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AN EXTENSION OF POSITIVITY FOR INTEGRALS OF BESSEL FUNCTIONS AND BUHMANN'S RADIAL BASIS FUNCTIONS

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Abstract. As to the Bessel integrals of type

$$\int_0^x (x^\mu - t^\mu)^\lambda t^\alpha J_\beta(t)dt \qquad (x > 0),$$

we improve known positivity results by making use of new positivity criteria for ${}_1F_2$ and ${}_2F_3$ generalized hypergeometric functions. As an application, we extend Buhmann's class of compactly supported radial basis functions.

1. Introduction

We consider the problem of determining positivity of the integrals

$$\int_0^x (x^\mu - t^\mu)^\lambda t^\alpha J_\beta(t)dt \qquad (x > 0) \tag{1}$$

for appropriate values of parameters $\mu, \lambda, \alpha, \beta,$ where J_β stands for the Bessel function of order β. For the sake of convergence and practical applications, it is common to assume $\mu > 0, \lambda \geq 0, \beta > -1, \alpha + \beta + 1 > 0$.

Owing to various applications, the problem has been studied by many authors over a long period of time and we refer to Askey [1] and Gasper [7] for historical backgrounds. Of our primary concern is the result of Misiewicz and Richards which states in a simplified version as follows.

Theorem A (Misiewicz and Richards [14]). *Let $\mathcal{A}$ be the set of parameters (β, α) defined by*

$$\mathcal{A} = \left\{\beta > -\frac{1}{2},\ -\beta - 1 < \alpha \leq \min\left(\beta,\ \frac{3}{2}\right)\right\}.$$

For $0 < \mu \leq 1$ and $\lambda \geq 1$, if $(\beta, \alpha) \in \mathcal{A}$, then

$$\int_0^x (x^\mu - t^\mu)^\lambda t^\alpha J_\beta(t)dt > 0 \qquad (x > 0).$$

An additional range of parameters α, β is also available. In fact, if $j_{\beta,2}$ denotes the second positive zero of J_β and $\alpha_*(\beta)$ the solution of

$$\int_0^{j_{\beta,2}} t^{\alpha_*(\beta)} J_\beta(t)dt = 0 \tag{2}$$

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for each β, then Misiewicz and Richards pointed out the above positivity holds true for $-1 < \beta < \frac{1}{2}, -\beta - 1 < \alpha < \alpha_*(\beta)$. As it is described in detail by Askey [2], however, the explicit nature of $\alpha_*(\beta)$ is still unknown and we shall exclude this range throughout.

In the special case $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$, the integrals of (1) reduce to the Fourier cosine transforms for which Kuttner proved its positivity:

Theorem B (Kuttner [12]). *For* $0 < \mu \le 1$ *and* $\lambda \ge 1$,

$$\int_0^x (x^\mu - t^\mu)^\lambda \cos t \, dt > 0 \qquad (x > 0).$$

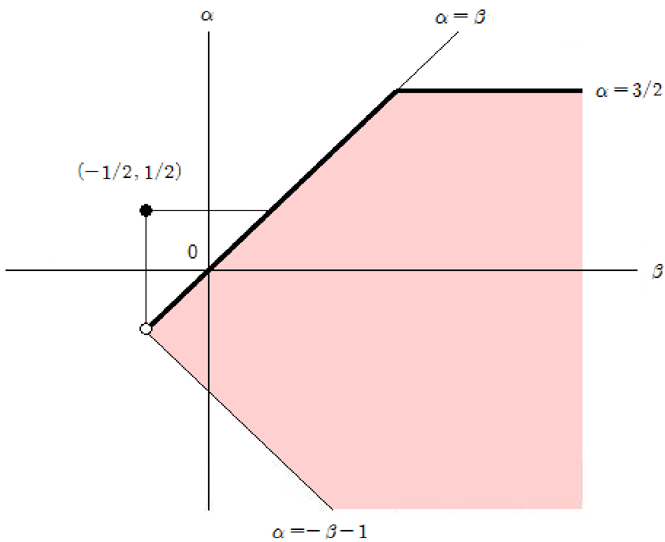

FIGURE 1. The positivity regions of Misiewicz-Richards (pink) and Kuttner (black dot)

The combined positivity region is depicted in Figure 1.

