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## Preface

This book is based on courses of lectures which I delivered at University of Tokyo, Nagoya University, Osaka University and Tokyo Institute of Technology between 1996 and 1998.

The purpose of the lectures was to discuss various properties of the Hilbert schemes of points on surfaces. This object is originally studied in algebraic geometry, but as it has been realized recently, it is related to many other branches of mathematics, such as singularities, symplectic geometry, representation theory, and even to theoretical physics. The book reflects this feature of Hilbert schemes. The subjects are analyzed from various points of view. Thus this book tries to tell the harmony between different fields, rather than focusing attention on a particular one.

These lectures were intended for graduate students who have basic knowledge on algebraic geometry (say chapter 1 of Hartshorne: "Algebraic Geometry", Springer) and homology groups of manifolds. Some chapters require more background, say spectral sequences, Riemannian geometry, Morse theory, intersection cohomology (perverse sheaves), etc., but the readers who are not comfortable with these theories can skip those chapters and proceed to other chapters. Or, those readers could get some idea about these theories before learning them in other books.

I have tried to make it possible to read each chapter independently. I believe that my attempt is almost successful. The interdependence of chapters is outlined in the next page. The broken arrows mean that we need only the statement of results in the outgoing chapter, and do not need their proof.

Sections 9.1, 9.4 are based on A. Matsuo's lectures at the University of Tokyo. His lectures contained Monster and Macdonald polynomials. I regret omitting these subjects. I hope to understand these by Hilbert schemes in the future.

The notes were prepared by T. Gocho and N. Nakamura. I would like to thank them for their efforts. I am also grateful to A. Matsuo and H. Ochiai for their comments throughout the lectures. A preliminary version of this book has been circulating since 1996. Thanks are due to all those who read and reviewed it, in particular to V. Baranovsky, P. Deligne, G. Ellingsrud, A. Fujiki, K. Fukaya, M. Furuta, V. Ginzburg, I. Grojnowski, K. Hasegawa, N. Hitchin, Y. Ito, A. King, G. Kuroki, M. Lehn, S. Mukai, I. Nakamura, G. Segal, S. Strømme, K. Yoshioka, and M. Verbitsky. Above all I would like to express my deep gratitute to M. A. de Cataldo for his useful comments throughout this book.

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# Introduction

Moduli spaces parametrizing objects associated with a given space $X$ are a rich source of spaces with interesting structures. They usually inherit some of the structures of $X$, but sometimes even more: they have more structures than $X$ has, or pull out hidden structures of $X$. The purpose of these notes is to give an example of these phenomena. We study the moduli space parametrizing 0-dimensional subschemes of length $n$ in a nonsingular quasi-projective surface $X$ over $\mathbb{C}$. It is called the Hilbert scheme of points, and denoted by $X^{[n]}$.

An example of a 0-dimensional subscheme is a collection of distinct points. In this case, the length is equal to the number of points. When some points collide, more complicated subschemes appear. For example, when two points collide, we get *infinitely near points*, that is a pair of a point $x$ and a 1-dimensional subspace of the tangent space $T_xX$. This shows the difference between $X^{[n]}$ and the $n$th symmetric product $S^nX$, on which the information of the 1-dimensional subspace is lost.

However, when $X$ is 1-dimensional, we have a unique 1-dimensional subspace in $T_xX$. In fact, the Hilbert scheme $X^{[n]}$ is isomorphic to $S^nX$ when $\dim X = 1$.

When $X$ is 2-dimensional, $X^{[n]}$ is smooth and there is a morphism $\pi\colon X^{[n]} \to S^nX$ which is a resolution of singularities by a result of Fogarty [**31**]. This presents a contrast to Hilbert schemes for $\dim X > 2$, which, in general, have singularities.

As we mentioned at the beginning, $X^{[n]}$ inherits structures from $X$. First of all, it is a scheme. It is projective if $X$ is projective. These facts follow from Grothendieck's construction of Hilbert schemes. A nontrivial example is a result by Beauville [**11**]: $X^{[n]}$ has a holomorphic symplectic form when $X$ has one. When $X$ is projective, $X$ has a holomorphic symplectic form only when $X$ is a $K3$ surface or an abelian surface by classification theory. We also have interesting noncompact examples: $X = \mathbb{C}^2$ or $X = T^*\Sigma$ where $\Sigma$ is a Riemann surface. These examples are particularly nice because of the existence of a $\mathbb{C}^*$-action, which naturally induces an action on $X^{[n]}$. (See Chapter 7.) Moreover, for $X = \mathbb{C}^2$, we shall construct a hyper-Kähler structure on $X^{[n]}$ in Chapter 3.

These structures of $X^{[n]}$, discussed in the first half of this note, are inherited from $X$. We shall begin to study newly arising structures in later chapters. They appear when we consider the components $X^{[n]}$ *all together*. We will encounter this phenomenon first in Göttsche's formula for the Poincaré polynomials in Chapter 6. Their generating function is given by

$$\sum_{n=0}^{\infty} q^n P_t(X^{[n]}) = \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1(X)}(1+t^{2m+1}q^m)^{b_3(X)}}{(1-t^{2m-2}q^m)^{b_0(X)}(1-t^{2m}q^m)^{b_2(X)}(1-t^{2m+2}q^m)^{b_4(X)}},$$

where $b_i(X)$ is the $i$th Betti number of $X$. Each individual term $P_t(X^{[n]})$ has no nice expression. It is even more apparent if we consider the generating function of Euler numbers: The answer is the power of the Dedekind $\eta$-function (the term $q^{1/24}$ is missing), which has a nice modular property. Recall that the $q$-expansion coefficients of modular forms often have number theoretic meaning, but structures appear after they are treated all together. In fact, the above formula appears in a very different context. It coincides with the character formula for a representation of products of the Heisenberg and Clifford algebras, a kind of infinite dimensional Lie (super)algebras. In Chapter 8 we shall pursue this line of idea. We shall construct a representation of products of the Heisenberg and Clifford algebras on the direct sum of homology groups of all components $\bigoplus_n H_*(X^{[n]})$ in a geometric way. In Chapter 9, we take our construction further. We shall construct a representation of an affine Lie algebra (or a structure of the vertex algebra) on the direct sum of homology groups. In these constructions, it is important to treat all $H_*(X^{[n]})$ ($n = 0, 1, \dots$) simultaneously, since operators of these algebras map $H_*(X^{[n]})$ to another $H_*(X^{[n']})$. Each $H_*(X^{[n]})$ is a weight space of a representation. Note that the characters of representations (or the string functions) of affine Lie algebras have the modular property, as observed by Kac-Peterson (see [**67**, Chapter 13]).

This appearance of modular forms and the affine Lie algebra surprises us very much since we usually think that they are connected with loops or elliptic curves. Recall that the affine Lie algebra is the central extension of the Lie algebra of loops in a finite dimensional Lie algebra, and plays an important role in conformal field theory. For example, the above mentioned modular property of its character can be explained if one interprets it as a partition function of the conformal field theory on an elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. Since it depends only on the complex structure of $E_\tau$, it is invariant under $\tau \mapsto -1/\tau$. However, we could not find either loops or elliptic curves in Hilbert schemes on surfaces. To make matters worse, our construction does not give any clue of the actual reason for the appearance of the affine Lie algebra. What we shall do here is only (1) to define operators on $\bigoplus_n H_*(X^{[n]})$ which correspond to generators of the Heisenberg/Clifford algebra, and (2) to check the defininig relations. We do not have a good explanation why the defining relations hold, though we can check them.

