

Preface

Alexandre Grothendieck conceived his definition of *motives* in the 1960s. By that time, it was already established that there exist several cohomology theories for, say, smooth projective algebraic varieties defined over a given field k , and A. Weil's brilliant insight about counting points over finite fields via the Lefschetz trace formula was validated.

With his characteristic passion for unification and “naturalness”, Grothendieck wanted to construct a universal cohomology theory (with, say, coefficients R) that had to be a functor h from the category $\text{Var}(k)$ of smooth k -varieties to an abelian tensor category $\text{Mot}(k)$ of “(pure) motives” (or $\text{Mot}(k)_R$, where R is a ring of coefficients), satisfying a minimal list of expected properties.

Grothendieck also suggested a definition of $\text{Mot}(k)$ and of the motivic functor. It consisted of several steps.

For the first step, one keeps objects of $\text{Var}(k)$, but replaces its morphisms by correspondences. This passage implies that morphisms $Y \rightarrow X$ now form an *additive group*, or even an R -module rather than simply a set. Moreover, correspondences themselves are not just cycles on $X \times Y$ but *classes* of such cycles modulo an “adequate” equivalence relation. The coarsest such relation is that of numerical equivalence, when two equidimensional cycles are equivalent if their intersection indices with each cycle of complementary dimension coincide. The finest one is the rational (Chow) equivalence, when equivalent cycles are fibres of a family parametrized by a chain of rational curves. The direct product of varieties induces the tensor product structure on the category.

The second step in the definition of the relevant category of pure motives consists in a formal construction of new objects (and relevant morphisms) that are “pieces” of varieties: kernels and images of projectors, *i.e.*, correspondences $p : X \rightarrow X$ with $p^2 = p$. This produces a *pseudo-abelian*, or *Karoubian* completion of the category. In this new category, the projective line \mathbb{P}^1 becomes the direct sum of (motive of) a point and the Lefschetz motive \mathbb{L} (intuitively corresponding to the affine line).

The third, and last step of the construction, is one more formal enhancement of the class of objects: they now include *all* integer tensor powers $\mathbb{L}^{\otimes n}$, not just non-negative ones, and tensor products of these with other motives. An important role is played by \mathbb{L}^{-1} which is called the Tate motive \mathbb{T} .

The first twenty-five years of the development of the theory of motives were summarised in the informative Proceedings of the 1991 Research Conference conference “Motives”, published in two volumes by the AMS in 1994.

By that time it was already clear that the richness of ideas and problems involved in this project resists any simple-minded notion of “unification”, and with time, the theory of motives was more and more resembling a Borgesian garden

of forking paths. Each strand of the initial project tended to unfold in its own direction, whereas the central stumbling stone on the Grothendieck visionary road, the Standard Conjectures, resisted and still resists all efforts.

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The book by Gonçalo Tabuada is a dense combination of a survey paper and a research monograph dedicated to the development of the theory of motives during the next twenty five years. The author contributed many important results and techniques in the theory in recent years. In this book, he focuses on the so-called “noncommutative motives”. I will make a few brief comments about the scope of this subject.

In very general terms, one can say that motivic constructions of the New Age start not only with smooth varieties but rather with triangulated categories and their enhancements, dg categories. Classical varieties fit there by supplying their derived and more general enhanced derived categories, such as categories of perfect complexes. Enhancement essentially means that morphisms rather than objects are treated as complexes, complexes modulo homotopy, etc. Hence the usual categorical framework is no longer sufficient: we must deal with 2-categories and eventually with categories of higher level.

Correspondences between such “varieties” are introduced using Morita-like constructions. Recall that in the basic Morita theory morphisms between non-necessarily commutative rings $A \rightarrow B$ are replaced with (A, B) -bimodules, and that the difference between commutative and noncommutative rings in this framework essentially vanishes because any commutative ring is Morita equivalent to the ring of matrices of any given order over it.

One of the first great surprises of this insight transplanted into (projective) algebraic geometry was Alexander Beilinson’s discovery (1983) that the derived category of coherent sheaves of a projective space can be described as a triangulated category made out of modules over a Grassmann algebra. In particular, a projective space became “affine” in some kind of noncommutative geometry! The development of Beilinson’s technique led to a general machinery describing triangulated categories in terms of exceptional systems and expanding the realm of candidates to the role of noncommutative motives.

Thus the abstract properties of the categories constructed in this way justify the intuition and terminology of “noncommutative geometry” which was one motivation for M. Kontsevich’s project of Noncommutative Motives and became the central subject of Tabuada’s book.

This shift of the viewpoint required much work to establish how much we lose by passing from the classical picture to the new one, and what we gain in understanding both the old and new universes of Algebraic Geometry. Some of these exciting results are surveyed in Tabuada’s monograph, and the reader who wants to focus on a particular strand of research will be able to follow the relevant original papers cited in the ample references list.

This stimulating book will be a precious source of information for all researchers interested in algebraic geometry.

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