The main purpose of the present paper is to improve Theorem A and Theorem B by extending positivity regions for $0 < \mu \le 1, \lambda \ge 1$ as well as by providing a positivity region for unrestricted $\mu > 0, \lambda \ge 0$.

As an application of our results, we shall improve in several directions the range of positive definiteness for Buhmann's class of compactly supported radial basis functions [4] which are of considerable interest in the theory of approximations and interpolations.

2. Positivity in the unrestricted case

For α, β satisfying $\beta > -1, \alpha + \beta + 1 > 0$, if we put

$$\Phi(x) = {}_1F_2\left(\begin{matrix} \frac{\alpha+\beta+1}{2} \\ \beta + 1, \frac{\alpha+\beta+3}{2} \end{matrix} \middle| -\frac{x^2}{4} \right), \tag{3}$$

then it is simple to evaluate by integrating termwise or by parts

$$\int_0^x t^\alpha J_\beta(t)dt = \frac{x^{\alpha+\beta+1}}{2^\beta\Gamma(\beta+1)(\alpha+\beta+1)}\,\Phi(x), \tag{4}$$

$$\int_0^x (x^\mu - t^\mu)^\lambda\, t^\alpha J_\beta(t)dt$$
$$= \frac{\mu\lambda x^{\mu\lambda+\alpha+\beta+1}}{2^\beta\Gamma(\beta+1)(\alpha+\beta+1)}\int_0^1 \Phi(xt)(1-t^\mu)^{\lambda-1} t^{\mu+\alpha+\beta}\,dt \tag{5}$$

for $\mu > 0,\, \lambda > 0$ and $x > 0$. Therefore positivity of (1) would follow once kernel Φ were shown to be positive in the case $\lambda = 0$ or nonnegative in the case $\lambda > 0$.

To investigate the sign of ${}_1F_2$ generalized hypergeometric function Φ, we shall make use of the following general criterion recently established by Cho and Yun [5], which will be applied subsequently in other occasions as well.

As it is standard, the Newton diagram associated to a finite set of planar points $\big\{(\alpha_i,\,\beta_i) : i = 1, \cdots, N\big\}$ refers to the closed convex hull containing

$$\bigcup_{i=1}^N \big\{(x,y)\in\mathbb{R}^2 : x \ge \alpha_i,\; y \ge \beta_i\big\}\,.$$

Lemma 2.1. (Cho and Yun, [5]) *For* $a > 0,\, b > 0,\, c > 0,$ *put*

$$\phi(x) = {}_1F_2\left(a\,; b, c\,; -\frac{x^2}{4}\right) \qquad (x > 0).$$

(i) *If* $\phi \ge 0,$ *then necessarily* $b > a,\; c > a,\; b + c \ge 3a + \frac{1}{2}\,.$
(i) *Let* P_a *denote the Newton diagram associated to*

$$\Lambda = \left\{\left(a+\frac{1}{2},\, 2a\right),\, \left(2a,\, a+\frac{1}{2}\right)\right\}.$$

If $(b,c)\in P_a\,,$ *then* $\phi \ge 0$ *and strict positivity holds unless* $(b,c)\in\Lambda.$

For the sake of presenting this paper in a self-contained way, we shall give a simplified proof in the appendix. Keeping in mind that the line segment joining two point of Λ is given by $c = 3a + 1/2 - b$ in the (b,c)-plane, it is straightforward to obtain the range for the positivity or nonnegativity of Φ by implementing Lemma 2.1.

