

## CHAPTER 2

### Tiling spaces and inverse limits

In Chapter 1 we defined a variety of tilings and tiling spaces using substitutions, cut-and-project methods, and local matching rules. In this chapter we begin to study the topology of such spaces, and relate tiling spaces to objects that are better-known, namely inverse limits of branched manifolds.

#### 2.1. Local structure

Suppose  $\Omega$  is a tiling space and  $T \in \Omega$  is a tiling. What does an  $\epsilon$ -neighborhood of  $T$  look like? If  $T'$  is in that  $\epsilon$ -neighborhood, then  $T$  and  $T'$  agree on  $B_{1/\epsilon}$ , up to a small translation. In other words, all the tilings in the neighborhood can be obtained by the following procedure:

- (1) Take the patch  $[B_{1/\epsilon}]_T$ .
- (2) Wiggle this patch by up to  $\epsilon$  in any direction. This gives our tiling space continuous degrees of freedom.
- (3) Extend the patch out to infinity. This involves discrete choices.

If, as is typical, there are infinitely many discrete choices to be made, the set of such choices will be uncountable – typically a Cantor set. In other words,  $\Omega$  looks locally like the product of a  $d$ -dimensional disk (if we are tiling  $\mathbb{R}^d$ ) and a Cantor set. Such spaces are old hat to people who work in ergodic theory and dynamical systems, but are pretty weird to those of us who learned topology in the context of smooth manifolds! Fortunately, inverse limit spaces provide plenty of examples with which to stretch our intuition.

#### 2.2. Inverse limit spaces

Suppose  $\Gamma_0, \Gamma_1, \dots$  are topological spaces, and for each non-negative integer  $n$ ,  $f_n : \Gamma_{n+1} \rightarrow \Gamma_n$  is a continuous map. Consider the product space  $\prod \Gamma_i$ , with the product topology. This can be viewed as a set of sequences  $(x_0, x_1, \dots)$  with each  $x_n \in \Gamma_n$ . We define the *inverse limit space*

$$\varprojlim(\Gamma, f) = \{(x_0, x_1, \dots) \in \prod \Gamma_n \mid \text{for all } n, x_n = f_n(x_{n+1})\}. \quad (2.1)$$

The spaces  $\Gamma_n$  are called *approximants* to the inverse limit since, if you know  $x_n$ , you automatically know  $x_0, \dots, x_{n-1}$ . We are working in the product topology, where two sequences  $(x_0, \dots)$  and  $(y_0, \dots)$  are close if their first  $N$  terms are close for a large value of  $N$ . If the sequences are close, then  $x_N$  and  $y_N$  are close. Conversely, if  $x_N$  and  $y_N$  are sufficiently

close, then by the continuity of  $f_{N-1}$ ,  $x_{N-1}$  and  $y_{N-1}$  are close, and likewise each  $x_i$  with  $i < N$  is close to  $y_i$ .

A simple example of an inverse limit space is the dyadic solenoid. Let each  $\Gamma_n$  be the unit circle  $\mathbb{R}/\mathbb{Z}$ , and let each  $f_n$  be multiplication by 2. That is,  $\Gamma_1$  is a circle that wraps twice around  $\Gamma_0$ ,  $\Gamma_2$  is a circle that wraps twice around  $\Gamma_1$ , and so on, as in Figure 2.1. A point in  $\varprojlim(\Gamma, f)$  is described by a point  $x_0$  on the circle  $\Gamma_0$  (a continuous degree of freedom), a choice of  $x_1$  from among the two preimages of  $x_0$ , a choice of  $x_2$  from among the two preimages of  $x_1$ , etc. Our space is locally the product of an interval and a countable number of 2-fold choices, i.e., the product of an interval and a Cantor set.

