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Preface

These are the notes to a course of lectures I gave at Munich in the fall of 1997 and again at Göteborg the following spring. I sought to give in seven lectures a unified exposition of four recent articles in the general area of quantum groups and knot polynomials. The four articles have established an interesting connection between the notion of Frobenius algebra or the more general Frobenius extension on the one hand and Hopf subalgebras [**27**], solutions of the Yang-Baxter Equation [**6**], the Jones polynomials $V_L(t)$ [**41**], and 2-dimensional topological quantum field theories [**1**] on the other hand. There are interesting possibilities for further interactions with the theories of Atiyah [**3**], Drinfel'd [**21**], Jones [**37**], Turaev [**94**] and Witten [**99**].

By a Frobenius algebra we mean a finite dimensional associative algebra A which has a nondegenerate linear functional ϕ. Equivalently, A is a Frobenius algebra if A and its dual A^* are isomorphic as left A-modules via a map $a \mapsto a\phi$. This particular characterization generalizes to a definition of Frobenius extension of rings: a finite projective ring extension $S \to A$ is Frobenius if A and its S-valued dual are isomorphic as A-S bimodules. Our purpose then is to elaborate on these ideas, providing a relatively thorough account of Frobenius theory based on the literature by Brauer, Eilenberg, Kasch, Nakayama, Morita, and others. We also study separability in relation to Frobenius theory, which is important to two of the recent articles, [**6**, **41**]. Then a unified exposition is given the four recent topics named above. As a bonus, an interaction of the recent topics and Frobenius theory is seen in several of the theorems and propositions.

In Chapters 1 and 2, and in Sections 4.1 and 7.1, we develop the theory of Frobenius extensions. In Section 4.2 we develop some more theory for Frobenius algebras over commutative rings; in Section 4.3, for the classical Frobenius algebras over fields; and in Section 6.1, for augmented Frobenius algebras over commutative rings. Sections 2.4, 2.5, 2.6, 5.3, and 7.1 study the relationship of Frobenius extensions with separability.

Chapter 3 elaborates on [**41**] by the author, and shows that a certain separable Frobenius extension of algebras with trace, which we introduce here as a Markov extension, has an endomorphism ring theorem and a key idempotent. We iterate this construction, generating a tower of algebras, and obtaining the characters for the braid groups that Jones obtained via II_1 subfactors. We thereby derive the Jones knot and link polynomials $V_L(t)$ [**37**] by a general method. The mechanism in building the tower of algebras is the endomorphism ring of the generic type of *relatively semisimple* pair given by a Markov extension of algebras over an algebraically closed field.

In Chapter 4 we study Frobenius algebras and prove a result of [**6**, Beidar-Fong-Stolin]. We show that the Frobenius element of the tensor-square of a Frobenius algebra A over a commutative ring k is a solution of the Yang-Baxter Equation, though not necessarily an invertible one. The proof depends on the Nakayama automorphism of a Frobenius algebra. The question of when a solution is invertible is postponed until Chapter 5. Chapter 4 ends with a sketch of [**1**, Abrams] where it is proven that 2-dimensional topological quantum field theories are equivalent to commutative Frobenius algebras.

In Chapter 5 we develop the theory of Azumaya algebras in a self-contained section. We study three characterizations of Azumaya algebras which provide a simplification in the proof of two theorems. First, we prove that the solutions from Chapter 4 of the Yang-Baxter equation are invertible solutions if and only if A is an Azumaya k-algebra. Second, Azumaya algebras with invertible Hattori-Stallings rank are strongly separable, symmetric Frobenius algebras, a theorem of [**16, 30**, DeMeyer, Hattori].

Based on [**45**, author-Stolin], Chapter 6 develops a Frobenius theory for a Hopf algebra H which is finite projective over a commutative ring with trivial Picard group. Certain key properties of norms and integrals together with modular functions are already given for augmented Frobenius algebras. We then study the basic theorem of Larson-Pareigis-Sweedler [**57, 75**] which informs us that H is an augmented Frobenius algebra. We find an explicit Frobenius system (Frobenius homomorphism with dual base) with Nakayama automorphism for H. Then we obtain other Frobenius systems for H by applying the antipode S, its compositional inverse S^{-1} and the automorphism S^2. The Frobenius theory in Chapter 1 which compares Frobenius systems and Nakayama automorphisms of the same algebra then yields two well-known Radford formulas involving S^2 and S^4. As a corollary to the first of these formulas, we show that Drinfeld's quantum double $D(H)$ is a unimodular, symmetric algebra.

In Chapter 7 we restrict our attention to finite dimensional Hopf algebras after making a study of β-Frobenius extensions in Section 7.1. Based on [**27**], Sections 7.2 and 7.3 show explicitly that a Hopf subalgebra K of a finite dimensional Hopf algebra H is a Frobenius extension H/K twisted on one side by an automorphism β of K. A β-Frobenius system for H/K is given in terms of a relative Nakayama automorphism and the Frobenius system studied in Chapter 6. We conclude with several examples of Frobenius extensions drawn from the Taft algebra and the quantum double.

No prior knowledge of Frobenius algebras, separable algebras, or Hopf algebras is assumed of the reader, although a look at two of the early chapters of [**15**, Curtis-Reiner] on Frobenius and separable algebras, and the first several chapters in [**90**, Sweedler] on Hopf algebras will put much of this material in perspective. An index is provided at the end of this paper. A brief historical account and guide to the literature are given in Appendix A.

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