Theorem 2.1. *Let* $\mathcal{R}$ *be the set of parameters* (β,α) *defined by*

$$\mathcal{R} = \{\beta > -1,\; -\beta - 1 < \alpha \le 0\} \cup \left\{\beta > 0,\; 0 < \alpha \le \min\left(\beta,\, \frac{1}{2}\right)\right\}.$$

For $\mu > 0,\, \lambda \ge 0$ *and* $(\beta,\alpha)\in\mathcal{R},$ *we have*

$$\int_0^x (x^\mu - t^\mu)^\lambda\, t^\alpha J_\beta(t)dt > 0 \qquad (x > 0)$$

unless $\lambda = 0,\, \alpha = \beta = 1/2.$ *In the exceptional case, it reduces to*

$$\int_0^x J_{\frac{1}{2}}(t)\sqrt{t}\,dt = \frac{2\sqrt{2}}{\sqrt{\pi}}\,\sin^2\left(\frac{x}{2}\right) \ge 0.$$

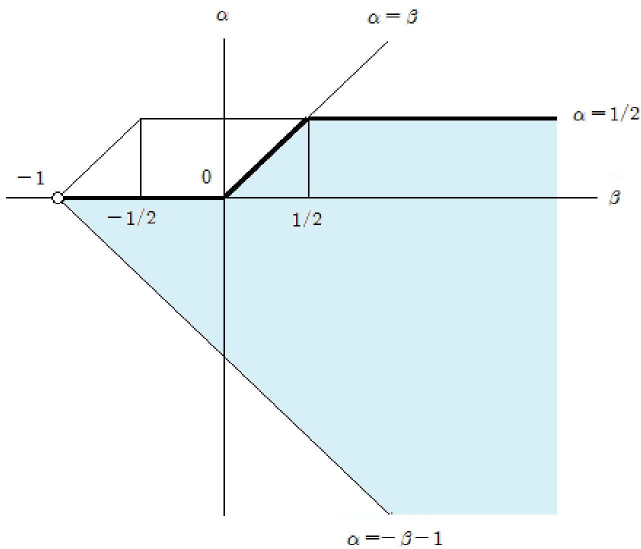

FIGURE 2. The positivity region in the unrestricted case $\mu > 0,\ \lambda \geq 0$.

Remark 2.1. Geometrically, $\mathcal{R}$ represents an infinite polygonal region depicted as in Figure 2. In the case $\lambda = 0$, it follows from an inspection on Lemma 2.1 that the necessity region for nonnegativity is given by $\{\beta > -1,\ -\beta - 1 < \alpha \leq 1/2,\ \alpha < \beta + 1\}$ so that our result does not cover the parallelogram defined by

$$\{0 < \alpha \leq 1/2,\ \beta < \alpha < \beta + 1\}.$$

Proof. For the positivity of Φ, we write $A = (\alpha + \beta + 1)/2$ and apply Lemma 2.1 with $a = A,\ b = A + 1,\ c = \beta + 1$. For $0 < A < 1/2$, Φ is positive when $\beta + 1 \geq 2A$, that is, $-\beta - 1 < \alpha < -\beta,\ \alpha \leq 0$. For $A \geq 1/2$, Φ is positive when $\beta + 1 \geq \max(2A - 1/2,\ A + 1/2)$, that is, $-\beta \leq \alpha \leq \min(\beta,\ 1/2)$. Combining, we obtain the stated region of positivity. □

In the special case $\mu = 2,\ \lambda > 0$, positivity region $\mathcal{R}$ of Theorem 2.1 can be improved considerably. As a matter of fact, if we observe

$$\int_0^x \left(x^2 - t^2\right)^{\lambda} t^{\alpha} J_\beta(t)dt = \frac{B\left(\lambda + 1, \frac{\alpha+\beta+1}{2}\right) x^{2\lambda+\alpha+\beta+1}}{2^{\beta+1}\Gamma(\beta + 1)} \Psi(x), \tag{6}$$

where B stands for the Euler's beta function and

$$\Psi(x) = {}_1F_2\left(\begin{array}{c} \frac{\alpha+\beta+1}{2} \\ \beta + 1,\ \lambda + 1 + \frac{\alpha+\beta+1}{2} \end{array} \middle| -\frac{x^2}{4} \right), \tag{7}$$

then it is routine to deduce from Lemma 2.1 the following result.