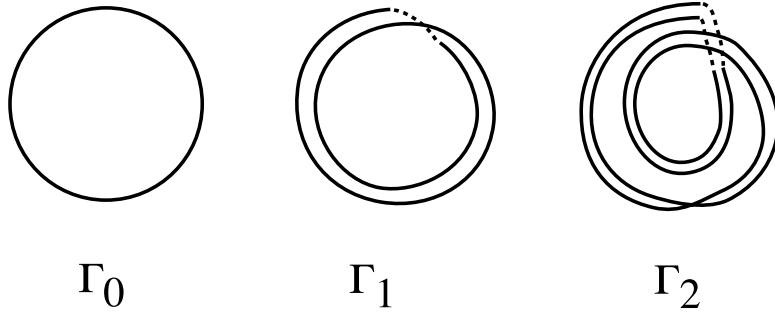


FIGURE 2.1. The first three approximants to the dyadic solenoid

### 2.3. Tiling spaces are inverse limits

In the past decade, there has been an explosion of work of the topology of tilings, most of which relies on constructions of tiling spaces as inverse limits of simpler spaces. This explosion was begun in 1998 by Anderson and Putnam [AP], who showed that substitution tiling spaces are inverse limits of branched manifolds. We will review the Anderson-Putnam construction later in this chapter.

Ormes, Radin and Sadun [ORS] generalized the Anderson-Putnam construction to allow for pinwheel-like tilings that have tiles pointing in arbitrary directions. Bellissard, Benedetti and Gambaudo [BBG] gave a very general construction that applied to the hull of any (simple) tiling, not just a substitution tiling. They then used this construction to prove powerful gap-labeling theorems. Independently, Franz Gähler [Gah] gave an exceptionally simple proof that hulls of simple tilings are inverse limit spaces. Benedetti and Gambaudo [BG] then generalized the Bellissard, Benedetti and Gambaudo construction, and Sadun [Sa2] independently generalized the Gähler construction, both efforts including pinwheel-like tilings and tilings of non-Euclidean spaces. Recent work by Frank and Sadun [FS] shows that even spaces without “finite local complexity” (see chapter 7) can be understood using inverse limits.

## 2.4. Gähler's construction

Let  $\Gamma_0$  be the set of all possible instructions for laying a tile at the origin, let  $\Gamma_1$  be instructions for laying that center tile plus a ring of tiles around it, and generally let  $\Gamma_n$  be instructions for laying down a center tile and  $n$  layers of tiles around it. The map  $f_n : \Gamma_{n+1} \rightarrow \Gamma_n$  is just the “forgetful map” that forgets the instruction for the outermost ring. The statement that  $x_n = f_n(x_{n+1})$  just says that the instructions  $x_n$  and  $x_{n+1}$  agree on the first  $n$  rings.

A point in the inverse limit is a nested sequence of consistent instructions for placing a tile at the origin and an infinite number of rings around it. In other words, it is instructions for laying out an entire tiling. The inverse limit, being the set of all such instructions, is naturally isomorphic to the tiling space itself.

The only problem with this argument is that sets of instructions aren't topological spaces! We need to show that  $\Gamma_n$  can be viewed as a geometric object whose topological properties are computable. We begin with  $\Gamma_0$ .

A point in  $\Gamma_0$  tells us how to place a tile at the origin. This involves a choice of tile, followed by a choice of a point in that tile to align with the origin. In other words,  $\Gamma_0$  contains one copy, and only one copy, of every kind of tile that is allowed.

But what happens if the origin lies on an edge? In giving instructions for the patch shown in Figure 2.2a, do we specify a point on the right edge of tile A or a point on the left edge of tile B? The answer is to identify the right edge of A with the left edge of B! That is,  $\Gamma_0$  is the union of one copy of each tile type, with some edges identified. If somewhere in some tiling, a tile of type  $A$  shares an edge with a tile of type  $B$ , then those two edges are identified.

The result is a branched surface. If more than one kind of tile can appear to the left of B (say, A and C), then the left edge of B will be identified with both the right edge of A and with the right edge of C, as in Figure 2.2b.

