

## PREFACE

Many important functions of mathematical physics have *integral representations*, i.e. are defined by integrals depending on parameters. Such functions include, in particular, the fundamental solutions of a majority of classical partial differential equations, Newton–Coulomb potentials, integral Fourier transforms, initial data of inverse tomography problems, hypergeometric functions, Feynman integrals, etc. The general construction of these integral representations is as follows. Suppose we have an analytic fiber bundle  $E \rightarrow T$ , an exterior differential form  $\omega$  on  $E$ , whose restrictions on the fibers are closed, and a family of integration cycles in these fibers, parametrized by the corresponding points of the base  $T$  and depending continuously on these points. Then the integral  $\omega$  along these cycles is a function on the base.

Analytic and qualitative properties of such functions depend on the *monodromy* of these cycles, i.e. on the natural action of the fundamental group of the base in the homology groups of the fibers: this action defines the ramification of the analytic continuation of our integral function.

The study of this action (which is a purely topological problem) allows us to answer questions on the analytic behavior of the integral function, in particular whether this function is single-valued or at least algebraic, what are the singular points of this function, and (partially) what is its asymptotics close to these points.

Ramification of integral functions arising in different problems is described by different (but having some common features) versions of the *Picard–Lefschetz theory*. Our book contains a list of such versions, including the classical local monodromy theory of singularities of functions and complete intersections, F. Pham’s generalized Picard–Lefschetz formulas, stratified version of the theory (studying in particular the monodromy of homology groups of hyperplane sections of singular varieties), and also twisted versions of all these theories (related to integrals of multivalued forms).

Using them, we study four famous classes of functions:

- *volume functions* arising in the Archimedes–Newton problem on integrable bodies;
- *Newton–Coulomb potentials*,
- *fundamental solutions of hyperbolic partial differential equations* (studied, in particular, in the lacuna theory of Hadamard–Petrovskii–Atiyah–Bott–Gårding), and
- *multidimensional (Gelfand–Aomoto) hypergeometric functions* generalizing the Gauß hypergeometric integral.

Some of main results described in the book are as follows.

1. Newton’s theorem on the algebraic nonintegrability of plane ovals (stating that the function on the space of lines in  $\mathbb{R}^2$ , associating with any line

the area cut by it from a convex domain with smooth boundary, cannot be algebraic) is extended to non-convex domains in  $\mathbb{R}^2$  and to convex domains in any even-dimensional space. In the odd-dimensional case we find numerous geometric obstructions to the algebraicity of such a function, showing that this algebraicity (taking place for a ball, accordingly to Archimedes) is a very exceptional situation.

2. In the theory of hyperbolic partial differential equations, for almost all operators with constant coefficients we prove the Atiyah–Bott–Gårding conjecture on the equivalence of the local regularity (“sharpness”) of the fundamental solution to the local topological Petrovskii condition; on the other hand we present a very degenerate operator, for which the conjecture fails. Also we find all domains of regularity close to all simple singularities of the wave front (in particular, to all singularities arising on generic fronts in  $\leq 7$ -dimensional spaces) and to many non-simple singularities. A combinatorial algorithm for finding such domains is described (its computer realization is available via <http://www.pdmi.ras.ru/~arnsem/papers>).
3. In potential theory, analyzing the results of Newton, Ivory and Arnold, we reduce the study of analytic properties of the Newtonian potential of an algebraic hypersurface to the monodromy theory of complete intersections, and find all cases when such potentials of generic hyperbolic hypersurfaces are algebraic.
4. In the Gelfand–Aomoto theory of general hypergeometric functions we calculate the exact number of linearly independent solutions of generalized hypergeometric equations for the most important types of such functions, and prove that all these solutions have integral representations. (This part is written following our joint work with I.M. Gelfand and A.V. Zelevinskiĭ.)
5. The “stratified” Picard–Lefschetz formulas are written explicitly; they greatly reduce the calculation of the ramification of homology groups of hyperplane sections of *singular* algebraic varieties and their complements.

The background of these results includes:

- standard Picard–Lefschetz theory (describing the ramification of cycles and integrals in smooth varieties),
- classification of singular points of smooth functions (essentially coinciding with the classification of singularities of wave fronts),
- stratification theory and intersection homology theory (in the terms of which the ramification of cycles on singular varieties is described),
- topological study of complete intersections (which is the natural tool for the study of the ramification of potentials),
- theory of plane arrangements,
- homology groups of local systems (to which the integration cycles of multi-valued differential forms belong, in particular of hypergeometric forms and of kernels of the Laplace operator), and
- generalizations of the Picard–Lefschetz theory for these homology groups.

The necessary information about all these subjects is collected in Chapters I, II, VI, and VIII.

**Acknowledgments.** I thank V.I. Arnold, who attracted my attention to the three problems of integral geometry studied in Chapters III–V and VII.

I thank W. Ebeling, A.M. Gabrielov, I.M. Gelfand, V.A. Ginzburg, V.V. Gorjunov, S.M. Gusein-Zade, V.P. Palamodov, A.V. Zelevinskiĭ, and B.Z. Shapiro for useful conversations.

In preparing this manuscript I was supported in part by grants from RFBR (project 01-01-00660), NWO-RFBR (047-008-005) and INTAS (00-0259).