Theorem 2.2. *Let $\mathcal{S}$ be the set of parameters (β, α) defined by*

$$\mathcal{S} = \{\beta > -1,\ -\beta - 1 < \alpha \leq 0\} \cup \left\{\beta > 0,\ \alpha \leq \min\left(\beta,\ \lambda + \frac{1}{2}\right)\right\}.$$

If $\lambda > 0$ *and* $(\beta, \alpha) \in \mathcal{S}$, *then*

$$\int_0^x \left(x^2 - t^2\right)^\lambda t^\alpha J_\beta(t)dt > 0 \qquad (x > 0)$$

unless $\alpha = \beta = \lambda + 1/2$ *for which the integral reduces to*

$$\int_0^x \left(x^2 - t^2\right)^\lambda t^{\lambda+\frac{1}{2}} J_{\lambda+\frac{1}{2}}(t)dt = \frac{\sqrt{\pi}\,\Gamma(\lambda+1)(2x^2)^{\lambda+\frac{1}{2}}}{2} J^2_{\lambda+\frac{1}{2}}\left(\frac{x}{2}\right) \geq 0.$$

Remark 2.2. In [7], Gasper also obtained a positivity region in this case. Our result, however, is an improvement in that the triangle with vertices $(0, 0)$, $(\lambda + 1/2, 0)$, $(\lambda + 1/2, \lambda + 1/2)$ is missing in Gasper's positivity region.

3. Positivity of ${}_2F_3$ hypergeometric functions

While Newton diagrams give positivity regions of ${}_1F_2$ hypergeometric functions, it appears that there are no such criteria available for ${}_2F_3$ hypergeometric functions. Our purpose here is to develop some criteria of positivity, which will be exploited later on.

As usual, we shall use Pochhammer's notation to denote

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1), \quad (\alpha)_0 = 1$$

for any real number α and positive integer k. We refer to Bailey [3], and Luke [13] for definitions and basic properties of generalized hypergeometric functions.

A basic feature on positivity is the following.

Lemma 3.1. *For positive real numbers* a, b, c, d, e, *suppose that*

$${}_2F_3\left(\begin{array}{c} a,\, b \\ c,\, d,\, e \end{array} \middle| -\frac{x^2}{4}\right) > 0 \qquad (x > 0).$$

Then for any $\delta \geq 0$, $\gamma \geq 0$, $\epsilon \geq 0$, *we also have*

$${}_2F_3\left(\begin{array}{c} a,\, b \\ c+\delta,\, d+\gamma,\, e+\epsilon \end{array} \middle| -\frac{x^2}{4}\right) > 0 \qquad (x > 0).$$

Proof. Assuming $\delta > 0$, we have

$$\begin{aligned} &{}_2F_3\left(\begin{array}{c} a,\, b \\ c+\delta,\, d,\, e \end{array} \middle| -\frac{x^2}{4}\right) \\ &= \frac{2}{B(c,\delta)} \int_0^1 {}_2F_3\left(\begin{array}{c} a,\, b \\ c,\, d,\, e \end{array} \middle| -\frac{x^2t^2}{4}\right) (1-t^2)^{\delta-1} t^{2c-1}\, dt > 0 \end{aligned}$$

and the other cases follow in the same manner or by symmetry. □

We next deal with ${}_2F_3$ hypergeometric functions of the form

$$\begin{aligned} \Omega(x) &= {}_2F_3\left(\begin{array}{c} a,\, a+\frac{1}{2} \\ c+1,\, a+b,\, a+b+\frac{1}{2} \end{array} \middle| -\frac{x^2}{4}\right) \\ &= \frac{1}{B(2a,\, 2b)} \int_0^1 {}_0F_1\left(c+1\,; -\frac{x^2t^2}{4}\right)(1-t)^{2b-1} t^{2a-1}\, dt \end{aligned} \tag{8}$$

with parameters satisfying $a > 0,\, b > 0,\, c > -1$.

