0. Introduction

This book is based on the notes "Constructions in Ergodic Theory" written in collaboration with E. A. Robinson, Jr.

0.1. Theorems and constructions and ergodic theory. Ergodic theory, which is sometimes also called measurable dynamics, is primarily concerned with the properties of measure-preserving (and, to a lesser extent, non-singular) transformations and flows of a standard measure space, usually called Lebesgue space, invariant up to a measure-preserving (correspondingly, measurable non-singular) conjugacy or up to natural weaker equivalence relations such as monotone (Kakutani) equivalence. Several central bodies of results in ergodic theory deal with description of particularly important equivalence classes (e.g. standard, loosely Bernoulli; see [**K1**], [**ORW**]), or the classification of special classes of systems (such as pure-point spectrum [**CFS**], [**HaK**] or Bernoulii [**O1, T**]). For a general overview of the subject see [**HaK**], Section 3.

There are other classes which are not *a priori* defined in isomorphism invariant terms where measurable structure exhibits certain rigidity and as a result classification up to isomorphism turns out to be accessible. Horocycle flows on surfaces of constant negative curvature [**Rat**] and other homogeneous unipotent systems are characteristic examples of this phenomenon which is even more pronounced for actions of higher–rank abelian groups (see [**KKS, KalSp, KitSch**]), and becomes prevalent for actions of groups which are themselves "rigid" such as higher rank semisimple Lie groups or lattices in such groups (see e.g. [**Z1, Fu**]).

However, there is a widely shared perception that in the classical case of measure-preserving automorphisms and flows, i.e. actions of $\mathbb{Z}$ and $\mathbb{R}$, most naturally defined classes of measure-preserving systems cannot be classified (K-systems, superficially look so close to Bernoulli, provide an outstanding example [**O2, OSh, OS2**]), and that most properties or combinations of properties which are not prohibited by some basic and mostly fairly elementary general conditions can in fact be realized. Thus, classical ergodic theory contains relatively few general "theorems" and plenty of interesting "examples" and "counter-examples".*)

This general observation explains why various constructions play such an important role in ergodic theory. A very large part of those constructions is combinatorial-approximational in nature. The general idea is to produce an approximate version of a desired property for an appropriate finite object and then concatenate such objects at various scales to produce a desirable system as a limit of some sort.

*)Two outstanding open problems which may yet turn into real "theorems" are the multiple mixing problem (for deep results connecting spectral and other properties with multiple mixing see [**Ho, Kal**]) and the simple Lebesgue spectrum problem where it still looks likely that absence of an example is due more to a lack of ingenuity than to deep structural reasons.

This constructive approach plays an even greater role in the questions concerning realizations of various particular conjugacy classes of measure-preserving systems or some systems with particular properties within more specialized frameworks. Premier examples of those frameworks are smooth or real-analytic systems on compact manifolds preserving an absolutely continuous or smooth measure. Other classes include abelian group extensions over some simple systems, interval exchange transformations, time-changes of some simple flows, etc. This remark provides a bridge between purely measurable ergodic theory considerations and other areas of dynamics.*) This interplay is quite evident in the second part of these notes.

Prevalence of examples and counterexamples over "theorems" somewhat curiously correlates with the fact (recognized by many experts) that there are no adequate up-to-date textbooks in ergodic theory, either at the introductory or at a more comprehensive level, a variety of books excellent in their own right notwithstanding [**CRW, Pa, Na, Pe, F2, H, W**]. Least of all, the present notes aspire to fill any of these gaps or even to be a seed for such a text.

0.2. A little history. In the prehistory of ergodic theory the 1930 work of L.G. Shnirelman [**LSh**] should be noted. Shnirelman constructed the first non-trivial example of complicated dynamical behavior unrelated to any topological complexity: perturbations of rotations of the disc with dense orbits. Although not area preserving the Schnirelman example was one of the chief sources for the approximation by conjugation constructions.

Soon afterwards J. von Neumann in the founding text of ergodic theory [**N**] provided the first examples of weakly mixing transformations and flows (see more detailed comments in Section 5.5 of Part II). In the 1940's P.R. Halmos and V.A. Rokhlin realized that combinatorial constructions together with the Baire category theorem can be used for proving both existence and genericity of measure preserving transformations with certain properties; in particular, existence of weakly mixing but not mixing transformations was first established indirectly by genericity arguments, (although von Neumann's original examples in fact have that property, see [**Kc1**]). See [**H**] for the proofs and discussion of their results and Section 2 below for an example of a more elaborate implementation of essentially the same scheme; more discussion can be found in Sections 6 and 7.

In the mid-1960's combinatorial constructions were brought explicitly into the context of ergodic theory by two independent and almost simultaneous developments:
(i) the "Chacon example" [**C1**] which still provides an interesting and challenging instance of a construction with good rescaling properties but with no apparent symmetry coming from a group structure, and
(ii) the method of periodic approximation introduced by a group of Moscow mathematicians (see [**KS1**] and references thereof)**), which served as the foundation for a systematic study of "Liouvillean" behavior in dynamics characterized by abnormally fast recurrence and accompanied by various forms of instability.

