Chapter IV
The Moduli Space

This chapter shows that the moduli space of an equisingularity class is never quasi-compact except in two particular cases $g = 1$, or $g = 2$ and $(n; \beta_1) = (4, 6)$.

1. Noncompactness of the moduli space for $g \geq 3$

Let $(n; \beta_1, \dots, \beta_g)$ be the characteristic of the equisingularity class $L(C)$. We will prove that its moduli space is not the quotient of a compact space by constructing a continuous mapping onto a noncompact space (in this case $\mathbb{C} - \{0\}$). In order to accomplish this we will introduce a new *analytic invariant*.

1.1. We consider a *canonical* form of the parametrization of C :

$$\begin{cases} x = t^n \\ y = t^{\beta_1} + A + b_2 t^{\beta_2} + B + b_3 t^{\beta_3} + \cdots \end{cases}$$

where A is a polynomial in t^{e_1} whose degree (in t) is strictly smaller than β_2, B is a polynomial in t^{e_2} whose degree is strictly smaller than β_3 etc.

Let $u \in \mathbb{C}$. We are already familiar with the transformation defined by:

$$\begin{cases} \tilde{t} = ut \\ \tilde{x} = \frac{x}{u^n} \\ \tilde{y} = \frac{y}{u^m} . \end{cases}$$

If $b_2 u^{\beta_2 - \beta_1} = 1$, the parametrization of C becomes:

$$\begin{cases} \tilde{x} = \tilde{t}^n \\ \tilde{y} = \tilde{t}^{\beta_1} + \tilde{A} + \tilde{t}^{\beta_2} + \tilde{B} + \tilde{b}_3 \tilde{t}^{\beta_3} + \dots \end{cases}$$

where $\tilde{A}$ and $\tilde{B}$ satisfy the same hypotheses as A and B. From now on, we will use parametrizations of the branches of $L(C)$ in this *canonical* form.

Proposition 1.2. *Let C and C' be two analytically isomorphic branches whose parametrizations are given by*

$$C : \begin{cases} x = t^n \\ y = t^{\beta_1} + A + t^{\beta_2} + B + b_3 t^{\beta_3} + \dots \end{cases}$$
$$C' : \begin{cases} x' = t^n \\ y' = t^{\beta_1} + A' + t^{\beta_2} + B' + b'_3 t^{\beta_3} + \dots \quad . \end{cases}$$

Then $b_3^{\beta_2-\beta_1} = (b_3')^{\beta_2-\beta_1}$.

We will use a "brute force" method to prove this proposition. While the reader takes a deep breath before beginning, we remark that we will have, once the proposition is proved, a continuous mapping of the moduli space of $L(C)$ onto $\mathbb{C}-\{0\}$ which assigns $b_3^{\beta_2-\beta_1}$ to an analytic type (in its canonical form). The moduli space is therefore not a quasi-compact space.

Proof. Let φ denote an isomorphism from $\mathcal{O}(C')$ onto $\mathcal{O}(C)$. Then φ extends in a natural way to an automorphism of $\mathbb{C}[[t]]$ where $\mathbb{C}[[t]]$ denotes the integral closure of both $\mathcal{O}(C)$ and $\mathcal{O}(C')$. Since φ preserves the valuations, one obtains the following by considering the valuations of the different terms:

$$\begin{aligned} &\varphi(x') = a_{11}x + a_{12}y + \cdots ,\\ (\text{since } \beta_1 \not\equiv 0 \pmod n)\quad &\varphi(y') = \qquad\quad a_{22}y + \cdots ,\\ &\varphi(t) = c_1 t + \cdots , \end{aligned}$$

where $a_{11}a_{22}c_1 \neq 0$.

Since $x' = t^n$, we have:

(i) $[\varphi(t)]^n = \varphi(t^n) = \varphi(x') = a_{11}x + a_{12}y + \cdots$;

similarly, $y' = t^{\beta_1} + A' + t^{\beta_2} + B' + b_3' t^{\beta_3} + \ldots$ implies

(ii) $\varphi(y') = \varphi(t^{\beta_1}) + \varphi(A') + \varphi(t^{\beta_2}) + \varphi(B') + b_3'\varphi(t^{\beta_3}) + \cdots = a_{22}y + \cdots$.

The proof of the proposition proceeds by identifying the coefficients of the powers of t in the equations (i) and (ii).

(i) $\varphi(t^n) = a_{11}x + a_{12}y + \cdots$. We first write $\varphi(t)$ in the form:

$$\varphi(t) = t\,(c_1 + c_{1+e_1}t^{e_1} + \cdots + c_{1+\rho_1 e_1}t^{\rho_1 e_1}) + c_\gamma t^\gamma + \cdots \tag{1.3}$$

where $\gamma = \beta_2 - n + 1$. Since $\gamma - 1 \not\equiv 0 \pmod{e_1}$, it follows that γ is the first possible exponent (after $1+\rho_1 e_1$) that could appear in the series for $\varphi(t)$. In the series for $[\varphi(t)]^n$, the first term whose exponent is not divisible by e_1 will be $nc_1^{n-1}c_\gamma t^{\beta_2}$. In $\varphi(x')$, this same term equals $a_{12}t^{\beta_2}$ because each power of x is divisible by t^{e_1}. One therefore has:

$$\begin{aligned} &nc_1^{n-1}c_\gamma = a_{12}\\ &\rho_1 = \left[\frac{\beta_2 - n}{e_1}\right] \qquad (\text{because } 1+\rho_1 e_1 < \gamma)\,. \end{aligned}$$

Moreover, $c_1^n = a_{11}$ (by identifying the terms of lowest degree).

We now write $\varphi(t)$ in a form that privileges the terms whose exponent is not divisible by e_2 :

$$\begin{aligned} \varphi(t) = t\big(c_1 + c_{1+e_1}t^{e_1} + \cdots + c_{1+\rho_1 e_1}t^{\rho_1 e_1} + c_{\beta_2+1-n}t^{\beta_2-n+1} + \cdots\\ \cdots + c_{1+\rho_2 e_2}t^{\rho_2 e_2}\big) + c_\delta t^\delta + \cdots , \end{aligned} \tag{1.4}$$

where, this time, $\delta = \beta_3 - n + 1$. The same calculation as above then shows:

$$\begin{aligned} &nc_1^{n-1}c_\delta = a_{12}b_3\\ &\rho_2 = \left[\frac{\beta_3 - n}{e_2}\right]. \end{aligned}$$