We apply Gasper's *sums of squares formula* ([7], (3.1)) to write

$$\Omega(x) = \Gamma^2(\nu+1)\left(\frac{x}{4}\right)^{-2\nu} \sum_{n=0}^\infty C(n,\nu) \frac{(2n+2\nu)}{n+2\nu} \frac{(2\nu+1)_n}{n!} J^2_{\nu+n}\left(\frac{x}{2}\right) \tag{9}$$

in which $C(n,\nu)$ denotes the terminating series defined by

$$C(n,\nu) = {}_5F_4\left(\begin{matrix} -n,\, n+2\nu,\, \nu+1,\, a,\, a+\frac{1}{2} \\ \nu+\frac{1}{2},\, c+1,\, a+b,\, a+b+\frac{1}{2}\end{matrix}\,\middle|\,1\right) \tag{10}$$

and ν is an arbitrary real number such that 2ν is not a negative integer.

Due to the interlacing property on the zeros of Bessel functions J_ν, $J_{\nu+1}$ (see Watson [16]), the positivity of Ω would follow instantly from formula (9) if $C(n,\nu) > 0$ for all nonnegative integers n and $\nu > -1/2$.

Our investigation on the sign of $C(n,\nu)$ will be carried out along the following steps. We recall that a ${}_pF_q$ generalized hypergeometric function is said to be *Saalschützian* if the sum of numerator parameters plus one is equal to the sum of denominator parameters.

Step 1. We choose $\nu > -1/2$ in such a unique way that each coefficient $C(n,\nu)$ becomes a Saalschützian terminating series, that is,

$$\nu = b + \frac{c}{2} - \frac{1}{4} \quad \text{with} \quad b + \frac{c}{2} + \frac{1}{4} > 0. \tag{11}$$

Step 2. In [8], Gasper discovered a summation formula which states

$$\begin{aligned} &{}_{p+2}F_{p+1}\left(\begin{matrix} -n,\, a_1,\, \cdots,\, a_{p+1} \\ b_1,\, \cdots,\, b_{p+1}\end{matrix}\,\middle|\,1\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(b_1+b_2-a_1-1)_k (b_1-a_1)_k (b_2-a_1)_k}{(b_1+b_2-a_1-1)_{2k}} \frac{(a_2)_k \cdots (a_{p+1})_k}{(b_1)_k \cdots (b_{p+1})_k} \\ &\times\ {}_{p+1}F_p\left(\begin{matrix} k-n,\, k+a_2,\, \cdots,\, k+a_{p+1} \\ 2k+b_1+b_2-a_1,\, k+b_3, \cdots,\, k+b_{p+1}\end{matrix}\,\middle|\,1\right). \end{aligned}$$

An application of this formula gives

$$\begin{aligned} C(n,\nu) &= {}_5F_4\left(\begin{matrix} -n,\, n+2b+c-\frac{1}{2},\, b+\frac{c}{2}+\frac{3}{4},\, a,\, a+\frac{1}{2} \\ b+\frac{c}{2}+\frac{1}{4},\, c+1,\, a+b,\, a+b+\frac{1}{2}\end{matrix}\,\middle|\,1\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{\left(2a+b-\frac{c}{2}-\frac{5}{4}\right)_k \left(a-\frac{c}{2}-\frac{3}{4}\right)_k \left(a-\frac{c}{2}-\frac{1}{4}\right)_k}{\left(2a+b-\frac{c}{2}-\frac{5}{4}\right)_{2k}} \\ &\times \frac{\left(n+2b+c-\frac{1}{2}\right)_k (a)_k \left(a+\frac{1}{2}\right)_k}{(a+b)_k (c+1)_k \left(a+b+\frac{1}{2}\right)_k \left(b+\frac{c}{2}+\frac{1}{4}\right)_k} A_k(a,b,c), \end{aligned} \tag{12}$$

