Preface

The *Geometry of Positive Definite Quadratic Forms* is a rich and old subject which arose in the arithmetic studies of quadratic forms. Through the seminal works of Minkowski and Voronoi a century ago, the geometric viewpoint became predominant. The study of arithmetical and inhomogeneous minima of positive definite quadratic forms turned into a study of lattice sphere packings and coverings. Lattices and, more generally, periodic (point) sets are by now widespread in mathematics and its applications. The important monograph "Sphere packings, Lattices and Groups" [**69**] by Conway and Sloane, with its over 100 pages of references, shows exemplarily the influence on other mathematical disciplines. This becomes particularly apparent for the 24-dimensional Leech lattice and its connections to number theory, group theory, coding theory and mathematical physics. Since the complexity of problems grows with the dimension, it is no surprise that over the past decades more and more computer support was used to study higher dimensional lattices and more general structures. Still, the Geometry of Positive Definite Quadratic Forms is an essential tool, not only in the study of lattice sphere packings and coverings.

One aim of this book is to give a nearly self-contained introduction to this beautiful subject. We present the known material with new proofs, which then admit natural generalizations. These extentions of the known theory were mainly targeted to support the study of extreme periodic sets. However, it turned out that the resulting new theory has other applications as well, as for example, the classification of totally real thin number fields. On the way, always an eye is kept on computability; algorithms are developed that allow computer assisted treatments. Using tools from combinatorial, from linear and from convex optimization, many difficult problems become accessible now. This is, for example, demonstrated in the search for new currently best-known lattice sphere coverings and in the classification of 8-dimensional perfect lattices, which previously was thought to be impossible with the known methods.

Although this book deals with classical topics which have been worked on extensively by numerous authors, it shows exemplarily how computers may help to gain new insights. On the one hand it is shown how computer assisted (sometimes heuristic) exploration helps to discover new exceptional structures. In many cases these would probably not have been found without a computer. On the other hand several computer assisted proofs are given, which deal with extraordinarily large data or involve large enumerations. It is shown how proofs can be obtained from numerical results, by postprocessing of roundoff solutions. All of these aspects of computer mathematics are nowadays supported by a growing functionality of computer algebra systems and by an increasing number of reliable small programs for specific purposes. In some cases one has to combine, to supplement and to improve on existing software tools. If solutions for basic tasks are obtained they should be made accessible to the growing community of computer enthusiastic mathematicians. Underlying many of the presented computational results are in particular two such programs: A program for rigorous determinant maximization (including semidefinite programming) allowing exact certified error bounds, and secondly, a program for polyhedral representation conversion under symmetries.

Computer assisted mathematical explorations and proofs are of increasing importance in many areas of modern mathematics. Even close to the topics of this book there have been amazing developments recently. An example is the proof by Hales [**129**], [**130**] of the famous Kepler conjecture. Several exciting results have been obtained in the context of *linear and semidefinite programming bounds* for spherical codes and point sets in Euclidean spaces. There is the new sphere packing bound by Cohn and Elkies [**60**], and based on it, the proof by Cohn and Kumar [**63**] (see also [**61**]) that the Leech lattice gives the best lattice sphere packing in 24 dimensions. There is the proof of Musin [**185**] showing that the kissing number in four dimensions is 24 (see [**197**] for an excellent survey). Shortly after, Bachoc and Vallentin [**6**], gave more general, new bounds on the size of spherical codes. Their works are followed by similar approaches for other problems, using semidefinite programming. As in some parts of this book, these works involve numerical computations which are then turned into mathematical rigorous proofs. Often numerical quests and subsequent mathematical analysis lead to new mathematical insights. A fascinating example is the study of *universally optimal point configurations*, recently invoked by Cohn and Kumar [**62**] (see also [**9**] and [**264**]). Although all of this is happening literally next door to the topics of this book, I decided to keep it focused as it is. Adequate treatments will hopefully fill other books in the near future. For now I encourage the reader to study the great original works.

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