(ii) $\varphi(y') = a_{22}y + x(a_{23}y + \cdots)$. We use (1.3) to rewrite both sides as series in t and look for the first term with exponent not divisible by e_1. Thus:

$$\begin{aligned}
\varphi(y') &= (\varphi(t))^{\beta_1} + \varphi(A) + (\varphi(t))^{\beta_2} + \cdots \\
(\varphi(t))^{\beta_1} &= c_1^{\beta_1}\, t^{\beta_1} + (\text{ polynomial in } t^{e_1}) + \beta_1 c_1^{\beta_1 - 1}\, c_{\beta_2 - n + 1}\, t^{\beta_1 + \beta_2 - n} + \cdots \\
\varphi(A) &= (\text{ polynomial in } t^{e_1}) + \text{ terms of degree } > \beta_1 + \beta_2 - n \\
(\varphi(t))^{\beta_2} &= c_1^{\beta_2}\, t^{\beta_2} + \text{ terms of degree } > \beta_2 \,.
\end{aligned}$$

One therefore sees that the first term in $\varphi(y')$ with exponent not divisible by e_1 is $c_1^{\beta_2}\, t^{\beta_2}$ (since $\beta_1 + \beta_2 - n > \beta_2$). In $a_{22}y + x(a_{23}y + \cdots)$, this term is $a_{22}t^{\beta_2}$. One therefore has $c_1^{\beta_2} = a_{22}$. Moreover, $c_1^{\beta_1} = a_{22}$ follows by identifying the terms of smallest degree.

We now use the expression (1.4) for $\varphi(t)$ and identify the first term of $\varphi(y')$ whose exponent is not divisible by e_2. A similar calculation shows that this is $b_3'\, c_1^{\beta_3}\, t^{\beta_3}$. In $a_{22}y + x(a_{23}y + \cdots)$, this term is written $a_{22}\, b_3\, t^{\beta_3}$. Thus, $a_{22}\, b_3 = b_3'\, c_1^{\beta_3}$. Summarizing, we have obtained:

$$\begin{aligned}
&c_1^{\beta_2 - \beta_1} = 1 \\
&b_3 = b_3'\, c_1^{\beta_3 - \beta_2} \,.
\end{aligned}$$

By raising both sides of the second equation to the power $\beta_2 - \beta_1$ and using the first equation, one concludes:

$$b_3^{\beta_2 - \beta_1} = (b_3')^{\beta_2 - \beta_1} \,.$$

2. The case $g = 2$

We will look for an analytic invariant in the case $g = 2$. We start by giving some preliminaries.

Definition 2.1. Let L be an equisingularity class of analytic branches with characteristic $(n; \beta_1, \ldots, \beta_g)$. One says that s is a permissible exponent for L if and only if there exists a curve C belonging to L with the following parametrization:

$$\begin{cases} x = t^n \\ y = \sum_{i>n} a_i t^i \qquad \text{with } a_s \neq 0 \,. \end{cases}$$

Thus, s is a permissible exponent if and only if $s = \beta_i$ for some $1 \leq i \leq g$, or $\beta_i < s < \beta_{i+1}$ and $s = \beta_i + k e_i$, or $s > \beta_g$.

We now assume $g \geq 2$, and consider the set of permissible exponents satisfying the following two conditions:

(a) $s > \beta_2$;

(b) s is not divisible by e_1.

This set is nonempty. Indeed, all integers $\geq \beta_2$ are permissible, and they can not all be divisible by e_1 which is strictly larger than 1 ($g \geq 2$).

We *define* β^* as the smallest element of this set. When g was at least 3, we had defined an analytic invariant by using the coefficient of t^{β_3}. In this case, we therefore compare β_3 and β^*.

$$\bullet\ \beta_3 = \beta^* \textit{ iff } \begin{cases} \beta_3 < \beta_2 + e_2\,, \\ \text{or} \\ \beta_2 + e_2 < \beta_3 < \beta_2 + 2e_2 \quad \text{and } \beta_2 + e_2 \equiv 0 \pmod{e_1}\,. \end{cases}$$

To see this, first note that $\beta_2 + e_2$ and $\beta_2 + 2e_2$ cannot both be divisible by e_1 since e_2 is not divisible by e_1. Since β_3 is never divisible by e_1, one sees that β^* can only belong to the set $\{\beta_3, \beta_2 + e_2, \beta_2 + 2e_2\}$. The conclusion follows immediately.

In the same manner, one also has (if $g \geq 3$):

- $\beta^* = \beta_2 + e_2$ *iff* $\beta_2 + e_2 < \beta_3$
  and in all the other cases where $\beta_2 + e_2 \not\equiv 0 \pmod{e_1}$;
  $\beta^* = \beta_2 + 2e_2$ if neither of the two preceding conditions is satisfied.

Remarks. - if $g = 2$, then $e_2 = 1$, and β^* equals $\beta_2 + 1$ or $\beta_2 + 2$.
- in all cases $\beta^* \leq \beta_2 + 2e_2$. □

Denote by b^* the coefficient of t^{β^*} in the parametrization in canonical form $x = t^n,\ y = t^{\beta_1} + \cdots + t^{\beta_2} + \cdots + b^*\, t^{\beta^*} + \cdots$ of a branch C in the class L.

Proposition 2.2. *Each of the following four conditions:*
(A) $g \geq 3$ *and* $\beta^* = \beta_3$
(B) $\beta^* = \beta_2 + e_2$
(C) $m_1 - n_1 > 1$
(D) $n_2 > 2$
implies that $(b^*)^{\beta_2 - \beta_1}$ *is an analytic invariant of* C.

Proof. (A) is proved in the preceding discussion. The three other parts are simple consequences of the following remarks.

Let C and C' be two analytically isomorphic branches of L whose parametrizations in canonical form are as follows:

$$\begin{aligned} C: &\quad x = t^n, \quad y = t^{\beta_1} + \cdots + t^{\beta_2} + \cdots + b^* t^{\beta^*} + \cdots \\ C': &\quad x' = t^n, \quad y' = t^{\beta_1} + \cdots + t^{\beta_2} + \cdots + (b'^*) t^{\beta^*} + \cdots\,. \end{aligned}$$

Let φ denote the analytic isomorphism of $\mathcal{O}(C')$ onto $\mathcal{O}(C)$:

$$\begin{aligned} \varphi(x') &= a_{11}x + a_{12}y + a_{13}x^2 + a_{14}xy + a_{15}y^2 + \cdots \\ \varphi(y') &= \phantom{a_{11}x + {}} a_{22}y + a_{23}x^2 + a_{24}xy + a_{25}y^2 + \cdots \end{aligned}$$