**Collaring.** Suppose we have a tiling  $T$ . We can label the tiles of  $T$  not only by their own type, but by the pattern of their nearest neighbors. Tiles with such labels are called *collared tiles*. In Figure 2.3, tiles A and B are the same as uncollared tiles, but are considered different as collared tiles.

Strictly speaking, relabeling the tiles gives us a different set of possible tiles, and hence a different tiling. Call this tiling  $T'$ , and let  $\Gamma_1(T) = \Gamma_0(T')$ . It is a branched manifold, as before. A point in  $\Gamma_1(T)$  describes how to place a collared tile at the origin, which is equivalent to describing how to place a tile at the origin together with all of its nearest neighbors. Likewise, for any  $n > 0$ , we let  $\Gamma_n(T) = \Gamma_{n-1}(T')$ .

As an example, we construct  $\Gamma_0$  and  $\Gamma_1$  for the Fibonacci tiling. There are two (uncollared) tile types,  $a$  and  $b$ , so  $\Gamma_0$  is the union of one  $a$  tile and one  $b$  tile, modulo identifications. Since  $a$  is always followed by  $b$ , the endpoint of the  $a$  tile is identified with the beginning of the  $b$  tile. Since  $b$  is

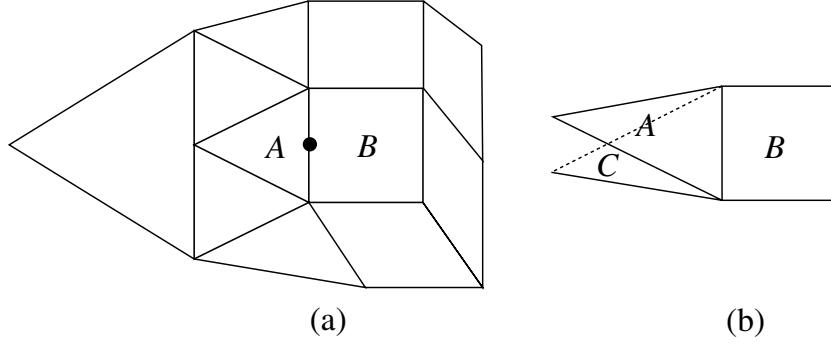


FIGURE 2.2. (a) What instructions do you give when the origin sits on an edge? (b) Gluing tiles yields a branched manifold.

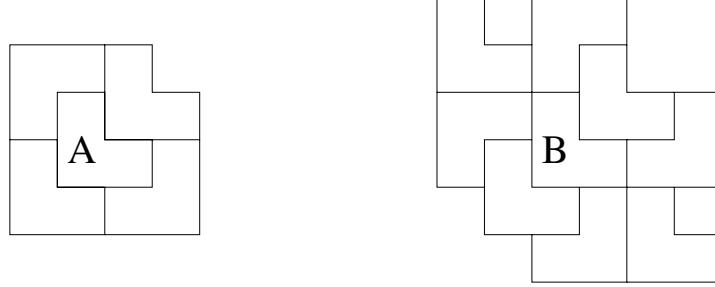


FIGURE 2.3. Collared tiles

sometimes followed by  $a$ , the endpoint of  $b$  is identified with the beginning of  $a$ . Finally, since  $b$  is sometimes followed by  $b$ , the beginning and end of  $b$  are identified. The result is a figure-eight, as shown in Figure 2.4.