where $A_k(a,b,c)$ denotes the Saalschützian series defined as

$$A_k(a,b,c) = {}_4F_3\left(\begin{matrix} k-n,\, k+n+2b+c-\frac{1}{2},\, k+a,\, k+a+\frac{1}{2} \\ 2k+2a+b-\frac{c}{2}-\frac{1}{4},\, k+b+\frac{c}{2}+\frac{1}{4},\, k+c+1\end{matrix}\,\middle|\,1\right).$$

Step 3. We next apply Whipple's transformation formula (Bailey [3], 7.2(1)),

$$\begin{aligned} {}_4F_3\left(\begin{matrix} -m,\, x,\, y,\, z \\ u,\, v,\, w\end{matrix}\,\middle|\,1\right) &= \frac{(v-z)_m (w-z)_m}{(v)_m (w)_m} \\ &\times\ {}_4F_3\left(\begin{matrix} -m,\, u-x,\, u-y,\, z \\ 1-v+z-m,\, 1-w+z-m,\, u\end{matrix}\,\middle|\,1\right), \end{aligned}$$

valid if it is Saalschützian, to decompose further

$$A_k(a,b,c) = \frac{\left(b+\frac{c}{2}+\frac{1}{4}-a\right)_{n-k}(c+1-a)_{n-k}}{\left(k+b+\frac{c}{2}+\frac{1}{4}\right)_{n-k}(k+c+1)_{n-k}} \times$$

$$(13) \qquad {}_4F_3\left(\begin{array}{c} k-n,\, k-n+2a-b-\frac{3c}{2}+\frac{1}{4},\, k+a+b-\frac{c}{2}-\frac{3}{4},\, k+a \\ k-n+a-b-\frac{c}{2}+\frac{3}{4},\, k-n+a-c,\, 2k+2a+b-\frac{c}{2}-\frac{1}{4} \end{array}\middle|1\right).$$

Step 4. From expansion formula (12), it is evident $C(n,\nu)>0$ if

$$(14) \qquad 2a > -b+\frac{c}{2}+\frac{5}{4},\; a \geq \frac{c}{2}+\frac{3}{4},\; b+\frac{c}{2}+\frac{1}{4}>0$$

and $A_k(a,b,c)>0$ for all k. By using an elementary inequality

$$\frac{(-m)_j(-m+\alpha)_j}{(-m+\beta)_j(-m+\gamma)_j} > 0, \quad j=0,1,\cdots,m,$$

is valid as long as $\alpha\leq 1,\,\beta<1,\,\gamma<1,$ we deduce from (13) $A_k(a,b,c)>0$ when

$$(15) \qquad 2a \leq b+\frac{3c}{2}+\frac{3}{4},\; a<b+\frac{c}{2}+\frac{1}{4},\; a<c+1.$$

Combining (14), (15), we may summarize what we have proved as follows.

Theorem 3.1. *For* $a>0,\,b>0,\,c>-1,$ *we have*

$$(16) \qquad \Omega(x) = {}_2F_3\left(\begin{array}{c} a,\, a+\frac{1}{2} \\ c+1,\, a+b,\, a+b+\frac{1}{2}\end{array}\middle|-\frac{x^2}{4}\right) > 0 \qquad (x>0)$$

if a,b,c *satisfy the following conditions simultaneously.*

$$\begin{cases} \dfrac{c}{2}+\dfrac{3}{4} \leq a < \min\left(c+1,\, b+\dfrac{c}{2}+\dfrac{1}{4}\right), \\ -b+\dfrac{c}{2}+\dfrac{5}{4} < 2a \leq b+\dfrac{3c}{2}+\dfrac{3}{4}. \end{cases}$$

(i) *In the boundary case* $a=b+\frac{c}{2}+\frac{1}{4}$, (16) *also holds true if*

$$\frac{1}{2} < b \leq \frac{c}{2}+\frac{1}{4},\; c>\frac{1}{2}.$$

(ii) *In the boundary case* $c=a-1,$ (16) *also holds true if*

$$b \geq \max\left[1,\, \frac{1}{2}\left(a+\frac{3}{2}\right)\right],\; (a,b)\neq\left(\frac{1}{2},\,1\right).$$

In the case $(a,b)=\left(\frac{1}{2},\,1\right)$, *we have* $\Omega(x)\geq 0$ *for* $x>0$.