Thus we are lead to wonder why we encounter objects, such as modular forms or affine Lie algebras. Formally, we have two possibilities: one is that these objects are so universal (like groups) that they appear everywhere. The other possiblity is that elliptic curves are hidden in the Hilbert schemes. The answer is not given in this book. In fact, we do not know which is correct at this moment, thus the question is left open for the future.

As we explained above, it is important to study the direct sum $\bigoplus_n H_*(X^{[n]})$, not individual $H_*(X^{[n]})$. If we want to see loops or elliptic curves in Hilbert schemes, we should look for it in the generating function of the Hilbert schemes:

$$\sum_{n=0}^{\infty} X^{[n]} q^n.$$

Moreover, this generating function should have some kind of modular properties. Unfortunately, there is no theory for generating functions (or generating spaces?) of manifolds at this moment. But the author believes the above object should exists.

It might be true that an appropriate language here is *string theory.* The relation between $X^{[n]}$ and $S^nX$ have many similarities with the relation between string and field theory. In fact, one of our motivation comes from physics. It is *duality.* Let us explain it briefly. In [**116**], Vafa and Witten have pointed out that the $S$-duality (or Montonen-Olive duality) conjecture implies that the generating function of Euler numbers of moduli spaces of instantons has a modular property. When the base manifold is a $K3$ surface, the Euler numbers of moduli spaces of instantons are the same as those of Hilbert schemes of points (strictly speaking, we must consider moduli spaces of stable *sheaves* instead of moduli spaces of instantons, which are usually noncompact). Then Göttsche's formula gives us the desired answer. Moreover, from results in [**93**], we can see that the homology groups of moduli spaces of sheaves on an ALE space (the minimal resolution of a simple singularity) form an integrable representation of an affine Lie algebra. Then the generating function of Euler numbers of moduli spaces is a character of the representation. Thus, as explained above, the modular property follows. More recently, Vafa [**115**] suggested that our Heisenberg algebra and the above affine Lie algebra could be understood in the frame work of the heterotic-type IIA duality. (See also Harvey-Moore [**53**].)

Let us briefly give some historical comments on the construction of the action of the Heisenberg/Clifford algebra on the homology groups of the Hilbert schemes, which lead the author to write these notes. Motivated by constructions of the lower triangular part of the quantized enveloping algebra by Ringel and Lusztig ([**108, 82**]), the author constructed integrable highest weight representations of the affine Lie algebra on the homology group of moduli spaces of torsion-free sheaves on an ALE space. (The construction [**93, 99**] was given in terms of moduli spaces of representations of quivers, whose identification with moduli spaces of torsion-free sheaves is proved by a modification of [**78**].) The generators of the affine Lie algebra (as a Kac-Moody algebra) are given by moduli spaces of parabolic sheaves regarded as correspondences in products of two moduli spaces. Thus it is important to treat moduli spaces with different Chern classes *all together.* Parabolic sheaves consist of pairs of sheaves which are isomorphic outside a given curve. Generators of the Heisenberg/Clifford algebra acting on homology groups of the Hilbert schemes (Chapter 8) are again given by certain correspondences. They consist of pairs of ideals which are isomorphic outside a point, which may move. Thus these two constructions have little differences, but also many common features. The author expects affine Lie algebras (or similar objects ?) acts on homology groups of moduli spaces of sheaves for general projective surfaces. This will be the subject of a further study.

This book is organized as follows. In the first chapter, we collect basic facts on the Hilbert schemes of points on surfaces, needed in later chapters. In Chapter 2, the Hilbert scheme (more generally moduli spaces of torsion free sheaves) on the affine plane $\mathbb{C}^2$ is shown to be isomorphic to a certain space of quadruple of matrices. This description is useful to study the Hilbert scheme. In Chapter 3, the Hilbert scheme of points on $\mathbb{C}^2$ is identified with a hyper-Kähler quotient of a certain quaternion vector space. In particular, it has a hyper-Kähler metric. The relationship between the moment map in symplectic geometry and geometric invariant theory is briefly discussed. In Chapter 4, we construct the minimal resolution of simple singularities, using the Hilbert scheme on $\mathbb{C}^2$. Some properties of the minimal resolution are discussed from this point of view. For example, the minimal resolution inherits the hyper-Kähler metric on the Hilbert scheme. In Chapter 5, we compute the

Poincaré polynomial of the Hilbert scheme of points on $\mathbb{C}^2$ using the natural torus action. The corresponding moment map is a natural Morse function on the Hilbert scheme. In Chapter 6, we generalize this formula to the case of a general surface $X$. In Chapter 7, the Hilbert scheme is studied when the base space is the cotangent bundle of a Riemann surface. In Chapter 8, we shall construct a representation of products of the Heisenberg algebras and the Clifford algebras on the homology group of the Hilbert scheme. Finally in Chapter 9, we shall study various homology classes arising from an embedded curve. They have close relation to symmetric functions and vertex operators.

There are other books on Hilbert schemes on surfaces such as [**62, 45**]. We try to avoid duplicating materials. Those readers who are interested in Hilbert schemes can use these books. There are large literature on sheaves on surfaces, since it becomes a big subject after Donaldson's results on the relation between moduli spaces of sheaves on a surface and differential topology of the underlying 4-manifold (see e.g., [**24**]). We refer to [**61**] for a survey from algebro-geometric point of view. Our treatements on the representation theory of affine Lie algebras and vertex algebras are very brief. The reader can consult, for example, Kac's book [**68**].

CHAPTER 1

# Hilbert scheme of points

In this chapter, we collect basic facts on the Hilbert scheme of points on a surface. We do not assume the field $k$ is $\mathbb{C}$ unless it is explicitly stated.

## 1.1. General Results on the Hilbert scheme

First, we recall the definition of the Hilbert scheme in general (not necessarily of points, nor on a surface). Let $X$ be a projective scheme over an algebraically closed field $k$ and $\mathcal{O}_X(1)$ an ample line bundle on $X$. We consider the contravariant functor $\mathcal{H}ilb_X$ from the category of schemes to the category of sets

$$\mathcal{H}ilb_X \colon [\text{Schemes}] \to [\text{Sets}],$$

which is given by

$$\mathcal{H}ilb_X(U) = \left\{ Z \subset X \times U \;\middle|\; \begin{array}{l} Z \text{ is a closed subscheme,} \\ \begin{array}{ccc} Z & \overset{i}{\hookrightarrow} & X \times U \\ \pi \downarrow & & \downarrow p_2 \\ U & = & U \end{array} \; : \pi \text{ is flat} \end{array} \right\}.$$

Namely, $\mathcal{H}ilb_X$ is a functor which associates a scheme $U$ with a set of families of closed subschemes in $X$ parametrized by $U$. Let $\pi\colon Z \to U$ be the projection. For $u \in U$, the Hilbert polynomial in $u$ is defined by

$$P_u(m) = \chi(\mathcal{O}_{Z_u} \otimes \mathcal{O}_X(m)),$$

where $Z_u = \pi^{-1}(u)$. Since $Z$ is flat over $U$, $P_u$ is independent of $u \in U$ if $U$ is connected. Conversely, for each polynomial $P$, let $\mathcal{H}ilb_X^P$ be the subfunctor of $\mathcal{H}ilb_X$ which associates $U$ with a set of families of closed subschemes in $X$ parametrized by $U$ which has $P$ as its Hilbert polynomial. Now the basic fact proved by Grothendieck is the following theorem.