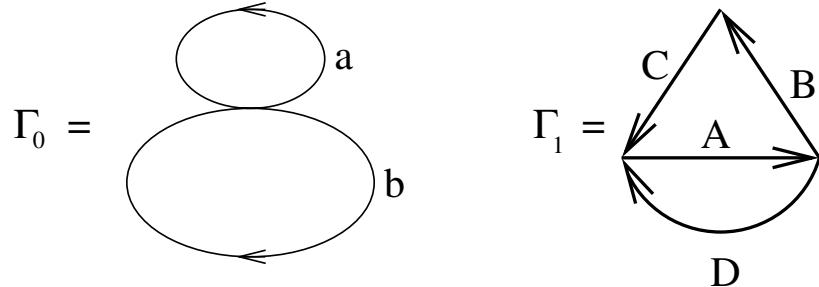


FIGURE 2.4. Two Gähler approximants for the Fibonacci tiling space

Suppose we have a Fibonacci tiling

$$T = \dots abbabbababbab \dots \quad (2.2)$$

Notice that each  $a$  tile is preceded and followed by a  $b$  tile. We therefore define the collared tile  $A = (b)a(b)$ , where the parentheses indicate the tiles before or after the one in question. There are three collared  $b$  tiles, namely  $B = (a)b(b)$ ,  $C = (b)b(a)$ , and  $D = (a)b(a)$ . (The patch  $bbb$  never appears in a Fibonacci tiling.) We can rewrite  $T$  as

$$T' = \dots ABCABCADABCA\dots \quad (2.3)$$

(The visible patch of  $T'$  is one tile shorter than that of  $T$ , since we do not know how to label the last visible  $b$  in  $T$  as a collared tile.) Since  $A$  is followed by  $B$  or  $D$ ,  $B$  is always followed by  $C$ , and  $C$  and  $D$  are always followed by  $A$ , we obtain the complex  $\Gamma_1$ , as shown in Figure 2.4.

Note that this construction is not specific to the tiling  $T$ . Another tiling  $\tilde{T}$  in the same space would have the same patches and the same set of collared tiles as  $T$ , and would yield the same complexes  $\Gamma_0$  and  $\Gamma_1$ . The approximants  $\Gamma_n$  describe the entire tiling *space*.

## 2.5. The Anderson-Putnam construction

Gähler's construction is simple, elegant, and conceptually powerful. It has been used to prove a variety of important theorems about tiling spaces. When it comes to computing topological invariants, however, it is almost useless. The approximants  $\Gamma_n$  and the bonding maps  $f_n$  depend on  $n$ , and can get successively more complicated as  $n$  increases.

With substitution tilings it is possible to avoid this problem. Anderson and Putnam defined an inverse limit structure in which all of the  $\Gamma_n$ 's were essentially the same and all of the  $f_n$ 's were essentially the same. We will first lay out the Anderson-Putnam (AP) construction in the easiest cases, where the substitution has a property called "forcing the border". We will then show how to modify the construction to handle all substitutions.

Suppose we have a substitution  $\sigma$  with a linear stretching factor of  $\lambda$ . Let  $\Gamma_0$  be exactly as before, with one copy of each tile type, stitched together at edges. Let  $\Gamma_1$  be a larger version of  $\Gamma_0$ , namely  $\Gamma_0$  stretched by a factor of  $\lambda$ . Similarly, let  $\Gamma_n$  be  $\Gamma_0$  expanded by a factor of  $\lambda^n$ . We view  $\Gamma_n$  as containing one copy of each supertile of level  $n$ .

The approximant  $\Gamma_0$  tells us how to place a tile at the origin. The approximant  $\Gamma_n$  tells us how to place a supertile of level  $n$  that contains the origin. There is a forgetful map  $f_n : \Gamma_{n+1} \rightarrow \Gamma_n$  that restricts attention to the level- $n$  supertile containing the origin, while forgetting about the rest of the level- $(n+1)$  supertile containing the origin.