Proof. It remains to prove positivity in the two boundary cases.

(i)We apply Whipple's transformation formula of Step 3 directly to obtain

$$C(n,\nu) = \frac{\left(b-\frac{1}{2}\right)_n(b)_n}{\left(2b+\frac{c}{2}+\frac{1}{4}\right)_n\left(2b+\frac{c}{2}+\frac{3}{4}\right)_n}$$
$$\times\; {}_4F_3\left(\begin{array}{c} -n,\, -n-2b+\frac{3}{2},\, -b+\frac{c}{2}+\frac{1}{4},\, b+\frac{c}{2}+\frac{3}{4} \\ -n-b+\frac{3}{2},\, -n-b+1,\, c+1\end{array}\middle|1\right),$$

which is easily seen to be positive under the former condition by the same reasonings as in Step 4. If $b=c/2+1/4,\, a=c+1/2,$ then Ω reduces to

$$\Omega(x) = {}_1F_2\left(\begin{array}{c} c+\frac{1}{2} \\ \frac{3}{2}\left(c+\frac{1}{2}\right),\, \frac{3}{2}\left(c+\frac{1}{2}\right)+\frac{1}{2}\end{array}\middle|-\frac{x^2}{4}\right)$$

and positivity with $c > 1/2$ follows by Lemma 2.1 (see also Fields and Ismail [6]).

(ii) In this case, it is easy to deduce again from Lemma 2.1

$$\Omega(x) = {}_1F_2\left(\begin{matrix} a + \frac{1}{2} \\ a + b,\ a + b + \frac{1}{2} \end{matrix} \,\middle|\, -\frac{x^2}{4} \right) > 0$$

under the stated condition. In the case $(a, b) = \left(\frac{1}{2},\ 1\right)$, Ω reduces to

$$\Omega(x) = \left[\frac{\sin(x/2)}{x/2}\right]^2 \geq 0.$$

□

In the special case $b = 1$, we obtain

Corollary 3.1. *For* $a > 0,\ c > -1,$ *we have*

$$ {}_2F_3\left(\begin{matrix} a,\ a + \frac{1}{2} \\ c + 1,\ a + 1,\ a + \frac{3}{2} \end{matrix} \,\middle|\, -\frac{x^2}{4} \right) > 0 \qquad (x > 0) \tag{17}$$

if a, c *satisfy one of the following conditions.*

(i) $$\begin{cases} \dfrac{c}{2} + \dfrac{3}{4} \leq a < \min\left(c + 1,\ \dfrac{c}{2} + \dfrac{5}{4}\right), \\ \dfrac{c}{2} + \dfrac{1}{4} < 2a \leq \dfrac{3c}{2} + \dfrac{7}{4}. \end{cases}$$

(ii) $a = \dfrac{c}{2} + \dfrac{5}{4},\ c \geq \dfrac{3}{2}.$ (iii) $c = a - 1,\ 0 < a \leq \dfrac{1}{2}.$

4. Improved results of Misiewicz and Richards

In the case $0 < \mu \leq 1 \leq \lambda$, the density $t \mapsto (1 - t^\mu)_+^\lambda$ is convex and nonincreasing on $(0, \infty)$. By using Williamson's characterization [19] on such monotone convex functions, Misiewicz and Richards [14] observed

$$\int_0^x (x^\mu - t^\mu)^\lambda\, t^\alpha J_\beta(t) dt = x^{\mu\lambda - 1} \int_0^1 K(xt)\, dG(t),$$
$$K(x) = \int_0^x (x - t) t^\alpha J_\beta(t) dt, \tag{18}$$

with a unique probability measure G, so that the positivity of (1) under consideration would follow once kernel K were shown to be nonnegative.