Theorem 1.1 (Grothendieck [**50**]). *The functor $\mathcal{H}ilb_X^P$ is representable by a projective scheme $\mathrm{Hilb}_X^P$.*

This means that there exists a universal family $\mathcal{Z}$ on $\mathrm{Hilb}_X^P$, and that every family on $U$ is induced by a unique morphism $\phi\colon U \to \mathrm{Hilb}_X^P$.

Moreover, if we have an open subscheme $Y$ of $X$, then we have the corresponding open subscheme $\mathrm{Hilb}_Y^P$ of $\mathrm{Hilb}_X^P$ parametrizing subschemes in $Y$. In particular, $\mathrm{Hilb}_Y^P$ is defined for a quasi-projective scheme $Y$.

The proof of this theorem is not given in this book. But we shall give a concrete description when $P$ is a constant polynomial and $X$ is the affine plane $\mathbb{A}^2$ (see Theorem 1.9). Once this is established, one can prove the representatibity of a similar functor for a nonsingular complex surface $X$ by a patching argument (see §1.5). In particular, we get $\mathrm{Hilb}_X^P$ as a complex manifold, and this is practically enough for our later purposes. Anyway, we do not need the proof of the theorem in this book.

DEFINITION 1.2. Let $P$ be the constant polynomial given by $P(m) = n$, for all $m \in \mathbb{Z}$. We denote by $X^{[n]} = \mathrm{Hilb}_X^P$ the corresponding Hilbert scheme and call it the Hilbert scheme of $n$ points in $X$.

Let $x_1, x_2, \ldots, x_n \in X$ be $n$ distinct points and consider $Z = \{x_1, x_2, \ldots, x_n\} \subset X$ as a closed subscheme. Since the structure sheaf of $Z$ is given by

$$\mathcal{O}_Z = \bigoplus_{i=1}^{n} \text{the skyscraper sheaf at } x_i \ ,$$

we have $\mathcal{O}_Z \otimes \mathcal{O}_X(m) = \mathcal{O}_Z$, for all $m \in \mathbb{Z}$, and hence $Z \in X^{[n]}$. This is the reason why $X^{[n]}$ is called the Hilbert scheme of $n$ points in $X$.

Let $S^nX$ be the $n$th symmetric product of $X$, i.e.

$$S^nX = \underbrace{X \times \cdots \times X}_{n \text{ times}} / \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ is the symmetric group of degree $n$. It parametrizes effective zero-dimensional cycles of degree $n$ in $X$. Its elements are written as a formal sum $\sum n_i[x_i]$, where $n_i \in \mathbb{N}$ with $\sum n_i = n$. Roughly speaking, $X^{[n]}$ is "the moduli space of $n$ points in $X$", but in general it is much more complicated and interesting than the symmetric product $S^nX$ as we shall see in the course of these lectures. In order to see the difference, we suppose $X$ to be nonsingular and consider $X^{[2]}$. As mentioned above, $\{x_1, x_2\}$ can be considered as a point in $X^{[2]}$ if $x_1$ and $x_2$ are distinct points. What happens when $x_1$ and $x_2$ collide ? For each point $x \in X$, a vector $v \neq 0 \in T_xX$ defines an ideal $\mathcal{J} \subset \mathcal{O}_X$ given by

$$\mathcal{J} = \{f \in \mathcal{O}_X \mid f(x) = 0, df_x(v) = 0\}. \tag{1.3}$$

Then $\mathcal{J}$ has colength 2, and hence $\mathcal{O}_X/\mathcal{J}$ defines a zero-dimensional subscheme $Z$ in $X^{[2]}$. We can regard $Z$ as a set of 2 points in $X$ infinitesimally attached to each other in the direction of $v$, and $X^{[2]}$ contains a set of this kind of infinitely near points in $X$. Actually, this gives a complete description of $X^{[2]}$. Namely,

$$X^{[2]} = \mathrm{Blow}_\Delta(X \times X)/\mathfrak{S}_2, \tag{1.4}$$

where $\mathrm{Blow}_\Delta(X \times X)$ is the blowup of $X \times X$ along the diagonal $\Delta$. We shall show this explicitly when $X$ is $\mathbb{A}^2$ later in this section.

When $\dim X = 1$, it is known that $X^{[n]} = S^nX$, and $X^{[n]}$ is actually the moduli space of $n$ points in $X$. This is roughly because we have only one direction in which two different points can collide, and hence only the positions of points and their multiplicities are relevant. For example, the Hilbert scheme of $n$ points in the affine line $\mathbb{A}$ is

$$\begin{aligned}
\mathbb{A}^{[n]} &= \{I \subset k[z] \mid I \text{ is an ideal, } \dim_k k[z]/I = n\} \\
&= \{f(z) \in k[z] \mid f(z) = z^n + a_1 z^{n-1} + \cdots + a_n,\ a_i \in k\} \\
&= S^n\mathbb{A}.
\end{aligned}$$

In general, the relation between $X^{[n]}$ and $S^nX$ is given by the following theorem.

THEOREM 1.5 ([**91**, 5.4]). *There exists a morphism*

$$\pi \colon X^{[n]}_{\mathrm{red}} \to S^nX$$

*defined by*

$$\pi(Z) = \sum_{x \in X} \operatorname{length}(Z_x)[x].$$

This morphism $\pi$ is called the *Hilbert-Chow morphism* which associates a closed subscheme with its corresponding cycle. For example, $\mathcal{O}_X/\mathcal{J} \in X^{[2]}$ is mapped to $2[x]$ in the example (1.3).

Let $\nu = (\nu_1, \dots, \nu_k)$ be a partition of $n$, namely, a finite sequence of nonincreasing positive integers $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k > 0$ with $\sum_{i=1}^{k} \nu_i = n$. For each partition $\nu$ of $n$, we define

$$S^n_\nu X = \left\{ \sum_{i=1}^{k} \nu_i [x_i] \in S^n X \;\middle|\; x_i \neq x_j \text{ for } i \neq j \right\}.$$

Then $S^n_\nu X$ has dimension $k \dim X$, where the number $k$ is called the length of $\nu$ and denoted by $l(\nu)$. We have the stratification of $S^n X$,

$$S^n X = \bigcup_\nu S^n_\nu X. \tag{1.6}$$

The open stratum $S^n_{(1,\dots,1)}$ is nonsingular if $X$ is.

NOTATION 1.7. For later purpose, we give another piece of notation for partitions.

(1) We may add an infinite string of zeros at the end of the partition $\nu$ as $\nu = (\nu_1, \dots, \nu_k, 0, 0, \dots)$. Thus a partition is an infinite sequence of non-increasing non-negative integers containing only finitely many non-zero terms.

(2) We may write a partition $\nu = (\nu_1 \geq \cdots \geq \nu_k)$ of $n$ as $\nu = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$ indicating the number of times each positive integer occurs in $\nu$, i.e. $\alpha_i = \#\{l \mid \nu_l = i\}$. In this notation, we may or may not write terms with $\alpha_i = 0$.

EXERCISE 1.8. (1) Suppose that $X$ is nonsingular with $\dim X = 1$. Show that $X^{[n]} = S^n X$ is nonsingular.

(2) Show that $S^n(\mathbb{P}^1) = \mathbb{P}^n$.