Figure 2.5 shows  $\Gamma_0$ ,  $\Gamma_1$  and  $\Gamma_2$  for the Fibonacci tiling. The map  $f_0$  sends the upper loop of  $\Gamma_1$  to the lower loop of  $\Gamma_0$ , and sends the lower loop of  $\Gamma_1$  to the upper loop of  $\Gamma_0$  followed by the lower loop. Likewise,  $f_1$  sends the upper loop of  $\Gamma_2$  to the lower loop of  $\Gamma_1$ , and the lower loop of  $\Gamma_2$  to the upper and lower loops of  $\Gamma_1$ . If we rescale the approximants  $\Gamma_n$  to make them all the same as  $\Gamma_0$ , then the maps  $f_n$  are all the same. Each  $f_n$

is just the substitution map  $\sigma : \Gamma_0 \rightarrow \Gamma_0$  that takes each tile, stretches it, subdivides it, and sends each piece to  $\Gamma_0$ .

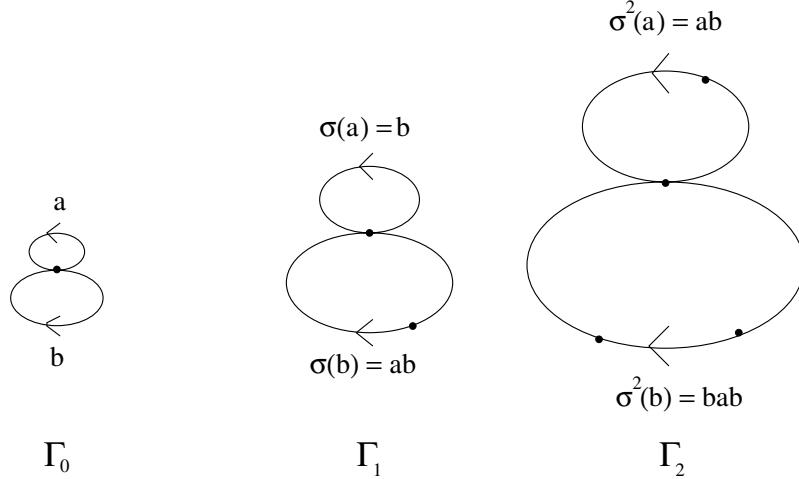


FIGURE 2.5. The first three Anderson-Putnam approximants to the Fibonacci tiling space

Topologically,  $\varprojlim(\Gamma, f)$  is the inverse limit of a single branched manifold  $\Gamma_0$  by a single map  $\sigma$ . A point in the inverse limit gives consistent instruction for placing the origin in a tile, that tile in a level-1 supertile, that supertile in a level-2 supertile, and so on forever. In other words, it tells how to put the origin in an infinite supertile. However, that infinite supertile may not cover the entire plane, as is seen in Figure 2.6! The infinite supertile can always be extended to a tiling of the entire plane, but that extension may not be unique.

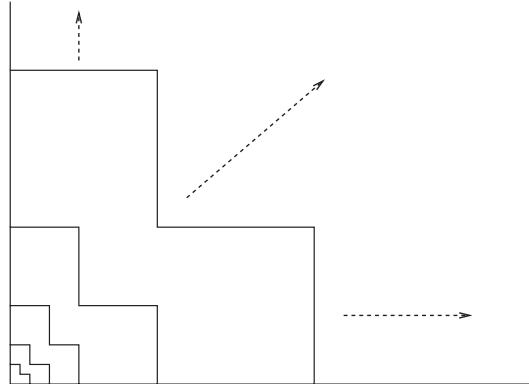


FIGURE 2.6. An infinite-order supertile may not cover the entire plane.

**Forcing the border.** A substitution is said to *force the border* if there exists a positive integer  $n$  such that any two level- $n$  supertiles of the same type have the same pattern of neighboring tiles. In that case, a level- $(n+k)$  supertile determines all of the level- $k$  supertiles around it, and an infinite supertile determines the entire plane. Points in the inverse limit  $\varprojlim(\Gamma, f)$  are then in 1-1 correspondence with tilings of the plane, and we have a topological isomorphism  $\varprojlim(\Gamma, f) \leftrightarrow \Omega_\sigma$ .

