

Preface

This book is intended as a very personal introduction to the topology of tiling spaces, with a target audience of graduate students who wish to learn about the interface of topology with aperiodic order. It isn't a comprehensive and cross-referenced tome about everything having to do with tilings. That would be too big, too hard to read, and way too hard to write! Instead, I have tried to lay out the subject as I see it, in a linear manner, with emphasis on those developments that I find to be the most interesting. In other words, this book is about how I think about tilings, and what inspires me to keep working in the area. My hope is that it will also inspire you.

“Interesting” is a subjective term, of course. Many subjects that others consider to be central are not covered here. For instance, you will find little about cut-and-project tilings in this book, despite the mass of work that has been done on them. I mean no disrespect to the practitioners of that field! I just can't do that subject justice, and am happy to leave its exposition to people who know it far better than I. (For cut-and-project tilings, I particularly recommend Bob Moody's review article [**Moo**] and, for the ambitious, the comprehensive monograph by Forest, Hunton and Kellendonk [**FHK**].)

By contrast, I love inverse limit structures, tiling cohomology, substitution tilings and the role of rotations. I love pattern-equivariant cohomology. I love tilings that don't have finite local complexity. In this book you'll see them all, in considerable detail.

Modern tiling theory developed from four very different directions. One direction was from logic. In the 1960s, Hao Wang and his students posed a variety of problems in terms of square tiles with marked edges (aka Wang tiles). Given a set of such tiles, can you determine whether it's possible to tile the plane in a way that the edges of adjacent tiles match? Wang's student Berger [**Ber**] proved the answer to be “no” in general, and in the process produced a set of tiles that would tile the plane but only nonperiodically. Berger's example was extremely complicated, but people quickly produced simpler examples.

The second ingredient was a good example for study and wonder. Roger Penrose produced a set of aperiodic tiles in the mid-70s that sparked intense interest and brought aperiodic tilings into popular culture. The Penrose tilings [**Pen**] aren't just mathematically interesting – they're pretty! They

also demonstrate a remarkable rotational symmetry; every pattern that appears somewhere in the tiling also appears rotated by 36 degrees, and with the same frequency. This “statistical symmetry” contradicts the long-held belief that only rotations by 60, 90, 120 and 180 degrees can appear in highly ordered structures.

The third ingredient came from physics, or you might say from materials science. In 1982, Shechtman and coworkers [SBGC] discovered a new class of solid, neither crystal nor amorphous, called *quasicrystals*. Quasicrystals have sharp diffraction patterns, long thought to be the hallmark of a periodic crystal, but some of these patterns have 8- or 10-fold rotational symmetry. It didn’t take people long to realize that quasicrystals are modeled well by aperiodic tilings, and in particular by 3-dimensional versions of the Penrose tiling and by several other cut-and-project tilings!

The fourth ingredient came from ergodic theory and dynamical systems, where substitution sequences had long been a subject of interest. Some of the simplest substitutions, like the Thue-Morse substitution, were defined over 100 years ago. However, it was only in the 1980s that people went from substitution subshifts to substitution tilings in one dimension, and from there to higher-dimensional substitution tilings. It didn’t hurt that the Penrose tiling could be realized in this way. Soon people discovered other interesting geometric “rep-tiles” and computed properties of the resulting tilings.

These trends came together in the 1990s. Tilings, including the Penrose tiling, were used to model quasicrystals. These tilings were in turn generated in a number of ways, including local matching rules, cut-and-project methods, and substitutions. The tilings were then studied as dynamical systems, and their dynamical properties were related to physical properties of the quasicrystals that they model.

Suppose you had a quasicrystal that was modeled by an aperiodic tiling. A physicist might ask the following questions about the quasicrystal.

- P1. What is the x-ray diffraction pattern of the material? This is equivalent to the Fourier transform of the autocorrelation function of the positions of the atoms. Sharp peaks are the hallmark of ordered materials, such as crystals and quasicrystals.
- P2. What are the possible energy levels of electrons in the material? The locations of the atoms determine a quasiperiodic potential, and the spectrum of the corresponding Schrödinger Hamiltonian has infinitely many gaps. What are the energies of these gaps, and what is the density of states corresponding to each gap?
- P3. Can you really tell the internal structure of the material from diffraction data? What deformations (either local or non-local) of the molecular structure are consistent with the combinatorics of the molecular bonds? Which of these are detectable from diffraction data?

Although the physicist is interested in a single quasicrystal (or a single tiling), mathematicians like to define *spaces*. From the tiling T we construct a space Ω_T of tilings that have the same properties of T . If T has desirable properties (like finite local complexity, repetitivity, and well-defined patch frequencies), then Ω_T has corresponding properties (compactness, minimality as a dynamical system, and unique ergodicity), and we can ask the following mathematical questions:

- M1. What is the topology of Ω_T ? What does the neighborhood of a point of Ω_T look like? What are the (Čech) cohomology groups of Ω_T ?
- M2. There is a natural action of the group \mathbb{R}^d of translations on Ω_T . This makes Ω_T into a dynamical system, with d commuting flows. What are the ergodic measures on Ω_T ? For each such measure, what is the spectrum of the generator of translations (think: momentum operator) on $L^2(\Omega_T)$? This is called the *dynamical spectrum* of Ω_T .
- M3. From the action of the translation group on Ω_T , one can construct a C^* algebra. What is the K-theory of this C^* -algebra?

Remarkably, each math question about Ω_T answers a physics question about a material modeled on T . M1 answers P3, M2 answers P1, and M3 (in large part) answers P2. Far from being a pointless mathematical abstraction, tiling spaces are important!

This book is the story of the first mathematical question, and the answers we have gleaned so far. In chapter 1 we consider a variety of interesting tilings and the construction of the corresponding tiling spaces. In chapter 2 we explore the local structure of Ω_T and its realization as an inverse limit space. In chapter 3 we introduce the Čech cohomology of Ω_T and show how the answer to the third physics question is tied to the first Čech cohomology. In chapter 4 we study the rotational properties of tilings — what made quasicrystals and the Penrose tiling so amazing in the first place! In chapter 5 we introduce “pattern-equivariant cohomology”, a beautiful realization of tiling cohomology in terms of properties of an individual tiling. In this way we come full circle, from tilings to tiling spaces and back to individual tilings.

The material in the first four chapters is basically set in place, as is some of the material of chapter 5. Chapters 6 and 7, however, are cutting-edge research. The reader may have some difficulty with these chapters, both because the concepts aren’t as neatly prescribed as the earlier topics, and because the calculations require more advanced algebraic topology.

Chapter 6 is devoted to “tricks of the trade”, recently developed calculational techniques that are powerful but are *not* generally known. Chapter 7 is about tilings without the simplifying assumption of finite local complexity. Until recently, such tilings were thought to be beyond our understanding, but that is rapidly changing.

Exercises for the reader are embedded in the text. I strongly recommend that you work these out carefully. If you run out of patience, you can look up the answers to most of these exercises in the appendix.

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