φ will also denote the extension to an automorphism to $\mathbb{C}[[t]]$:

$$\varphi(t) = t\bigl(c_1 + c_{1+e_1}t^{e_1} + \cdots + c_{1+\rho_1 e_1}t^{\rho_1 e_1}\bigr) + c_\gamma t^\gamma + \cdots\,,$$

where $\gamma = \beta_2 - n + 1$, and $c_1 \neq 0$. By comparing $\varphi(x')$ and $(\varphi(t))^n$ one obtains (see the proof of Proposition 1.2):

$$\begin{aligned} c_1^n &= a_{11} \\ n c_\gamma c_1^{n-1} &= a_{12}\,. \end{aligned}$$

Lemma 2.3. *Let σ be the smallest integer such that $c_{1+\sigma e_1} \neq 0$. Then $\sigma \geq \inf\{n_1, m_1 - n_1\}$.*

Proof. The first power of t after t^n that appears in $\varphi(x')$ is greater than or equal to $\inf\{\beta_1, 2n\}$. In

$$(\varphi(t))^n = t^n\big(c_1 + c_{1+\sigma e_1}t^{\sigma e_1} + \cdots\big)^n + nt^{n-1}\big(c_1 + c_{1+\sigma e_1}t^{\sigma e_1} + \cdots\big)^{n-1} c_\gamma t^\gamma + \cdots ,$$

this same power is $n + \sigma e_1$. (Indeed, $\gamma + n - 1 = \beta_2$, and β_2 is strictly larger than β_1). One therefore has $n + \sigma e_1 \geq \inf\{\beta_1, 2n\}$, that is, $\sigma \geq \inf\{m_1 - n_1, n_1\}$. $\square$

We now compare the two expressions for $\varphi(y')$:

$$\varphi(y') = a_{22}y + a_{23}x^2 + a_{24}xy + \cdots \quad \text{and} \quad \varphi(y') = \varphi(t^{\beta_1} + \cdots) = (\varphi(t))^{\beta_1} + \cdots .$$

By examining the coefficients of t^{β_1}, the calculation in §1 shows that $a_{22} = c_1^{\beta_1}$ and $a_{22} = c_1^{\beta_2}$. Thus, $c_1^{\beta_2 - \beta_1} = 1$.

We now look at the expressions for the coefficient of t^{β^*} in these two series (where the terms with exponent divisible by e_1 do not require greater precision):

$$\begin{aligned}
\varphi(y') = {} & \text{polynomial in } t^{e_1} \\
& + a_{22}t^{\beta_2} + a_{22}b^* t^{\beta^*} + a_{24}(t^{\beta_2 + n} + \cdots) \\
& + 2a_{25}t^{\beta_1+\beta_2} + \text{ terms of higher degree} \\
\varphi(t^{\beta_1} + \cdots) = {} & \text{polynomial in } t^{e_1} + c_1^{\beta_2}\, t^{\beta_2} + c_1^{\beta^*}\, (b'^*)\, t^{\beta^*} \\
& + \beta_1 c_1^{\beta_1 - 1}\, c_\gamma t^{\beta_1+\beta_2-n} + \beta_2 c_1^{\beta_2-1}\, c_\gamma t^{2\beta_2 - n} \\
& + \sum_{\sigma \leq \alpha \leq \rho_1} \beta_2 c_1^{\beta_2 - 1}\, c_{1+\alpha e_1}\, t^{\beta_2 + \alpha e_1} + \cdots .
\end{aligned}$$

We know that $\beta^* > \beta_2$, but we do not know the position of β^* relative to $\beta_1 + \beta_2 - n$ and $\beta_2 + \sigma e_1$. In fact, we will show:

each of the hypotheses B, C, D implies that $\beta^* < \beta_1 + \beta_2 - n$ and $\beta^* < \beta_2 + \sigma e_1$.

This will therefore show that

$$c_1^{\beta^*}\, (b'^*) = a_{22} b^* .$$

Hypothesis (B) ($\beta^* = \beta_2 + e_2$):

• We know that $m > n$, therefore, $m_1 - n_1 \geq 1$. As a result, one has:

$$e_2 < e_1 \leq e_1(m_1 - n_1) = \beta_1 - n \quad \text{and} \quad \beta^* = \beta_2 + e_2 < \beta_2 + \beta_1 - n .$$

• On the other hand $e_2 < e_1 \leq \sigma e_1$.

Thus, $\beta^* < \beta_2 + \sigma e_1$.

Hypothesis (C) ($m_1 - n_1 \geq 2$):

• $2e_2 < 2e_1 \leq (m_1 - n_1)e_1 = \beta_1 - n$. According to the definition of β^*, one has $\beta^* \leq \beta_2 + 2e_2$, and therefore, $\beta^* < \beta_2 + \beta_1 - n$.

• By Lemma 2.3, $\sigma \geq \inf\{n_1, m_1 - n_1\}$ since $g \geq 2$ insures that $n_1 \geq 2$ and $\sigma \geq 2$. Thus, $\beta^* \leq \beta_2 + 2e_2 < \beta_2 + 2e_1 \leq \beta_2 + \sigma e_1$.

Hypothesis (D) ($n_2 > 2$):

- $2e_2 < n_2e_2 \leq (m_1 - n_1)n_2e_2 = \beta_1 - n$, therefore $\beta^* \leq \beta_2 + 2e_2 < \beta_2 + \beta_1 - n$.
- $2e_2 < n_2e_2 = e_1 \leq \sigma e_1$.

Thus, $\beta^* \leq \beta_2 + 2e_2 < \beta_2 + \sigma e_1$.

To summarize, the preceding shows that each of the three hypotheses (B), (C), (D) implies that $c_1^{\beta^*}(b'^*) = a_{22}b^*$. Since $a_{22} = c_1^{\beta_1}$ and $c_1^{\beta_2-\beta_1} = 1$, one can now conclude:

$$(b'^*)^{\beta_2-\beta_1} = (b^*)^{\beta_2-\beta_1} .$$

Remark 2.4. The only cases that are not included in the proposition that we have just proved are those where one has simultaneously:

$$\begin{cases} \beta^* = \beta_2 + 2e_2 \\ m_1 - n_1 = 1 \\ n_2 = 2 \text{ (if } g \geq 2\text{, one cannot have } n_2 = 1). \end{cases}$$

With the additional hypothesis that $n_1 > 2$, we will now prove that $(b^*)^{\beta_2-\beta_1}$ is still an *analytic invariant*.

We consider two analytically isomorphic branches C and C' of the equisingularity class $L(C)$. Let φ be the automorphism of $\mathbb{C}[[t]]$ that induces the automorphism of $\mathcal{O}(C')$ onto $\mathcal{O}(C)$. We write $\varphi(t) = t(c_1 + c_{1+\tau}t^\tau + \cdots)$, where $c_{1+\tau}$ denotes the first nonzero coefficient after c_1 in the series expression for $\varphi(t)$.