This phenomenon is best illustrated with the example of the half-hex tiling. In the half-hex tiling, each tile pairs off with another tile to form a hexagon. Knowing the position and type of one tile actually tells you about two: that tile and its partner. Knowing the position and type of a level-1 supertile tells you about the partner of that supertile, and also about the partners of the unpaired tiles within the two supertiles. Figure 2.7 shows, in dotted lines, the tiles determined by a half-hex tile, a level-1 supertile and a level-2 supertile. The level-2 supertile determines *all* of its immediate neighbors, so the half-hex substitution forces the border with  $n = 2$ .

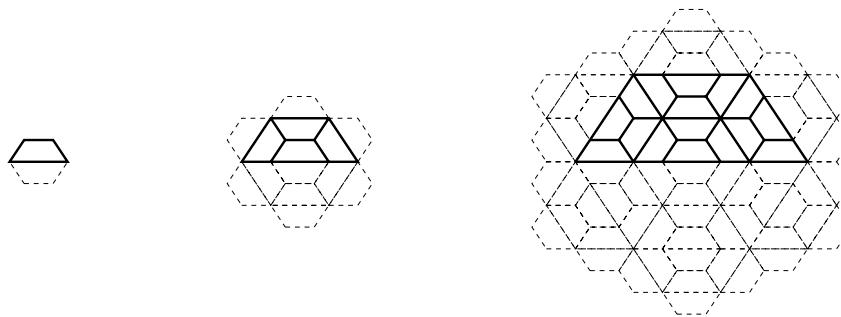


FIGURE 2.7. In bold face, a half-hex tile, an order-1 supertile, and an order-2 supertile. In dotted lines, the nearby tiles that these determine.

Among 1-dimensional substitutions, Thue-Morse does not force the border. The Fibonacci and period-doubling substitutions force the border on one side only; since all supertiles end in  $b$ , there is a  $b$  to the left of every supertile, but there is no way to determine the tile to the right of a general supertile. By contrast, the substitution  $\sigma(a) = abaab$ ,  $\sigma(b) = aabab$  does force the border, since all supertiles begin with  $a$  and end with  $b$ .

Among 2-dimensional substitutions, the Penrose substitution forces the border. In fact, the term “force the border” was coined by Kellendonk [Kel1] in his work on the Penrose tilings. The chair substitution does not force the border. The tiles along the edges of a large supertile are determined, but the tiles at the corner are not, as illustrated in Figure 2.8. The arrow and table substitutions also fail to force the border.

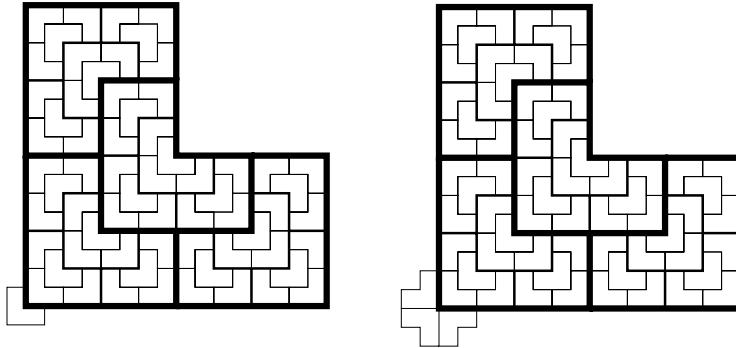


FIGURE 2.8. There are two ways to extend a high-order chair supertile around the southwest corner.

**The Anderson-Putnam collaring trick.** There is a simple trick, introduced by Anderson and Putnam [AP], for describing an arbitrary substitution tiling space by a substitution that forces the border. Just rewrite the substitution in terms of collared tiles!

