to a countable equivalence relation and the analysis in [**38**] of *essentially* F_σ equivalence relations induced by S_∞ all being reducible to countable equivalence relations. The broader results of §3.2 came later.

Indeed even after the development of turbulence, 3.18 was not the first proof of non-reducibility to S_∞-actions. Earlier it was shown by a transfinite induction along the rank of the corresponding *Scott sentence* associated to $\theta(x)$ that for any generically F_2-ergodic E we must in general further obtain that E is generically $E^Y_{S_\infty}$-ergodic. This alternative proof was not only shorter, but also more informative. Its disadvantage was in being less than completely elementary. Moreover the proof presented at 3.18 turns out to be useful in the construction from §9.2.

All well and good. Later an offhand reference was noticed in [**20**] to an unpublished proof of the density ideal's coset equivalence relation not admitting classification by countable models. Trivially this ideal is Polishable, and thus Harvey Friedman and Lee Stanley answered a principal question of [**4**] before it was ever even asked.

In early 1997 Friedman visited Los Angeles and made available the argument, which should eventually be published in [**19**]. In fact it was indeed very close to the afore mentioned induction along the Scott sentence.

3.4.3. Borel dichotomy theorems. A towering influence has been exerted on the theory of Borel equivalence relations by [**30**]:

THEOREM 3.30. *(Harrington, Kechris, Louveau) Let E be a Borel equivalence relation. Then exactly one of the following two things must happen:*

1. $E \leq_B \mathrm{id}(\mathbb{R})$*;*
2. $E_0 \sqsubseteq_c E$*.*

This is the paradigm *dichotomy theorem*: Either an equivalence relation is smooth, or its non-smoothness is explained by the embeddability of E_0. The result fails for general $\underset{\sim}{\Sigma}^1_1$ equivalence relations, even those induced by the continuous actions of Polish groups; but in [**37**] it is noted that either $E_0 \sqsubseteq_c E^X_G$ or there is UBMC (see 3.12) reduction of E^X_G to $2^{<\omega_1}$.

In view of 3.30 and a handful of similar discoveries early in the analysis of equivalence relations, one might hope that understanding turbulence, or any notion of non-classifiability, should ultimately rest on establishing the correct battery of dichotomy theorems. A definition helps to formalize this idea.

DEFINITION 3.31. An orbit equivalent relation is *minimum* turbulent if it is turbulent and Borel reducible to every other turbulent orbit equivalent. An orbit equivalent relation E_G^X is *minimal* turbulent if for all

$$E \leq_B E_G^X$$

either:

1. E admits classification by countable models; or
2. $E_G^X \leq_B E$.

[**33**] indicated that the structure here is more complicated than the smooth\non-smooth dichotomy.

THEOREM 3.32. *There is no minimum turbulent orbit equivalence relation.*

The proof of this result required showing $\mathbb{R}^{\mathbb{N}}/l^1$ and $\mathbb{R}^{\mathbb{N}}/c_0$ are $\leq_B$-incomparable, and that the first of these is *minimal*. So at least we obtain:

THEOREM 3.33. $\mathbb{R}^{\mathbb{N}}/l^1$ *(equivalently,* $2^{\mathbb{N}}/I_2$*) is minimal turbulent.*[3]

In view of these developments it seemed natural at the time of the first draft of this manuscript to conjecture that $\mathbb{R}^{\mathbb{N}}/c_0$ is also minimal turbulent. However Mike Oliver in [**70**] showed that the large mass of $\leq_B$ incomparable Borel equivalence relations (all of which are easily seen to be turbulent) from [**61**] may be reduced to $\mathbb{R}^{\mathbb{N}}/c_0$, and thus one has:

THEOREM 3.34. *(Oliver)* $\mathbb{R}^{\mathbb{N}}/c_0$ *(equivalently,* $2^{\mathbb{N}}/I_d$*) is not minimal turbulent.*[4]

Of course given 3.33 and the happy glow of Harrington, Kechris, Louveau we might still hope for a simple basis of turbulence. But in 1998 this was refuted by Ilijas Farah:

THEOREM 3.35. *(Farah; see* [**16**]*) There is no finite or even countable collection of turbulent orbit equivalence relation*

$$E_{G_1}^{X_1}, ... E_{G_k}^{X_k}, ...$$

such that for any turbulent orbit equivalence relation E_G^X *there is some* $k \in \mathbb{N}$ *with*

$$E_{G_k}^{X_k} \leq_B E_G^X.$$

And then the final stake through the heart:

THEOREM 3.36. *(Farah; see* [**17**]*) There is a turbulent orbit equivalence relation which is not* $\leq_B$ *above any minimal turbulent orbit equivalence relation.*

In short, we will never be able to produce any kind of dichotomy theorem along the lines of [**30**] and this avenue may never lead to any insight regarding when an equivalence relation admits classification by countable models.

Instead of continually revising downwards the goals for dichotomy theorems, this manuscript tries to suggest another approach. Rather than attempting to understand various divisions of classifiable versus non-classifiable in terms of a small or finite basis of unruly examples, we may analyze smoothness and classifiability by countable structures by isolating the dynamic properties necessary for their

[3] As noted in [**33**], Kechris has shown $2^{\mathbb{N}}/I_2 \leq_B \mathbb{R}^{\mathbb{N}}/l^1 \leq_B 2^{\mathbb{N}}/I_2$.

[4] Oliver also observes that $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B 2^{\mathbb{N}}/I_d \leq_B \mathbb{R}^{\mathbb{N}}/c_0$.

failure. To this end chapter 9 shows that the presence of a turbulent action is a necessary condition for an orbit equivalence relation to deny having at least a UBMC assignment of HC sets as complete invariants, just as the role of a properly generically ergodic action is always implicit in a group action that refuses a UBMC classification by bounded subsets of the countable ordinals.