Lemma 2.5. *If* $\tau \neq 0$, *then* $\tau \geq n$.

Proof. By definition, $(\varphi(t))^n = \varphi(x') \in \mathcal{O}(C)$. Since $c_1t^n = c_1x$ also belongs to $\mathcal{O}(C)$, it follows that $(\varphi(t))^n - c_1t^n = nc_1^{n-1}c_{1+\tau}t^{n+\tau} + \cdots$ is an element of $\mathcal{O}(C)$. The valuation (in t) of $(\varphi(t))^n - c_1t^n$ is therefore an element of the semigroup $\Gamma = v(\mathcal{O}(C))$. Thus, $n + \tau \in \Gamma$.

We know that the generators of the semigroup Γ are (Ch. II, thm. 3.9):

$$\begin{aligned} &\overline{\beta}_0 = n \\ &\overline{\beta}_1 = \beta_1 = n + e_1 \quad \text{(since } m_1 - n_1 = 1 \text{)} \\ &\overline{\beta}_2 = \beta_2 + (n_1 - 1)\beta_1 \\ &\ldots\ldots \end{aligned}$$

We also know that $\overline{\beta}_2 = \beta_2 + (n_1 - 1)\beta_1 > \beta_2 + \beta_1 > 2n$. The other $\overline{\beta}_q$ are larger than $\overline{\beta}_2$, and thus larger than $2n$. To prove the lemma, it therefore suffices to show that $\tau + n \neq \overline{\beta}_1$, that is, $\tau \neq e_1$. To prove this, we first prove a general lemma on the reduction of a parametrization of a branch that extends the result of Chapter III, §2.3. Recall that this showed that if in (III.2.5) one has $\nu_\rho + n \equiv 0 \pmod{m}$, then C is isomorphic to a branch with parametrization given by (III.2.7) (see also [Z2]).

Lemma 2.6. *Let* C *have the parametrization:*

$$x = t^n, \qquad y = t^{\beta_1} + \sum_{i>\beta_1} a_it^i.$$

Assume $\lambda > \beta_1$ is such that $\lambda + n = b\beta_1$ with $b \in \mathbb{Z}_+$. Then there exists a branch C' analytically isomorphic to C with parametrization:

$$\begin{cases} x' = t^n \\ y' = t^{\beta_1} + \sum_{\beta_1 < i < \lambda} a_i t^i + \sum_{j > \lambda} a'_j t^j \, . \end{cases}$$

Proof. The proof is analogous to that given in Chapter III §2.3, where we showed that under the hypothesis $\nu_\rho + n \equiv 0 \pmod m$, there exists an analytic transformation that changes the parametrization (III.2.5) into that of (III.2.7). Since $b \neq 0$, we introduce an automorphism φ of $\mathbb{C}[[t]]$ such that $(\varphi(t))^n = \varphi(x') = x + ay^{b-1}$, where a is a complex number that will be specified below. One therefore has:

$$(\varphi(t))^n = t^n + a(t^{\beta_1} + \cdots)^{b-1} = t^n + at^{\beta_1(b-1)} + \cdots \, ,$$

which implies $\varphi(t) = t + \frac{a}{n} t^{\beta_1(b-1)-n+1} + \cdots$. That is,

$$\varphi(t) = t + \frac{a}{n} t^{\lambda - \beta_1 + 1} + \cdots$$

and

$$\varphi^{-1}(t) = t - \frac{a}{n} t^{\lambda - \beta_1 + 1} + \cdots \, .$$

We then calculate $\varphi^{-1}(y) = y'$, $\; y' = [\varphi^{-1}(t)]^{\beta_1} + \sum_{i > \beta_1} a_i [\varphi^{-1}(t)]^i$:

$$\begin{aligned} (\varphi^{-1}(t))^{\beta_1} &= t^{\beta_1} - \beta_1 \frac{a}{n} t^{\lambda} + \quad \text{terms of degree larger than } \lambda \\ (\varphi^{-1}(t))^{i} &= t^{i} - i \, \frac{a}{n} \, t^{\lambda + i - \beta_1} + \quad \text{terms of degree larger than } \lambda + i - \beta_1 \, . \end{aligned}$$

Since $\lambda + i - \beta_1 > \lambda$ for $i > \beta_1$, this shows:

$$y' = t^{\beta_1} + \sum_{\beta_1 < i < \lambda} a_i t^i + (a_\lambda - \beta_1 \frac{a}{n}) t^\lambda + \cdots \, .$$

Thus, by choosing $a = \frac{n a_\lambda}{\beta_1}$, one obtains the automorphism φ of $\mathbb{C}[[t]]$ that maps $\mathcal{O}(C')$ onto $\mathcal{O}(C)$, where the parametrization of C' has the desired form. □

Completion of proof of Lemma 2.5 Assuming the hypotheses in Remark 2.4, $\beta_1 + e_1$ is clearly an exponent that satisfies the condition of (2.6) because $\beta_1 + e_1 + n = 2\beta_1$ (since $m_1 - n_1 = 1$). One can therefore suppose that the exponent $\beta_1 + e_1$ does not appear in the parametrization in canonical form (see Definition 1.1) of C and C'. For example, for C' one has:

$$\begin{cases} x' = t^n \\ y' = t^{n+e_1} + \sum_{\rho \geq 3} c_{n+\rho e_1} t^{n+\rho e_1} + t^{\beta_2} + \cdots \, . \end{cases}$$

We now assume that $\tau = e_1$ and will show that this leads to a contradiction by comparing the two expressions for $\varphi(y')$:

$$\begin{aligned} \varphi(y') &= a_{22} y + a_{23} x^2 + \cdots \\ &= a_{22} \big(t^{n+e_1} + \sum_{\rho \geq 3} c_{n+\rho e_1} t^{n+\rho e_1} + \cdots \big) + \text{ terms of order } \geq 2n \, . \end{aligned}$$

In this case, $2n = n + n_1 e_1 \geq n + 3e_1$ because $n_1 > 2$ (our additional hypothesis). Therefore the term t^{n+2e_1} *does not appear* in the series $\varphi(y')$. On the other hand, since $\varphi(t) = t\big(c_1 + c_{1+e_1}t^{e_1} + \cdots\big)$,

$$\begin{aligned}\varphi(y') &= (\varphi(t))^{n+e_1} + \sum_{\rho\geq 3} c_{n+\rho e_1}\,(\varphi(t))^{n+\rho e_1} + \cdots \\ &= t^{n+e_1}\big(c_1^{n+e_1} + (n+e_1)c_1^{n+e_1-1}\,c_{1+e_1}t^{e_1} + \cdots\big) + \text{terms of order } \geq n+3e_1\,,\end{aligned}$$

and we find a term of order $n+2e_1$ with coefficient $(n+e_1)c_1^{n+e_1-1}c_{1+e_1} \neq 0$. This is a contradiction. Thus, $\tau \neq e_1$, which completes the proof of Lemma 2.5. $\square$