## 1.2. Hilbert scheme of points on the plane

In the previous section we have consider $X^{[n]}$ when $\dim X = 1$. In the rest of this book, we study, from various points of view, the next simplest case, namely the case when $\dim X = 2$. As a starting point, we now study the case when $X$ is the affine plane $\mathbb{A}^2$. This is the case which provides us a very good toy model for the subjects of these lectures. In fact, $(\mathbb{A}^2)^{[n]}$ has the following handy description.

THEOREM 1.9. *Let*

$$\widetilde{H} \overset{\text{def.}}{=} \left\{ (B_1, B_2, i) \;\middle|\; \begin{array}{ll} \text{(i)} & [B_1, B_2] = 0 \\ \text{(ii)} & \textit{(stability) There exists no proper subspace } S \subsetneq k^n \\ & \textit{such that } B_\alpha(S) \subset S \ (\alpha = 1, 2) \textit{ and } \operatorname{im} i \subset S \end{array} \right\}$$

*where $B_\alpha \in \operatorname{End}(k^n)$ and $i \in \operatorname{Hom}(k, k^n)$. Define an action of $\mathrm{GL}_n(k)$ on $\widetilde{H}$ by*

$$g \cdot (B_1, B_2, i) = (g B_1 g^{-1}, g B_2 g^{-1}, g i),$$

*for $g \in \mathrm{GL}_n(k)$, and consider the quotient space $H \overset{\text{def.}}{=} \widetilde{H}/\mathrm{GL}_n(k)$. Then $H$ is a nonsingular variety and represents the functor $\mathcal{H}ilb^P_X$ for $X = \mathbb{A}^2$, $P = n$.*

Before starting the proof, we explain how data $(B_1, B_2, i)$ arise from a (closed) point in $(\mathbb{A}^2)^{[n]}$. From the definition, a point in $(\mathbb{A}^2)^{[n]}$ is an ideal $I \subset k[z_1, z_2]$ such that $\dim_k(k[z_1, z_2]/I) = n$. Then we can define an $n$-dimensional vector space $V$ by $k[z_1, z_2]/I$. The multiplication by $z_\alpha \bmod I$ $(\alpha = 1, 2)$ give endomorphisms $B_\alpha \in \operatorname{End}(V)$. We also have a homomorphism $i \in \operatorname{Hom}(k, V)$ by $i(1) = 1 \bmod I$. It follows that $[B_1, B_2] = 0$, and the stability condition also holds since 1 multiplied by products of $z_1$'s and $z_2$'s span the whole of $k[z_1, z_2]$.

Conversely, if we have $(B_1, B_2, i)$ as in the theorem, we can define the map $\phi\colon k[z_1, z_2] \to k^n$ by $\phi(f) = f(B_1, B_2)i(1)$. Since $\operatorname{im}\phi$ is $B_\alpha$-invariant and contains $\operatorname{im} i$, it must be $k^n$ by the stability condition. Hence $\phi$ is surjective, and if we define $I = \ker\phi$, $I$ is an ideal in $k[z_1, z_2]$ and $\dim_k(k[z_1, z_2]/I) = n$. It is also clear that these two maps are mutually inverse to each other. Thus the quotient space $H$ is isomorphic to the set of codimension $n$ ideals of $k[z_1, z_2]$, as a set.

The second condition will be explained from geometric invariant theory in Chapter 3, Lemma 3.25. This is the reason why it is called *stability*.

Proof of Theorem 1.9. First we show that the differential of the map $(B_1, B_2, i) \mapsto [B_1, B_2]$ has constant rank. The cokernel of the differential is given by

$$\{\xi | \operatorname{tr}(\xi([B_1, \delta B_2] + [\delta B_1, B_2]) = 0 \text{ for any } \delta B_1,\, \delta B_2\} = \{\xi | [\xi, B_1] = [\xi, B_2] = 0\},$$

where $\xi$ is an endomorphism of $k^n$. Then the assignment $\xi \mapsto \xi(i(1))$ define a map from the cokernel to $k^n$. Conversely if we have $v \in k^n$, we can give $\xi$ by setting

$$\xi(B_1^l B_2^m i(1)) \overset{\text{def.}}{=} B_1^l B_2^m v \qquad l, m \in \mathbb{Z}_{\geq 0}.$$

Since $B_1^l B_2^m i(1)$ $(l, m \in \mathbb{Z}_{\geq 0})$ span $k^n$ by the stability condition, this gives a well-defined endomorphism of $k^n$. This is the inverse of the above map. In particular, the cokernel has dimension $n$. This implies that the variety $\widetilde{H}$ is nonsingular.

Next prove that the stabilizer of $\mathrm{GL}_n(k)$-action at $(B_1, B_2, i)$ is trivial. If $g \in \mathrm{GL}_n(k)$ stabilizes $(B_1, B_2, i)$, i.e. $gB_1g^{-1} = B_1$, $gB_2g^{-1} = B_2$, $gi = i$, the kernel of $g - \mathrm{id}$ contains $\operatorname{im} i$ and is invariant under $B_1$, $B_2$. Thus the stability condition implies $g = \mathrm{id}$.

Hence the quotient space $H = \widetilde{H}/\mathrm{GL}_n(k)$ has a structure of nonsingular variety such that the map $\widetilde{H} \to H$ is a principal étale fiber bundle for the group $\mathrm{GL}_n(k)$ by Luna's slice theorem [**81**]. Moreover, it carries a flat family $\mathcal{H} \to H$ of 0-dimensional subschemes of $\mathbb{A}^2$ given by the natural surjection

$$k[z_1, z_2] \ni f(z_1, z_2) \longmapsto f(B_1, B_2)i(1) \in k^n.$$

Finally, let us check the universality of $\mathcal{H} \to H$. We need to show that if $\pi\colon Z \to U$ is a flat family of 0-dimensional subschemes of $\mathbb{A}^2$ of length $n$, then there exists a unique morphism $\phi\colon U \to H$ such that the pull-back $\phi^*\mathcal{H} \to U$ is $Z \to U$.

Let $\pi\colon Z \to U$ be a such a family. Then $\pi_*\mathcal{O}_Z$ is a locally free sheaf of rank $n$. As above, we define $B_1$, $B_2$ from multiplication of coordinate functions $z_1$, $z_2$, and $i$ from the constant polynomial 1. Then, $B_1$, $B_2$ are commuting $\mathcal{O}_U$-linear endomorphisms of $\pi_*\mathcal{O}_Z$, and $i$ is a homomorphism $\mathcal{O}_U \to \pi_*\mathcal{O}_Z$.

Fix an open covering $\bigcup_\lambda U_\lambda$ of $U$ and trivializations of the restriction of $\pi_*\mathcal{O}_Z$ to $U_\lambda$. Then the above $(B_1, B_2, i)$ defines morphisms $U_\lambda \to \widetilde{H}$. If we compose them with $\widetilde{H} \to H$, they glue together to define a morphism $\phi\colon U \to H$. By construction $\phi^*\mathcal{H}$ is $Z$. The uniqueness of such a homomorphism is also clear. □

REMARK 1.10. (1) For $k = \mathbb{C}$, we will give another argument to show the smoothness of $H$ in Corollary 3.42. The essential point is the observation that $H$ is a hyper-Kähler quotient. Then we will observe that the smoothness of $\widetilde{H}$ is a formal consequence of the free-ness of the $\mathrm{GL}_n(k)$-action, and that the smoothness of $H$ follows from the existence of a slice for a compact Lie group action, instead of Luna's slice theorem. Thus the proof will become natural and elementary.

