# Preface

The present text is an extended and updated version of my lecture notes *Residuen und Dualität auf projektiven algebraischen Varietäten* (Der Regensburger Trichter **19** (1986)), based on a course I taught in the winter term 1985/86 at the University of Regensburg. I am grateful to David Cox for helping me with the translation and transforming the manuscript into the appropriate LaTeX $2_\varepsilon$ style, to Alicia Dickenstein and David Cox for encouragement and critical comments and for enriching the book by adding two sections, one on applications of algebraic residue theory and the other explaining toric residues and relating them to the earlier text.

The main objective of my old lectures, which were strongly influenced by Lipman's monograph [**71**], was to describe local and global duality in the special case of irreducible algebraic varieties over an algebraically closed base field $k$ in terms of differential forms and their residues. Although the dualizing sheaf of a $d$-dimensional algebraic variety $V$ is only unique up to isomorphism, there is a canonical choice, the sheaf $\omega_{V/k}$ of regular $d$-forms. This sheaf is an intrinsically defined subsheaf of the constant sheaf $\Omega^d_{R(V)/k}$, where $R(V)$ is the field of rational functions on $V$. We construct $\omega_{V/k}$ in § 9 after the necessary preparation. Similarly, for a closed point $x \in V$, the stalk $(\omega_{V/k})_x$ is a canonical choice for the dualizing (canonical) module $\omega_{\mathscr{O}_{V,x}/k}$ studied in local algebra. We have the residue map

$$\mathrm{Res}_x : H^d_x(\omega_{V/k}) \longrightarrow k$$

defined on the $d$-th local cohomology of $\omega_{V/k}$. The local cohomology classes can be written as generalized fractions

$$\begin{bmatrix} \omega \\ f_1, \dots, f_d \end{bmatrix}$$

where $\omega \in \omega_{\mathscr{O}_{V,x}/k}$ and $f_1, \dots, f_d$ is a system of parameters of $\mathscr{O}_{V,x}$. Using the residue map, we get the Grothendieck residue symbol

$$\mathrm{Res}_x \begin{bmatrix} \omega \\ f_1, \dots, f_d \end{bmatrix}.$$

For a projective variety $V$ the residue map at the vertex of the affine cone $C(V)$ induces a linear operator on global cohomology

$$\int_V : H^d(V, \omega_{V/k}) \longrightarrow k$$

called the integral. The local and global duality theorems are formulated in terms of $\mathrm{Res}_x$ and $\int_V$. There is also the residue theorem stating that "the integral is the sum of all of the residues." Specializing to projective algebraic curves gives the usual residue theorem for curves plus a version of the Serre duality theorem expressed in terms of differentials and their residues. Basic rules of the residue calculus are formulated and proved, and later generalized to toric residues by David

Cox. Because of the growing current interest in performing explicit calculations in algebraic geometry, we hope that our description of duality theory in terms of differential forms and their residues will prove to be useful.

The residues

$$\operatorname{Res}_x \begin{bmatrix} \omega \\ f_1, \ldots, f_d \end{bmatrix}$$

can be considered as intersection invariants, and by a suitable choice of the regular $d$-form $\omega$, a residue can have many geometric interpretations, including intersection multiplicity, angle of intersection, curvature, and the centroid of a zero-dimensional scheme. The residue theorem then gives a global relation for these local invariants. In this way, classical results of algebraic geometry can be reproved and generalized. It is part of the culture to relate current theories to the achievements of former times. This point of view is stressed in the present notes, and it is particularly satisfying that some applications of residues and duality reach back to antiquity (theorems of Apollonius and Pappus). Alicia Dickenstein gives applications of residues and duality to partial differential equations and problems in interpolation and ideal membership.

Since the book is introductory in nature, only some aspects of duality theory can be covered. Of course the theory has been developed much further in the last decades, by Lipman and his coworkers among others. At appropriate places, the text includes references to articles that appeared after the publication of Hartshorne's *Residues and Duality* [**38**]; see for instance the remarks following Corollary 11.9 and those at the end of § 12. These articles extend the theory of the book considerably in many directions. This leads to a large bibliography, though it is likely that some important relevant work has been missed. For this, I apologize.

The students in my course were already familiar with commutative algebra, including Kähler differentials, and they knew basic algebraic geometry. Some of them had profited from the exchange program between the University of Regensburg and Brandeis University, where they attended a course taught by David Eisenbud out of Hartshorne's book [**39**]. Similar prerequisites are assumed about the reader of the present text. The section by David Cox requires a basic knowledge of toric geometry.

I want to thank the students of my lectures who insisted on clearer exposition, especially Reinhold Hübl, Martin Kreuzer, Markus Nübler and Gerhard Quarg, all of whom also later worked on algebraic residue theory, much to my benefit and the benefit of this book. Thanks are also due to the referees for their suggestions and comments and to Ina Mette for her support of this project.

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