The automorphism of $\mathbb{C}[[t]]$ that induces the isomorphism $\mathcal{O}(C') \to \mathcal{O}(C)$ is therefore of the form:

$$\varphi(t) = t\big(c_1 + c_{1+\tau}t^{\tau} + \cdots\big)$$

with $\tau \geq n$ if $\tau \neq 0$. One therefore has $(\varphi(t))^n = t^n\big(c_1^n + nc_1^{n-1}\,c_{1+\tau}\,t^{\tau} + \cdots\big)$ with $n + \tau \geq 2n > n + e_1 = \beta_1$ (since $n_1 > 1$ and $m_1 - n_1 = 1$).

Since t^{β_1} is not a term of $(\varphi(t))^n = \varphi(x')$, this means that in the series expression $\varphi(x') = a_{11}x + a_{12}y + \cdots$, one has

$$a_{12} = 0.$$

Let $\gamma = \beta_2 - n + 1$. Using (1.3), one knows that

$$\varphi(t) = tA + c_\gamma t^\gamma + \text{ terms of degree } > \gamma,$$

where A denotes a polynomial in t^{e_1}. One then checks that $a_{12} = nc_1^{n-1}\,c_\gamma$. Since c_1 is always nonzero, $c_\gamma = 0$. Thus, $\varphi(t) = t\big(c_1 + c_{1+\tau}t^\tau + \cdots\big) + \text{ terms of order } > \gamma$, where $\tau \geq n$, $\tau > \gamma$, or τ is divisible by e_1.

Using a calculation that is by now well known, we now identify the two expressions for $\varphi(y')$ starting with:

$$y' = t^{\beta_1} + \sum_{\substack{\beta_1 < \alpha < \beta_2 \\ \alpha \equiv 0 \pmod{e_1}}} a'_\alpha t^\alpha + t^{\beta_2} + \cdots + b'^*\,t^{\beta^*} + \cdots .$$

- $\varphi(t^{\beta_1}) = t^{\beta_1}\big(c_1^{\beta_1} + \text{series in } t^{e_1} \text{ of valuation } \geq n\big) + \text{ terms of order } > \beta_1 + \beta_2 - n$.

Since $\alpha > \beta_1$ and $\alpha \equiv 0 \pmod{e_1}$, the following must also hold:

- $\varphi(t^\alpha) = t^\alpha\big(c_1^\alpha + \text{ series in } t^{e_1} \text{ of valuation } \geq n\big)$
  $+ \text{ terms of order } > \alpha + \beta_2 - n > \beta_1 + \beta_2 - n\,;$
- $\varphi(t^{\beta_2}) = t^{\beta_2}\big(c_1^{\beta_2} + \text{ series in } t^{e_1} \text{ of valuation } \geq n\big)$
  $+ \text{ terms of order } > 2\beta_2 - n > \beta_1 + \beta_2 - n\,;$
- $\varphi(t^{\beta^*}) = c_1^{\beta^*}\,t^{\beta^*} + \text{ terms of order } > \beta^*$.

In addition:

$$\begin{aligned}\beta_1 + \beta_2 - n &= \beta_2 + e_1, &\qquad (m_1 - n_1 = 1)\\ \beta_1 + \beta_2 - n &= \beta_2 + 2e_2 = \beta^* &\qquad (n_2 = 2)\,.\end{aligned}$$

Now use the second expression $\varphi(y') = a_{22}y + a_{23}x^2 + \cdots$, and expand out the right side in powers of t. By identifying the terms in the two series one obtains

$$c_1^{\beta_1} = a_{22}\,, \quad c_1^{\beta_2} = a_{22}\,, \quad b'^*\,c_1^{\beta^*} = a_{22}b^*\,.$$

From this, it now follows that $(b'^*)^{\beta_2 - \beta_1} = (b^*)^{\beta_2 - \beta_1}$.

3. Compactness of the moduli space of an equisingularity class with characteristic $(4;6,\beta_2)$

It remains to show that the moduli space for a branch with characteristic $(4,6,\beta_2)$ is a single point. We will use the following proposition.

Proposition 3.1. *Assume the branch C is parametrized as follows:*

$$\begin{cases} x = t^n \\ y = t^m + a_\lambda t^\lambda + \sum_{i>\lambda} a_i t^i\,, & \textit{where } m<\lambda \ \textit{ and } a_\lambda \neq 0. \end{cases}$$

Let j be an integer such that

(a) $j>\lambda$;
(b) $a_j \neq 0$;
(c) $j-\lambda = sn+\sigma m, \quad s,\sigma \geq 0$.

Then there exists a branch C', analytically isomorphic to C, with the following parametrization:

$$\begin{cases} x' = t^n \\ y' = t^m + a_\lambda t^\lambda + \sum_{i>\lambda} a'_i t^i\,, \end{cases}$$

where

$$a'_i = a_i \quad \textit{if } i<j, \qquad \textit{and} \qquad a'_j = 0.$$

In other words, if λ is the first exponent after m in the series for y, and $j-\lambda = sn+\sigma m$, then one can eliminate the term $a_j t^j$, leave unchanged all terms of order $<j$, and continue to preserve the analytic type.

Proof. The isomorphism between C and C' corresponds to an automorphism of $\mathbb{C}[[t]]$ that we will make explicit. Consider first the automorphism φ that satisfies:

$$x_1 = \varphi(t^n) = x - ncx^{s+1}y^\sigma$$

where c is a complex number that remains to be determined. Set $\overline{t} = \varphi(t)$.

By this transformation, one obtains a branch C_1 (analytically equivalent to C) where:

$$C_1 : \begin{cases} x_1 = \overline{t}^n \\ y_1 = \varphi^{-1}(y)\,. \end{cases}$$

To determine the series expansion in $\overline{t}$ for y_1 one proceeds as follows. First, note that

$$\begin{aligned} \varphi(t^n) = \overline{t}^n &= t^n\big[1 - nct^{ns}\big(t^m + a_\lambda t^\lambda + \sum_{i>\lambda} a_i t^i\big)^\sigma\big] \\ &= t^n\big[1 - nct^{ns+\sigma m} - nc\sigma a_\lambda t^{ns+(\sigma-1)m+\lambda} + \cdots\big] \\ &= t^n\big[1 - nct^{j-\lambda} - nc\sigma a_\lambda t^{j-m} + \cdots\big]\,. \end{aligned}$$

The terms of degree *larger than* $j-m$ do not interest us. We calculate t as a function of $\overline{t}$ by using an elementary lemma on series.