(2) The above proof gives the representability of the functor $\mathcal{H}ilb_X^P$ in the special case when $X = \mathbb{A}^2$, $P = n$ without invoking Theorem 1.1. In a preliminary version of this book, we use Theorem 1.1 and then identify $H$ with $\mathrm{Hilb}_X^P$ as a set. The above refinement is due to M.A. de Cataldo and L. Migliorini [**22**].

(3) As shown in the proof of the theorem, the ideal $I$ corresponding to $(B_1, B_2, i)$ is given by

$$I = \{f(z) \in k[z_1, z_2] \mid f(B_1, B_2)i(1) = 0\}.$$

Note that this can be written as

$$I = \{f(z) \in k[z_1, z_2] \mid f(B_1, B_2) = 0\},$$

by the stability condition.

(4) The above description gives the tangent space of $(\mathbb{A}^2)^{[n]}$ at the point corresponding to $(B_1, B_2, i)$. Let us consider the complex

$$\mathrm{Hom}(k^n, k^n) \xrightarrow{d_1} \begin{matrix} \mathrm{Hom}(k^n, k^n) \\ \oplus \\ \mathrm{Hom}(k^n, k^n) \\ \oplus \\ k^n \end{matrix} \xrightarrow{d_2} \mathrm{Hom}(k^n, k^n), \tag{1.11}$$

where $d_1(\xi) = \begin{pmatrix} [\xi, B_1] \\ [\xi, B_2] \\ \xi i \end{pmatrix}$, $d_2 \begin{pmatrix} C_1 \\ C_2 \\ I \end{pmatrix} = [B_1, C_2] + [C_1, B_2]$. The homomorphism $d_1$ is the derivative of the $\mathrm{GL}_n(k)$-action, and $d_2$ is that of the map $(B_1, B_2, i) \mapsto [B_1, B_2]$. We have shown that the cokernel of $d_2$ has dimension $n$. It is also easy to see the kernel of $d_1$ is trivial by the stability condition. The above construction of $H$ implies that the tangent space is the middle cohomology group of the above complex. Note also that it has dimension $2n$.

Now let us examine some examples.

EXAMPLE 1.12. (1) First, we consider the case of $n = 1$. We write $B_1 = \lambda, B_2 = \mu \in k$. From the stability condition, $i$ must be non-zero. Hence we may assume $i = 1$, after applying the action of $\mathrm{GL}_1(k)$ if necessary. The corresponding ideal is given by

$$I = \{f(z) \in k[z_1, z_2] \mid f(\lambda, \mu) = 0\}.$$

Therefore it corresponds to the point $(\lambda, \mu) \in k^2$, and this gives the description $(\mathbb{A}^2)^{[1]} \cong \mathbb{A}^2$.

(2) Next, we consider the case when $n = 2$. Suppose either $B_1$ or $B_2$ has two distinct eigenvalues. It is easy to see that we can assume $B_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and

$B_2 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ with $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$, and we can also assume that $i(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by the stability condition. The corresponding ideal is given by

$$I = \{f(z) \in k[z_1, z_2] \mid f(\lambda_1, \mu_1) = f(\lambda_2, \mu_2) = 0\}.$$

Therefore it corresponds to two distinct points in $\mathbb{A}^2$.

(3) Suppose both $B_1$ and $B_2$ have only one eigenvalue each. We can make $B_1$ and $B_2$ into upper triangular matrices simultaneously. We cannot have $B_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $B_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$ since then $S = \operatorname{im} i$ violates the stability condition. Thus we have $B_1 = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}$ and $B_2 = \begin{pmatrix} \mu & \beta \\ 0 & \mu \end{pmatrix}$ for some $(\alpha, \beta) \in k^2 \setminus \{(0,0)\}$. It is easy to see that we may assume $i(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The corresponding ideal is given by

$$\begin{aligned} I &= (\beta(z_1 - \lambda) - \alpha(z_2 - \mu), (z_1 - \lambda)^2, (z_2 - \mu)^2) \\ &= \left\{ f(z) \in k[z_1, z_2] \,\middle|\, f(\lambda, \mu) = (\alpha \frac{\partial f}{\partial z_1} + \beta \frac{\partial f}{\partial z_2})(\lambda, \mu) = 0 \right\}. \end{aligned}$$

Therefore it corresponds to a set of two points at $(\lambda, \mu) \in k^2$ infinitesimally attached to each other in the direction $\alpha \frac{\partial}{\partial z_1} + \beta \frac{\partial}{\partial z_2}$. This type of the ideal is parametrized by $[\alpha : \beta]$, the homogeneous coordinates in the projective tangent space $\mathbb{P}(T_{(\lambda,\mu)}\mathbb{A}^2)$. Hence we have an explicit identification $(\mathbb{A}^2)^{[2]} \cong \operatorname{Blow}_\Delta(\mathbb{A}^2 \times \mathbb{A}^2)/\mathfrak{S}_2$.

(4) It is also easy to describe the Hilbert-Chow morphism $\pi \colon (\mathbb{A}^2)^{[n]} \to S^n(\mathbb{A}^2)$. Let $[(B_1, B_2, i)] \in (\mathbb{A}^2)^{[n]}$. Since $[B_1, B_2] = 0$, we can make $B_1$ and $B_2$ simultaneously into upper triangular matrices as

$$B_1 = \begin{pmatrix} \lambda_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \mu_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mu_n \end{pmatrix}.$$

The Hilbert-Chow morphism is given by $\pi(B_1, B_2, i) = \{(\lambda_1, \mu_1), \dots, (\lambda_n, \mu_n)\}$. If all $(\lambda_i, \mu_i)$'s are distinct, $B_1$ and $B_2$ must be semisimple. It follows that $\pi$ is an isomorphism away from the diagonal. The complication along the diagonal comes from a point $[(B_1, B_2, i)] \in (\mathbb{A}^2)^{[n]}$ where either $B_1$ or $B_2$ is not semisimple. We have already encountered this complication in the example of $n = 2$ above.

Suppose $k = \mathbb{C}$. The description in Theorem 1.9 can be understood as a holomorphic symplectic quotient. (Later in Chapter 3, we shall identify it with a hyper-Kähler quotient. So we do not give details here.) Thus we have a holomorphic symplectic form $\omega$ on $(\mathbb{C}^2)^{[n]}$, i.e. an element in $H^0((\mathbb{C}^2)^{[n]}, \Omega^2_{(\mathbb{C}^2)^{[n]}})$ which is nondegenerate at every point. In the next section, we shall generalize this result: $X^{[n]}$ has a holomorphic symplectic form if $X$ does.

As an application of the existence of the holomorphic symplectic form, we could give an estimate of the dimension of fibers of the Hilbert-Chow morphism $\pi \colon (\mathbb{C}^2)^{[n]} \to S^n(\mathbb{C}^2)$ as follows. The parallel translation of $\mathbb{C}^2$ gives us the factorization $(\mathbb{C}^2)^{[n]} = \mathbb{C}^2 \times ((\mathbb{C}^2)^{[n]}/\mathbb{C}^2)$ into a product. In the description in Theorem 1.9, a point in $(\mathbb{C}^2)^{[n]}/\mathbb{C}^2$ corresponds to $(B_1, B_2, i)$ with $\operatorname{tr}(B_1) = \operatorname{tr}(B_2) = 0$. We regard $\pi^{-1}(n[0])$ as a subvariety of $(\mathbb{C}^2)^{[n]}/\mathbb{C}^2$. Then we have the following:

THEOREM 1.13. *The subvariety $\pi^{-1}(n[0])$ is isotropic with respect to the holomorphic symplectic form on $(\mathbb{C}^2)^{[n]}/\mathbb{C}^2$. In particular,* $\dim \pi^{-1}(n[0]) \le n-1$. *Moreover, there exists at least one $(n-1)$-dimensional component.*

In Chapter 5, we shall see that $\pi^{-1}(n[0])$ have exactly one $(n-1)$-dimensional irreducible component.