We illustrate the method with the Fibonacci substitution  $\sigma(a) = b$ ,  $\sigma(b) = ab$  and the Fibonacci collared tiles  $A = (b)a(b)$ ,  $B = (a)b(b)$ ,  $C = (b)b(a)$  and  $D = (a)b(a)$ . Substituting an  $a$  that is preceded and followed by a  $b$  gives a  $b$  that is preceded and followed by the word  $ab$ . In terms of collared tiles, that is an  $A$  preceded by a  $B$  and followed by an  $A$ . Applying the same reasoning to all four collared tiles we get

$$\begin{aligned}
 \sigma(A) &= (ab)b(ab) = (B)C(A) \\
 \sigma(B) &= (b)ab(ab) = AD(A) \\
 \sigma(C) &= (ab)ab(b) = (D)AB \\
 \sigma(D) &= (b)ab(b) = AB \\
 \sigma^2(A) &= (AD)AB(C) \\
 \sigma^2(B) &= (B)CAB(C) \\
 \sigma^2(C) &= (AB)CAD(A) \\
 \sigma^2(D) &= (C)CAD(A)
 \end{aligned} \tag{2.4}$$

This rewritten substitution does not force the border with  $n = 1$ . We know the *uncollared* tile that precedes  $\sigma(B)$ , namely  $b$ , but we do not know the *collared* tile that precedes  $\sigma(B)$  (or those that precede  $\sigma(D)$  or follow  $\sigma(C)$  or  $\sigma(D)$ ). However, it does force the border with  $n = 2$ .

**EXERCISE 2.1.** *Prove that a substitution applied to collared tiles always forces the border. Do this first in one dimension, and then in higher dimensions.*

We summarize the situation with the theorem proved by Anderson and Putnam:

**THEOREM 2.1.** *Let  $\Gamma_0$  be the cell complex obtained from stitching one copy of each (uncollared) tile, and let  $\tilde{\Gamma}_0$  be the similar cell complex obtained*

by stitching one copy of each collared tile. (In the Gähler construction this was called  $\Gamma_1$ .) In both cases we denote the bonding map by  $\sigma$ , as it is induced by the substitution.  $\Omega_\sigma$  is always homeomorphic to  $\varprojlim(\tilde{\Gamma}_0, \sigma)$ . If  $\sigma$  forces the border, then  $\Omega_\sigma$  is also homeomorphic to  $\varprojlim(\Gamma_0, \sigma)$ .

In principle, we could take the inverse limit of a cell complex built from twice-collared tiles, or 17-times collared tiles. However, that's just computational overkill. In fact, collaring once is often overkill.

The Fibonacci substitution forces the border on the left, so we don't need to collar our Fibonacci tiles on the left, just on the right. Likewise, the chair tiling forces the border along the edges, but not at the corners. To construct the chair tiling space, we need only collar our tiles in a way that preserves information about those corners. By choosing a suitable “partial collaring” scheme, one can often simplify computations considerably. We will revisit the question of efficient collaring in Chapter 6.

**EXERCISE 2.2.** Relabel the tiles in the Fibonacci tiling so they are collared on the right but not on the left. Construct the partially collared Anderson-Putnam complex (call it  $\Gamma_{1/2}$ ). What does it look like as a topological space? Describe explicitly the map  $\sigma : \Gamma_{1/2} \rightarrow \Gamma_{1/2}$ .

**EXERCISE 2.3.** Compute the Anderson-Putnam complexes  $\Gamma_0$  and  $\tilde{\Gamma}_0$  for the Thue-Morse substitution and the maps  $\sigma$  on each one. Unlike for the Fibonacci substitution, the two spaces are not homotopy equivalent, and the inverse limit of  $\Gamma_0$  is not homeomorphic to the inverse limit of  $\tilde{\Gamma}_0$ .

**EXERCISE 2.4.** Devise an efficient collaring scheme for the arrow substitution. The fully collared complex  $\tilde{\Gamma}_0$  contains  $13 \times 4 = 52$  cells (13 patterns in 4 orientations), but it is possible to make do with only  $6 \times 4 = 24$ .