Lemma 3.2.
(a) *Let u be a series of the form*

$$u(t) = 1 + At^q + Bt^{hq+r} + \cdots , \quad 0 < r < q .$$

Let v be a series such that $v(t)^n = u(t)$. Then $v(t)$ has the following form:

$$v(t) = 1 + a_1t^q + a_2t^{2q} + \cdots + a_ht^{hq} + a_{h+1}t^{hq+r} + \cdots$$

with $a_1 = \frac{A}{n}$, $a_{h+1} = \frac{B}{n}$.
(b) *Define*

$$\overline{t} = t\bigl(1 + a_1t^q + a_2t^{2q} + \cdots + a_ht^{hq} + a_{h+1}t^{hq+r} + \cdots\bigr) .$$

Then $t = \overline{t}\bigl(1 + b_1\overline{t}^{\,q} + b_2\overline{t}^{\,2q} + \cdots + b_h\overline{t}^{\,hq} + b_{h+1}\overline{t}^{\,hq+r} + \cdots\bigr)$ *where* $b_1 = -a_1 = -\frac{A}{n}$, *and* $b_{h+1} = -a_{h+1} = -\frac{B}{n}$.

Proof of Lemma.
(a) $u(t) = v(t)^n = 1 + n(a_1t^q + \cdots) + \binom{n}{2}(a_1t^q + \cdots)^2 + \cdots + (a_1t^q + \cdots)^n$. By identifying the coefficients of the terms t^{iq} $(i = 1, \ldots , h)$ in the two series, one finds:

$$\begin{aligned}
&A = na_1\\
&0 = na_2 + \binom{n}{2}a_1^2\\
&0 = na_3 + \binom{n}{2}2a_1a_2 + \binom{n}{3}a_3\\
&\cdots\cdots\cdots\\
&0 = na_i + Q_i(a_1, \ldots , a_{i-1}) ,
\end{aligned}$$

where Q_i is a quasihomogeneous polynomial in $(a_1, \ldots , a_{i-1})$ if a_j has weight j

$$\begin{aligned}
&\cdots\cdots\cdots\\
&0 = na_h + Q_h(a_1, \ldots , a_{h-1})\\
\text{and finally}\qquad &B = na_{h+1} .
\end{aligned}$$

Thus, one sees that for given A and B, the system has *one and only one* solution in $(a_1, \ldots , a_{h+1})$. Moreover, $a_1 = \frac{A}{n}$ and $a_{h+1} = \frac{B}{n}$.
(b) Let

$$\begin{aligned}
&\overline{t} = t\bigl(1 + a_1t^q + a_2t^{2q} + \cdots + a_ht^{hq} + a_{h+1}t^{hq+r} + \cdots\bigr)\\
\text{and}\qquad &t = \overline{t}\bigl(1 + b_1\overline{t}^{\,q} + \cdots + b_{h+1}\overline{t}^{\,hq+r} + \cdots\bigr) .
\end{aligned}$$

By substituting the expression for $\overline{t}$ into the second series, one finds:

$$\begin{aligned}
t = (t + a_1t^{q+1} + \cdots + a_{h+1}t^{hq+r+1} + \cdots) + b_1(t + a_1t^{q+1} + \cdots)^{q+1} + \cdots\\
+ b_{h+1}(t + a_1t^{q+1} + \cdots)^{hq+r+1} + \cdots .
\end{aligned}$$

By identifying the coefficients of the terms in t^{iq+1}, this gives:

$$\begin{aligned}
&0 = a_1 + b_1\\
&0 = a_2 + (q+1)a_1b_1 + b_2a_2\\
&0 = a_3 + (q+1)a_2b_1 + (2q+1)b_2a_1\\
&\cdots\cdots\cdots\\
&0 = a_i + Q^{(i)}(a_1,\dots,a_{i-1},b_1,\dots,b_{i-1})\\
&\qquad\qquad\qquad \text{where } Q^{(i)} \text{ is a polynomial}\\
&\cdots\cdots\cdots\\
&0 = a_h + Q^{(h)}(a_1,\dots,a_{h-1},b_1,\dots,b_{h-1})\\
\text{and finally,}\quad &0 = a_{h+1} + b_{h+1}\,.
\end{aligned}$$

The system admits a unique solution in $(b_1,\dots,b_{h+1})$ for a given $(a_1,\dots,a_{h+1})$. In particular, $b_1 = -a_1$ and $b_{h+1} = -a_{h+1}$. □

Completion of proof of (3.1). One applies the lemma to finish the proof of Proposition 3.1 by setting:

$$A = -nc, \quad B = -nc\,\sigma\,a_\lambda, \quad q = j-\lambda, \quad j-m = h(j-\lambda)+r\,.$$

This gives:
• if $r \neq 0$,

$$t = \overline{t}\big(1 + b_1\overline{t}^{\,j-\lambda} + b_2\overline{t}^{\,2(j-\lambda)} + \cdots + b_h\overline{t}^{\,h(j-\lambda)} + b_{h+1}\overline{t}^{\,j-m} + \cdots\big)\,,$$

where $b_1 = c$ and $b_{h+1} = c\,\sigma a_\lambda$.
• if $r = 0$, then j is an element of the semigroup Γ, and we can then apply Proposition 1.2 of ch. III.

One can now calculate $y_1 = \varphi^{-1}(y)$ explicitly in terms of $\overline{t}$ by starting with

$$y = t^m + a_\lambda t^\lambda + \sum_{i>\lambda} a_i t^i\,,$$

and then substituting for t the series in $\overline{t}$ that we have just calculated. This gives:

$$\begin{aligned}
y_1 = {}&\overline{t}^{\,m}\big(1 + b_1\overline{t}^{\,(j-\lambda)} + b_2\overline{t}^{\,2(j-\lambda)} + \cdots + b_h\overline{t}^{\,h(j-\lambda)} + b_{h+1}\overline{t}^{\,(j-m)} + \cdots\big)^m\\
&+ a_\lambda\overline{t}^{\,\lambda}\big(1 + b_1\overline{t}^{\,(j-\lambda)} + \cdots\big)^\lambda + \sum_{i>\lambda} a_i\overline{t}^{\,i}\big(1 + b_1\overline{t}^{\,(j-\lambda)} + \cdots\big)^i\,.
\end{aligned}$$

By grouping together all terms of the same degree, it follows that

$$\begin{aligned}
y_1 = {}&\overline{t}^{\,m} + a_\lambda\overline{t}^{\,\lambda} + \sum_{\lambda<i<j} a_i\overline{t}^{\,i} + (mb_{h+1} + \lambda a_\lambda b_1)\overline{t}^{\,j}\\
&+ mb_1\overline{t}^{\,m+(j-\lambda)} + \sum_{2\le i\le h}\big(\text{ terms in } \overline{t}^{\,m+i(j-\lambda)}\big)\\
&+ \text{ terms of order } > j.
\end{aligned}$$

As a result, the coordinate change $t \to \bar{t}$ has introduced "parasite" terms in the expression for y_1 of order $m + i(j-\lambda),\ i = 1, \dots, h$. Thus, y_1 does not yet satisfy the conditions of Proposition 3.1. One will now try to eliminate these terms but preserve the analytic type of the initial branch C.