These developments were followed in the late 1960's and early 1970's by, on the one hand, a systematic use of the "cutting and stacking" constructions in ergodic

*)For an attempt to classify large parts of modern dynamics from a unified structural point of view see [**HaK**]

)The role of V.I. Oseledets at the early stages must be emphasized theory to produce a variety of often stunning counterexamples both with zero entropy (rank one mixing, minimal self–joinings, etc; see e.g. [Rud, JR2**]) and with positive entropy, (many nonisomorphic K–automorphisms with the same entropy, etc, see e.g. [**OSh**]), and, on the other hand, by the development of the approximation by conjugations constructions [**AK**] which provided the first general method for constructing smooth dynamical systems with interesting and controlled ergodic properties on a broad class of manifolds [**AK**]; see Section 8 for a more detailed discussion.

0.3. The story and purpose of these notes. The present work is based on our notes "Constructions in Ergodic Theory" which were mostly written in 1982-83. The first two of the projected four parts were finished at the time while the third and fourth parts were left unfinished by reasons explained below. The present text includes expanded and updated versions of Parts I and II and the section of Part IV which deals with combinatorial measurable (as opposed the smooth) setting.*)

We deal primarily with zero-entropy constructions. The origins of these constructions can be found in [**KS1**] and [**AK1**]; their prominent feature is a presence of an exact or approximate group structure, although in the former case it may be hidden. This distinguishes our constructions from "cutting and stacking" type constructions which tend to have less symmetry. Our main goal is to provide a detailed and well-illustrated presentation of methods which have been applied to a variety of problems and can be applied to many more. The nature of those methods, especially the approximation by conjugation method developed in [**AK1**], is such that they allow almost unlimited variations and any attempt to provide an all-encompassing framework would exclude some interesting applications. In the hindsight it was the struggle to find a proper framework, both elegant and sufficiently comprehensive according to the understanding at the time, for the treatment of realization of various ergodic properties in smooth and analytic categories (Part III) and for the approximation by conjugation method (Part IV) which impeded completion of the original project .

Part II is dedicated to cohomological constructions. While the developments of the last decade especially those dealing with actions of groups other than $\mathbb{Z}$ and $\mathbb{R}$ changed the face and to a certain extent even the basic perception of the area the program outlined and illustrated in Part II of "Constructions in ergodic theory" has proved to be fundamentally sound. An updated version of this part appeared as [**KR**]; with further additions of discussion of some recent results it is included into the present work as Part II.

The program of Part III has been advanced in two directions: (i) constructions of Bernoulli geodesic flows and other Hamiltonian systems in a large variety of setting including all compact manifolds in dimension two and three [**KB**] and (ii) the development of the theory of stable ergodicity since mid-1990's with the Dolgopyat–Pesin construction of a Bernoulli system with nonvanishing Lyapunov exponents on any manifold [**DP**] as one of the crowning achievements. While constructions and methods discussed in Part III still form the basis of the subject it seems clear that its up-to-date presentation should take a different form and with the prevalence of

*)Parts of the material included in this work originated from work presented and discussed in Moscow at the seminar run by D.V. Anosov and the author at the Steklov Institute (MIAN) and later at the Central Economics-Mathematics Institute (CEMI).

positive entropy constructions would probably not look quite in place in the context of the present work.

A number of recent developments revived the interest in the approximation by conjugations method. The central new observation is possibility of mixing within a modified context of the method [**Fa2**]. The treatment proposed in the Part IV of "Constructions in Ergodic Theory" has become outdated. We decided to keep the section which deals with combinatorial aspects of the method in the measurable setting since it is closely related to contents of Part I. It appears as Section 8. The recent paper [**FaK**] contains both an up-to-date survey of the approximation by conjugations method and proofs of a number of new representative results.

These notes may serve various purposes. On the one hand, most of the material is presented in details, with attention to motivation and key ideas but also with sufiicient details. Required background consists mostly of the material usually covered in the first–year graduate courses in mahtematics in most US universities, namely standard real analysis, point set topology, measure theory, functional analysis, including basic theory of Banach and Hilbert spaces and spectral theory for bounded unitary operators in a Hilbert space, and basic theory of Fourier series. We tried to make these parts fairly elementary and self–contained, often at the expense of generality, emphasizing methods rather than most general results. Familiarity with basic ergodic theory is quite useful, with Section 3 of [**HaK**] containing most necessary facts and some proofs. Thus these notes may be used as a basis of a graduate course roughly at the second year level as well as for independent study.

On the other hand, we present a survey of a certain area. We mention many results and concepts without detailed definitions and proofs. Those parts may be omitted by a student on the first reading. They are provided with references with the exception of cases where detailed treatments were not published. Those parts may be of interest to specialists as well as to more advanced students.

0.4. Acknowledgements. The original notes "Constructions in Ergodic Theory" were written in collaboration of E. A. Robinson, Jr., then a Ph.D. student at the University of Maryland. His help was invaluable; without his participation these notes would not have appeared. At the first stage of revision in the mid-1990's A. Mezhirov, then a Ph.D. student at the Pennsylvania State University, provided considerable help in improving and updating the text. The text was carefully checked by A. Windsor whose comments and criticism helped to correct a number of errors and improved presentations. I. Ugarcovici provided valuable help with the final typesetting of the text.

Over the course of twenty years a fairly large number of mathematicians who read various versions of "Constructions of Ergodic Theory" made numerous observations, suggestions and corrections. While it would be difficult to mention everyone by name the author expresses deep gratitude to all of them.

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