PROOF. Let us consider the torus action on $\mathbb{C}^2$ given by

$$\Phi_{t_1,t_2}\colon (z_1,z_2) \mapsto (t_1 z_1, t_2 z_2) \qquad \text{for } (t_1,t_2) \in \mathbb{C}^* \times \mathbb{C}^*.$$

This action lifts to $(\mathbb{C}^2)^{[n]}$ and $\pi^{-1}(n[0])$ is preserved under the resulting action. We use the same notation $\Phi_{t_1,t_2}$ for the lifted action. As $t_1$, $t_2$ goes to $\infty$, any point $Z$ in $\pi^{-1}(n[0])$ converges to a fixed point of the torus action. In particular, if $Z$ is a nonsingular point of $\pi^{-1}(n[0])$ and $v$, $w$ are tangent vectors at $Z$, $(\Phi_{t_1,t_2})_*(v)$, $(\Phi_{t_1,t_2})_*(w)$ converge as $t_1$, $t_2 \to \infty$. (See the proof of Proposition 7.1 for detail.) On the other hand, if we pull back the holomorphic symplectic form $\omega$ by $\Phi_{t_1,t_2}$ it gets multiplied by $t_1 t_2$. Hence

$$t_1 t_2 \omega(v,w) = \omega((\Phi_{t_1,t_2})_*(v), (\Phi_{t_1,t_2})_*(w)).$$

When $t_1$, $t_2$ goes to $\infty$, the above converges only when $\omega(v,w)=0$. Thus $\pi^{-1}(n[0])$ is isotropic.

Now we give an $(n-1)$-dimensional component explicitly. In the description in Theorem 1.9, it is given by

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \\ \text{\Large 0} & & & & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & a_1 & a_2 & \dots & a_{n-1} \\ & 0 & a_1 & \dots & a_{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & a_1 \\ \text{\Large 0} & & & & 0 \end{pmatrix},\ i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

where $a_1, \dots, a_{n-1}$ are parameters. As an ideal, it is give by

$$\mathcal{J} = (z_1^n, z_2 - (a_1 z_1 + a_2 z_1^2 + \cdots + a_{n-1} z_1^{n-1})).$$

In fact, we can give another proof of $\dim \pi^{-1}(n[0]) \le n-1$ by using the hyper-Kähler structure of $(\mathbb{C}^2)^{[n]}/\mathbb{C}^2$ as follows. Let $C$ be an irreducible component of $\pi^{-1}(n[0])$. Since $\pi^{-1}(n[0])$ is compact, $C$ defines a homology class $[C] \in H_{2\dim C}((\mathbb{C}^2)^{[n]}/\mathbb{C}^2)$. Since $(\mathbb{C}^2)^{[n]}/\mathbb{C}^2$ has a Kähler metric, $[C]$ is nonzero. On the other hand, $(\mathbb{C}^2)^{[n]}/\mathbb{C}^2$ is an affine algebraic variety with respect to another complex structure $J$ (see Theorem 3.45). Hence $H_k((\mathbb{C}^2)^{[n]}/\mathbb{C}^2) = 0$ for $k > \dim_{\mathbb{C}} (\mathbb{C}^2)^{[n]}/\mathbb{C}^2 = 2(n-1)$. This shows $\dim C \le n-1$. □

EXERCISE 1.14. Show that the middle cohomology group of the complex (1.11) is equal to $\operatorname{Hom}_{k[z_1,z_2]}(I, k[z_1,z_2]/I)$ if $I$ is the ideal corresponding to data $(B_1, B_2, i)$. This space is the Zariski tangent space of $(\mathbb{A}^2)^{[n]}$ at $I$.

## 1.3. Hilbert scheme of points on a surface

In these lectures, we are interested in the case of $\dim X = 2$, and in the following we assume that $X$ is nonsingular and $\dim X = 2$ unless otherwise stated. We assume $k = \mathbb{C}$ in this section. In this case, $X^{[n]}$ has especially nice properties as the next theorem shows.

THEOREM 1.15 (Fogarty [**31**]). *Suppose $X$ is nonsingular and $\dim X = 2$, then the followings hold.*

(1) *$X^{[n]}$ is nonsingular, and has dimension $2n$.*
(2) *$\pi \colon X^{[n]} \to S^n X$ is a resolution of singularities.*

PROOF. (1) Let $Z \in X^{[n]}$ and $\mathcal{J}_Z$ the corresponding ideal. Suppose $\pi(Z) = \sum_i \nu_i[x_i]$, where the points $x_i$ are pairwise distinct. Let $Z = \bigsqcup_i Z_i$ be the corresponding decomposition. Then locally (in the classical topology) $X^{[n]}$ decomposes into a product $\prod_i X^{[\nu_i]}$. Thus it is enough to show that $X^{[n]}$ is nonsingular at $Z$ when $Z$ is supported at a single point. But in this case, we may replace $X$ by the affine plane $\mathbb{A}^2$, hence we are done by Theorem 1.9.

(2) The nonsingular locus of $S^n X$ is the open stratum $S^n_{(1,\dots,1)}X$. Suppose $\pi(Z) = \{x_1, \dots, x_n\} \in S^n_{(1,\dots,1)}X$, for $Z \in X^{[n]}$. Since $\operatorname{length}(Z_{x_i}) = 1$, for $i = 1, \dots, n$, we must have

$$\mathcal{O}_Z = \bigoplus_{i=1}^{n} \text{ the skyscraper sheaf at } x_i,$$

and hence $Z = \{x_1, \dots, x_n\} \in X^{[n]}$. This shows that $\pi|_{\pi^{-1}(S^n_{(1,\dots,1)})X}$ is an isomorphism. It remains to show $\overline{\pi^{-1}(S^n_{(1,\dots,1)}X)} = X^{[n]}$, which follows from $\dim(X^{[n]} \setminus \pi^{-1}(S^n_{(1,\dots,1)}X)) < 2n$. It was shown that $\dim \pi^{-1}(n[x]) \le n-1$ in Theorem 1.13. If $C$ is contained in the stratum $S^n_\nu X$ for $\nu = (n_1, n_2, \dots, n_k)$, then

$$\pi^{-1}(C) \cong \pi^{-1}(n_1[x_1]) \times \cdots \times \pi^{-1}(n_k[x_k]).$$

Hence we have $\dim \pi^{-1}(C) \le n-k$. It implies that $\dim \pi^{-1}(S^n_\nu X) \le n+k < 2n$, unless $\nu = (1, \dots, 1)$. (A more precise result will be proved in Lemma 6.10.) □

Let us give (a sketch of) another proof of the first statement independent of Theorem 1.9. It is less elementary, but is more natural than the above proof. First observe that $X^{[n]}$ is connected. (This was shown in the above proof of Theorem 1.15(2). See [**31**, 2.3] or [**61**, 4.5.10] for the proof independent of Theorem 1.9.) Then it is enough to show that the dimension of the Zariski tangent space at every point $Z \in X^{[n]}$ is equal to $2n$, for then the closure of $\pi^{-1}(S^n_{(1,\dots,1)}X)$ is nonsingular and there can be no other irreducible components.