We will first deal with the term $mb_1\bar{t}^{\,m+(j-\lambda)}$, which has the same valuation as $x_1^s y_1^{\sigma+1}$. Indeed, $v\big(x_1^s\, y_1^{\sigma+1}\big) = sn + (\sigma+1)m = m + j - \lambda$. Expanding out in $\bar{t}$ one has

$$\begin{aligned} x_1^s y_1^{\sigma+1} &= \bar{t}^{\,ns}\big[\bar{t}^{\,m}\big(1 + b_1\bar{t}^{\,(j-\lambda)} + \cdots\big) + a_\lambda \bar{t}^{\,\lambda} + \cdots\big]^{\sigma+1} \\ &= \bar{t}^{\,ns}\big[\bar{t}^{\,m(\sigma+1)}\big(1 + b_1\bar{t}^{\,(j-\lambda)} + \cdots\big)^{\sigma+1} + (\sigma+1)\, a_\lambda \bar{t}^{\,m\sigma+\lambda} + \cdots\big]\,. \end{aligned}$$

From this, it follows that

$$\begin{aligned} x_1^s\, y_1^{\sigma+1} = \bar{t}^{\,m+j-\lambda} + \sum_{i\geq 2}\big(\text{ terms in } \bar{t}^{\,m+i(j-\lambda)}\big) + (\sigma+1)\, a_\lambda \bar{t}^{j} \\ + \text{ terms of order } > j\,. \end{aligned}$$

We remark that $x_1^s\, y_1^{\sigma+1}$ belongs to $\mathcal{M}$, the maximal ideal of the local ring of the branch. Therefore, there exists $\xi_1 \in \mathcal{M}$ such that

$$\begin{aligned} mb_1\bar{t}^{\,m+j-\lambda} = \xi_1 - m(\sigma+1)b_1 a_\lambda \bar{t}^{\,j} + \sum_{i\geq 2}\big(\text{ terms in } \bar{t}^{\,m+i(j-\lambda)}\big) \\ + \text{terms of order } > j\,. \end{aligned}$$

Similarly, $v\big(x_1^{2s}\, y_1^{2(\sigma+1)-1}\big) = 2sn + 2(\sigma+1)m - m = m + 2(j-\lambda)$, and

$$x_1^{2s}\, y_1^{2(\sigma+1)-1} = \bar{t}^{\,m+2(j-\lambda)} + \sum_{i\geq 3}\big(\text{ terms in } \bar{t}^{\,m+i(j-\lambda)}\big) + \text{ terms of order } > j\,.$$

Therefore, each term in $\bar{t}^{\,m+i(j-\lambda)}$, $i \geq 2$, has the form $\xi_i + \text{terms of order} > j$, where $\xi_i \in \mathcal{M}$. From this it follows that

$$y_1 = \bar{t}^{\,m} + a_\lambda\bar{t}^{\,\lambda} + \sum_{\lambda<i<j} a_i\bar{t}^{\,i} + a'_j\,\bar{t}^{\,j} + \xi + \text{terms of order} > j,\ \text{with } \xi \in \mathcal{M},$$

$$\text{and} \quad a'_j = mb_{h+1} + \lambda\, a_\lambda b_1 + a_j - m(\sigma+1)\, b_1\, a_\lambda\,.$$

Recall now that $b_1 = c$ and $b_{h+1} = c\,\sigma a_\lambda$. One then chooses the constant c in order that $a'_j = 0$. An elementary check shows $a'_j = 0$ if

$$c = -\frac{a_j}{(\lambda - m)a_\lambda}\,.$$

Setting $y' = y_1 - \xi$, one finally obtains a branch C' analytically equivalent to C (ch. III, prop. 1.2), where C' has the form asserted in the statement of the proposition. This completes the proof. □

3.3. Let us now return to the study of the moduli space of a branch with characteristic $(4; 6, \beta_2)$. Writing $\beta_2 = 2s + 1$, the *generators* of the semigroup Γ are $4, 6,$

and $\overline{\beta}_2 = (n_1-1)\beta_1+\beta_2 = 6+2s+1 = 2s+7$. Thus, Γ contains all the even integers starting with 4, as well as all the integers larger than $c = 2s+10$. If C is parametrized by the equations

$$C : \begin{cases} x = t^4 \\ y = t^6 + A + t^{2s+1} + \sum_{i>\beta_2} a_i t^i \,, \end{cases}$$

where A is a sum of monomials in t of even degree between 8 and $2s+1$, we will show with the help of the preceding proposition:

there exists a branch C', analytically equivalent to C, whose parametrization is:

$$C' : \begin{cases} x = t^4 \\ y = t^6 + t^{2s+1} + a_{2s+3}t^{2s+3} \,, \end{cases}$$

where the coefficient a_{2s+3} is the same for both branches C and C'.

A monomial of A has either the form $\alpha\, t^{4\sigma}$, $\sigma \geq 2$, or $\alpha\, t^{4\sigma+6}$, $\sigma \geq 1$. One can eliminate such terms by the isomorphisms:

$$y \to y - \alpha\, x^{\sigma} \,, \qquad y \to y - \alpha\, x^{\sigma} y \,.$$

Such transformations do not affect the coefficients of t^{2s+1} and t^{2s+3}. Moreover, we have already seen (ch. III, prop. 1.2) how to eliminate the other terms whose degrees are elements of Γ without affecting the analtyic type. As a result, there exists a branch $\tilde{C}$, $\tilde{C} \cong C$ that is parametrized as follows:

$$\tilde{C} : \begin{cases} x = t^4 \\ y = t^6 + t^{2s+1} + a_{2s+3}\, t^{2s+3} + a'_{2s+5}\, t^{2s+5} + a'_{2s+7}\, t^{2s+7} + a'_{2s+9}\, t^{2s+9} \,. \end{cases}$$

One then uses Proposition 3.1 with $\lambda = 2s+1$, $j = 2s+9$. It is clear that $j-\lambda$ $(=8)$ belongs to Γ. Thus, one obtains a branch $\tilde{C}_1$, $\tilde{C}_1 \cong \tilde{C}$ for which:

$$y = t^6 + t^{2s+1} + a_{2s+3}\, t^{2s+3} + a'_{2s+5}\, t^{2s+5} + a'_{2s+7}\, t^{2s+7} + \eta$$

where $v(\eta) \geq 2s+10 = c$. Thus, η belongs to the conductor $\mathfrak{C}$ of the local ring of the branch.

Applying once again the same Proposition 3.1, one can eliminate the terms $a'_j\, t^j$ for $j = 2s+5, 2s+7$. It is then easy to conclude[1] that every branch with characteristic $(4;6,2s+1)$ is analytically isomorphic to a branch C' with the parametrization:

$$C' : \begin{cases} x = t^4 \\ y = t^6 + t^{2s+1} + bt^{2s+3} \qquad b \in \mathbb{C} \,. \end{cases}$$

We will now show that all these branches are isomorphic to the branch obtained by setting $b = 0$:

$$C : \begin{cases} x = t^4 \\ y = t^6 + t^{2s+1} \,. \end{cases}$$

To do so we will construct an analytic automorphism which transforms C to C' whose coefficient b is arbitrary.

Let a denote an arbitrary constant. Let φ denote an automorphism of $\mathbb{C}[[t]]$ that satisfies the following two conditions:

(a) $\varphi(t) = \overline{t} = t + at^3 + at^{2s-2} + \cdots$;

(b) $\varphi(x) \in \mathcal{O}(C)$.

---

[1] See Chapter III, prop. 1.2.

We remark that condition (a), by itself, implies

$$\begin{aligned}\varphi(x) = \varphi(t^4) &= t^4 + 4at^6 + 6a^2t^8 + 4a^3t^{10} + a^4t^{12} + 4at^{2s+1} + 12a^3t^{2s+5} + \cdots \\ &= x + 4ay + 6a^2x^2 + a^4x^3 + 4a^3xy + \cdots .\end{aligned}$$

By an appropriate choice for the coefficients of t^j, $j > 2s-2$, one now defines φ so that

$$\varphi(t^4) = x + 4ay + 6a^2x^2 + a^4x^3 + 4a^3xy\,.$$

Let C' be the branch defined by

$$\begin{cases} x' = \overline{t}^4 \\ y' = \varphi^{-1}(y)\,. \end{cases}$$

One has $C' \cong C$ and:

$$\begin{aligned} y' &= \overline{t}^{\,6} + \cdots + a'_{2s+1}\,\overline{t}^{\,2s+1} + a'_{2s+3}\,\overline{t}^{\,2s+3} + \cdots \\ \overline{t}^{\,6} &= \varphi(t^6) = \left[t + at^3 + at^{2s-2} + \cdots\right]^6 = t^6 + \cdots + 6at^{2s+3} + \cdots \\ \overline{t}^{\,2s+1} &= \varphi(t^{2s+1}) = t^{2s+1} + \cdots + (2s+1)at^{2s+3} + \cdots \\ \overline{t}^{\,2s+3} &= \varphi(t^{2s+3}) = t^{2s+3} + \cdots . \end{aligned}$$

Substituting these expressions into the series (in $\overline{t}$) for y', one must obtain $y = t^6 + t^{2s+1}$. This requires that

$$a'_{2s+1} = 1\,, \qquad a'_{2s+3} = -(2s+7)\,a\,.$$

By the preceding discussion, we know that the branch C' is isomorphic to:

$$C'' : \begin{cases} x'' = t^4 \\ y'' = t^6 + t^{2s+1} - (2s+7)\,at^{2s+3}\,, \end{cases}$$

where a is an arbitrary constant. This completes the proof. ☐

In conclusion, we have shown that the moduli space of a branch with characteristic $(4; 6, 2s+1)$ consists of a single point.

Before ending chapter IV, we give one additional technical lemma that will be useful in chapters V and VI.

LEMMA 3.5. *Let $\overline{t}$ be a power series in t whose lowest order term equals t. Suppose that*

$$\overline{t}^n = t^n + n\sum_{\alpha\geq 0} B_\alpha t^{\mu+n-1+\alpha}$$

where μ is an integer ≥ 2. Then:

(a) $\overline{t} = t + \sum_{\alpha=0}^{\mu-2} B_\alpha\, t^{\mu+\alpha} +$ *terms of order* $> 2\mu - 2$;

(b) $t = \overline{t} - \sum_{\alpha=0}^{\mu-2} B_\alpha\, \overline{t}^{\,\mu+\alpha} +$ *terms of order* $> 2\mu - 2$.

PROOF. Since $\overline{t}^n - t^n$ has order $\geq \mu + n - 1$, while $\overline{t} - \epsilon t$ has order 1 in t as long as $\epsilon \neq 1$, the order in t of $\overline{t} - t$ is necessarily greater than or equal to μ. We may then write $\overline{t} = t + \sum_{\alpha \geq 0} B^*_\alpha \, t^{\mu+\alpha}$. It follows that

$$\overline{t}^n = t^n + n \sum_{\alpha \geq 0} B^*_\alpha \, t^{\mu+n-1+\alpha} + \text{ terms of order } \geq n - 2 + 2\mu.$$

By comparing this with the hypothesis, we obtain part (a) of the lemma.

If we now set $t = \overline{t} + \sum_{\alpha \geq 0} \overline{B}^*_\alpha \, \overline{t}^{\,\mu+\alpha}$ (note that the order of $t - \overline{t}$ in $\overline{t}$ is also $\geq \mu$), then we obtain the following identity:

$$t = \bigl(t + \sum_{\alpha \geq 0} B^*_\alpha \, t^{\mu+\alpha}\bigr) + \overline{B}^*_0 \bigl(t + \sum_{\alpha \geq 0} B^*_\alpha \, t^{\mu+\alpha}\bigr)^{\mu} + \cdots + \overline{B}^*_{\mu-2} \bigl(t + \sum_{\alpha \geq 0} B^*_\alpha \, t^{\mu+\alpha}\bigr)^{2\mu-2} + \cdots .$$

Assume that $\alpha \leq \mu - 2$. We then identify the coefficient of $t^{\mu+\alpha}$ on each side of this equation. This gives

$$0 = B^*_\alpha + \overline{B}^*_\alpha \, .$$

Combining this with part (a) completes the proof of part (b). □