From the exact sequence

$$0 \to \mathcal{J}_Z \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}_Z = \mathcal{O}_Z \to 0,$$

we have

$$\begin{aligned} 0 &\to \operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \to \operatorname{Hom}(\mathcal{J}_Z, \mathcal{O}_Z) \\ &\to \operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \to \operatorname{Ext}^1(\mathcal{J}_Z, \mathcal{O}_Z) \\ &\to \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_Z) \to \operatorname{Ext}^2(\mathcal{J}_Z, \mathcal{O}_Z) \to 0. \end{aligned}$$

Since the Euler characteristic $\sum_{i=0}^{2}(-1)^i \dim \operatorname{Ext}^i(\mathcal{J}_Z, \mathcal{O}_Z)$ is independent of $Z$, it is enough to check that $\dim \operatorname{Ext}^i(\mathcal{J}_Z, \mathcal{O}_Z)$ is independent of $Z$ only for $i = 1$ and 2. Since $\operatorname{Ext}^i(\mathcal{O}_X, \mathcal{O}_Z) \cong H^i(X, \mathcal{O}_Z) \cong H^i(X, \mathcal{O}_Z(n)) = 0$ for sufficiently large $n \in \mathbb{N}$ and $i \geq 1$ by the Serre vanishing theorem, we have

$$\begin{cases} \operatorname{Ext}^1(\mathcal{J}_Z, \mathcal{O}_Z) \cong \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \\ \operatorname{Ext}^2(\mathcal{J}_Z, \mathcal{O}_Z) = 0. \end{cases}$$

By the Serre duality theorem, $\operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$ and $\operatorname{Ext}^0(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X) = \operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X)$ are dual to each other. More precisely, we take a resolution $\mathcal{F}^\bullet$ of $\mathcal{O}_Z$ by locally free sheaves and apply the duality theorem [**52**] for the complex $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{F}^\bullet)$. (See e.g., [**87**, Proof of (1.15)] or [**61**, Chapter 10] for more detail.)

Since $\operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X) \cong \operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \cong \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_Z)$, we have dim $\operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = n$. This shows that $\dim \operatorname{Hom}(\mathcal{J}_Z, \mathcal{O}_Z) = 2n$. Note that we also proved $\operatorname{Hom}(\mathcal{J}_Z, \mathcal{O}_Z) \cong \operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$. (Note that we checked $\dim \operatorname{Hom}(\mathcal{J}_Z, \mathcal{O}_Z) = 2n$ in Remark 1.10(4) using Theorem 1.9.)

REMARK 1.16. $\pi\colon X^{[n]} \to S^n X$ can be considered as an analogy of the Springer resolution $\pi\colon T^*\mathrm{flag} \to \mathcal{N}$ for the nilpotent variety $\mathcal{N}$. This analogy will become clearer in Chapter 6.

## 1.4. Symplectic structure

Assume $k = \mathbb{C}$ and that $X$ has a holomorphic symplectic form $\omega$, i.e. $\omega$ is an element in $H^0(X, \Omega_X^2)$ which is nondegenerate at every point $x \in X$. Usually, the moduli space inherits a nice property of the base space, and in the case of $X^{[n]}$, this is the case as the next theorem shows.

THEOREM 1.17 (Fujiki ($n = 2$) [**35**], Beauville ($n \geq 2$) [**11**]). *Suppose $X$ has a holomorphic symplectic form $\omega$. Then $X^{[n]}$ also has a holomorphic symplectic form.*

PROOF. We give the proof following [**11**]. Let $S^n_* X$ be the subset of $S^n X$ consisting of $\sum \nu_i [x_i]$ ($x_i$ distinct) with $\nu_1 \leq 2$, $\nu_2 = \cdots = \nu_k = 1$. Its inverse image by the Hilbert-Chow morphism $\pi\colon X^{[n]} \to S^n X$ (resp. the quotient map $X^n \to S^n X$) is denoted by $X^{[n]}_*$ (resp. $X^n_*$). Let us denote by $\Delta \subset X^n$ the "big diagonal" consisting of elements $(x_1, \dots, x_n)$ with $x_i = x_j$ for some $i \neq j$. Then $\Delta \cap X^n_*$ is smooth of codimension 2 in $X^n_*$, where the codimension can be estimated by Theorem 1.13. Moreover, generalizing (1.4), we have the following commutative diagram

$$\begin{array}{ccc} \operatorname{Blow}_\Delta(X^n_*) & \xrightarrow{\ \eta\ } & X^n_* \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow \\ X^{[n]}_* & \xrightarrow{\ \pi\ } & S^n_* X, \end{array}$$

where $\eta\colon \operatorname{Blow}_\Delta(X^n_*) \to X^n_*$ denotes the blow-up of $X^n_*$ along $\Delta$, and $\rho$ is the map given by taking the quotient by the action of $\mathfrak{S}_n$. It is a covering ramified along the exceptional divisor $E$ of $\eta$.

The holomorphic symplectic form $\omega$ on $X$ induces one on $X^n$, which we still denote by $\omega$. Its pull-back $\eta^*\omega$ is invariant under the action of $\mathfrak{S}_n$, hence defines a holomorphic 2-form $\varphi$ on $X_*^{[n]}$ with $\rho^*\varphi = \eta^*\omega$. Then we have

$$\operatorname{div}(\rho^*\phi^n) = \rho^* \operatorname{div}(\varphi^n) + E.$$

On the other hand, the left hand side is equal to

$$\operatorname{div}(\eta^*\omega^n) = \eta^* \operatorname{div}(\omega^n) + E = E.$$

Therefore we have $\operatorname{div}\varphi^n = 0$, hence $\varphi$ is a holomorphic symplectic form on $X_*^{[n]}$. Now, $X^{[n]} \setminus X_*^{[n]}$ is of codimension 2 in $X^{[n]}$, hence $\varphi$ extends to the whole $X$ as a holomorphic form by the Hartogs theorem. We still have $\operatorname{div}\varphi^n = 0$ in $X^{[n]}$, hence $\varphi$ is non-degenerate. □

Note that the above construction is "local". It works for a quasi-projective surface $X$. For a projective surface, Mukai obtained more general results. (This result will not be used in the rest of the book.)

THEOREM 1.18 (Mukai [**87**]). *Let $X$ be a $K3$ surface or an abelian surface and $\mathcal{M}_{r,c_1,c_2}$ be the moduli space of stable sheaves on $X$ with fixed rank $r$ and Chern classes $c_1, c_2$, then $\mathcal{M}_{r,c_1,c_2}$ has a holomorphic symplectic form.*

The symplectic form on the moduli space $\mathcal{M}_{r,c_1,c_2}$ is described as follows. Let $\mathcal{E}$ be a stable sheaf on $X$, then the tangent space of $\mathcal{M}_{r,c_1,c_2}$ at $\mathcal{E}$ is given by $T_{\mathcal{E}}\mathcal{M}_{r,c_1,c_2} = \operatorname{Ext}^1(\mathcal{E},\mathcal{E})$, and the symplectic structure is defined by

$$\operatorname{Ext}^1(\mathcal{E},\mathcal{E}) \times \operatorname{Ext}^1(\mathcal{E},\mathcal{E}) \xrightarrow{\text{Yoneda product}} \operatorname{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{\text{tr}} H^2(X,\mathcal{O}_X) \cong \mathbb{C}.$$

In the last part, we use the fact that $X$ is a $K3$ surface or an abelian surface. For the proof of the non-degeneracy and the closedness, we refer to [**87**]. In Chapter 3, we shall show that the framed moduli space of torsion free sheaves on $\mathbb{C}^2$ has a holomorphic symplectic form.

The relation between the two theorems is as follows. Let $\mathcal{E}$ be a rank 1 torsion free sheaf on $X$, then its double dual $\mathcal{E}^{\vee\vee}$ is locally free. There exists a natural inclusion $\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee}$ and the cokernel $\mathcal{E}/\mathcal{E}^{\vee\vee}$ defines an element in $X^{[n]}$. If we associate $\mathcal{E}$ with $(\mathcal{E}^{\vee\vee}, \mathcal{E}^{\vee\vee}/\mathcal{E})$, then we have an identification of the component of $\mathcal{M}_{1,c_1,c_2}$ with $\operatorname{Pic}^0(X) \times X^{[n]}$. Note that if $X$ is a $K3$ surface, then $\operatorname{Pic}^0(X)$ is a point.

In Chapter 3, we shall explain the hyper-Kähler structure, which is closely related with the holomorphic symplectic structure. Actually, a hyper-Kähler manifold has a holomorphic symplectic structure as we shall explain in Chapter 3. The converse is also true if $X$ is compact.

PROPOSITION 1.19. *Let $X$ be a compact Kähler manifold which admits a holomorphic symplectic structure. Then $X$ has a hyper-Kähler structure.*

We use some differential geometric techniques in the following proof. The readers who are not familiar with those are advised to skip the proof. The proof will not be used later.

PROOF. Since $X$ has a holomorphic symplectic form $\omega$, we can identify $TX$ and $T^*X$ by using $\omega$. This implies $c_1(X) = 0$. By the Calabi conjecture proved by Yau [**122**], there exists a Ricci-flat Kähler metric on $X$. Since $g$ is Ricci-flat, the Bochner-Weitzenböck formula gives

$$\Delta|\omega|^2 = |\nabla\omega|^2,$$

where $\Delta$ is the Laplacian and $\nabla$ is the Levi-Civita connection. Integrating both sides over $X$, we have $\nabla\omega \equiv 0$, which means that $\omega$ is parallel. This shows that the holonomy group is contained in $\mathrm{SU}(2n) \cap \mathrm{Sp}(n,\mathbb{C}) = \mathrm{Sp}(n)$, where $n = \frac{1}{2}\dim_{\mathbb{C}} X$. Since a hyper-Kähler manifold can be defined as a Riemannian manifold whose holonomy group is contained in $\mathrm{Sp}(n)$, this completes the proof. □

It follows that $X^{[n]}$ has a hyper-Kähler metric if $X$ is a $K3$ surface or an abelian surface. Unfortunately, Yau's solution to the Calabi conjecture is an existence theorem, so it does not provide us an explicit description of the metric. Compare with Corollary 3.42.

EXERCISE 1.20. Compare Beauville's symplectic form and Mukai's symplectic form on the open set $\pi^{-1}(S^n_{(1,\dots,1)}X)$, where $S^n_{(1,\dots,1)}X$ is the open stratum of $S^nX$ and $\pi$ is the Hilbert-Chow morphism.

## 1.5. The Douady space

Although we assumed $X$ to be projective, it is known that the Hilbert schemes can be generalized to the case $X$ is a complex analytic space. This was done by Douady and the corresponding objects are called Douady spaces [**25**].

Many results in this chapter can be generalized to Douady spaces. The following are some of them. Results in later chapters may also generalized, though we may not mention explicitly. First of all, the Douady space of $n$ points in $X$, still denoted by $X^{[n]}$ is a complex space. (Since we may not have an ample line bundle and cannot define the Hilbert polynomial, the definition must be modified. But it is straightforward.) The Hilbert-Chow morphism $\pi\colon X^{[n]} \to S^nX$ is still defined as a holomorphic map. Fogarty's result (Theorem 1.15) clearly holds from our proof. Beauville's symplectic form can be defined for the Douady space of a complex surface with a holomorphic symplectic structure.

It is also known that $X^{[n]}$ has a Kähler metric if $X$ is compact and has a Kähler metric. This can be proved using a result of Varouchas [**118**]. (See [ibid.] for detail.) First introduce a notion of a Kähler morphism. If $X \to$ point is a Kähler morphism, then $X$ is called a Kähler space, and if $X$ is nonsingular, this is equivalent to the existence of a Kähler metric. Then Varouchas showed that $S^nX$ is a Kähler space in this sense. Then apply the following result to the Hilbert-Chow morphism $X^{[n]} \to S^nX$: If $Y \to Z$ is a Kähler morphism and $Z$ is a Kähler space, then any relatively compact subset $Y'$ of $Y$ is a Kähler space. This result can be applicable since the Hilbert-Chow morphism is projective and any projective morphism is a Kähler morphism.

In fact, M.A. de Cataldo and L. Migliorini [**22**] give a construction of the Douady spaces and the morphism $\pi\colon X^{[n]} \to S^nX$ (they call the Douady-Barelet morphism) by a completely different argument based on Theorem 1.9. We would like to sketch their argument. Consider the bi-disk $\Delta = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_\alpha| < 1\}$. By Theorem 1.9 and the description of the Hilbert-Chow morphism (see Example 1.12(4)), we can construct the Douady space $\Delta^{[n]}$ as

$$\begin{aligned}&\{Z \in (\mathbb{C}^2)^{[n]} \mid \pi(Z) \in S^n(\Delta)\}\\ = &\left\{((B_1,B_2,i) \bmod \mathrm{GL}_n(\mathbb{C})) \in (\mathbb{C}^2)^{[n]} \;\middle|\; \begin{array}{l}\text{the absolute values of the eigenval-}\\ \text{ues of } B_1,\, B_2 \text{ are smaller than } 1\end{array}\right\}.\end{aligned}$$

For a nonsingular complex surface $X$, we consider its $n$th symmetric product $S^nX$ and its stratification $S^nX = \bigsqcup S^n_\nu X$. For $\sum_i \nu_i[x_i] \in S^n_\nu X$, we take a collection of coordinate neighborhoods $\Delta_{x_i}$ of $x_i$ such that

(a) they are pairwise disjoint,
(b) each $\Delta_{x_i}$ is biholomorphic to the bi-disk $\Delta \subset \mathbb{C}^2$.

Consider the complex manifold $\prod_i (\Delta_{x_i})^{[\nu_i]}$. In this way we have a set of charts which glue by the universal property of the Douady space for $\Delta$ and get a complex manifold $X^{[n]}$. By construction $X^{[n]}$ carries a universal family $\mathcal{Z} \to X^{[n]}$ and represents the functor $\mathcal{H}ilb^P_X$ for $P = n$. This is a construction of the Douady space for $X$. The local Douady-Barlet morphisms $\prod_i (\Delta_{x_i})^{[\nu_i]} \to \prod_i S^{\nu_i}(\Delta_{x_i})$ also glue and define a global proper map $\pi\colon X^{[n]} \to S^nX$.