In view of the well-known Bessel identity (Watson [16]),

$$J_\beta(x) = \frac{1}{\Gamma(\beta + 1)} \left(\frac{x}{2}\right)^\beta {}_0F_1\left(\beta + 1\,; -\frac{x^2}{4}\right), \quad \beta > -1, \tag{19}$$

it is simple to modify (8) to evaluate

$$K(x) = \frac{B(\alpha + \beta + 1, 2) x^{\alpha + \beta + 2}}{2^\beta\, \Gamma(\beta + 1)}\, {}_2F_3\left(\begin{matrix} \frac{\alpha+\beta+1}{2},\ \frac{\alpha+\beta+2}{2} \\ \beta + 1,\ \frac{\alpha+\beta+3}{2},\ \frac{\alpha+\beta+4}{2} \end{matrix} \,\middle|\, -\frac{x^2}{4} \right) \tag{20}$$

and hence the problem of positivity reduces to the nonnegativity question on the ${}_2F_3$ hypergeometric functions defined in (20).

Our improvement of Theorem A reads as follows.

Theorem 4.1. *Let $\mathcal{P}$ be the set of parameters (β, α) defined by*

$$\mathcal{P} = \left\{\beta > -1,\ -\beta - 1 < \alpha \le \min\left[\beta + 1,\ \frac{1}{2}\left(\beta + \frac{3}{2}\right),\ \frac{3}{2}\right]\right\}.$$

For $0 < \mu \le 1 \le \lambda$ and $(\beta, \alpha) \in \mathcal{P}$, we have

$$\int_0^x (x^\mu - t^\mu)^\lambda\, t^\alpha J_\beta(t)dt > 0 \qquad (x > 0).$$

Remark 4.1. In Figure 3, the trapezoid with vertices

$$(-1, 0),\ (-1/2, -1/2),\ (3/2, 3/2),\ (-1/2, 1/2)$$

is newly added to the positivity region of Misiewicz and Richards which corresponds to the infinite polygon bounded by $\alpha = -\beta - 1,\ \alpha = \beta,\ \alpha = \frac{3}{2}$.

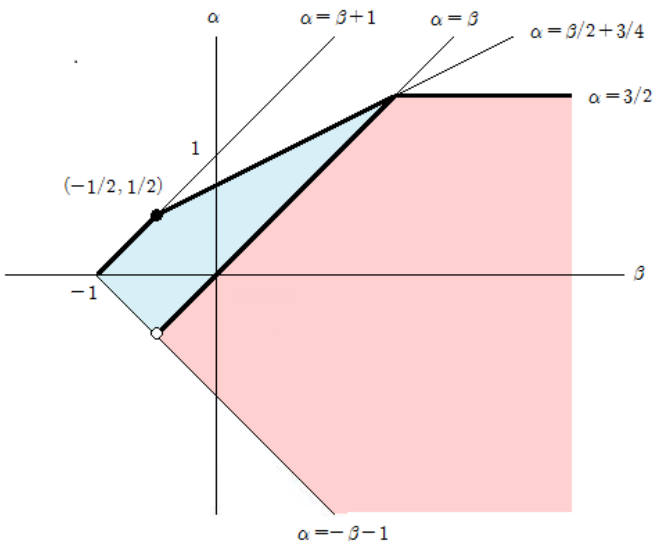

FIGURE 3. The positivity region in the case $0 < \mu \le 1 \le \lambda$ which improves the ones of Misiewicz and Richards (pink) and Kuttner (black dot).

Proof. The ${}_2F_3$ hypergeometric functions of (20) are of type (17) with

$$a = \frac{\alpha + \beta + 1}{2}, \quad c = \beta.$$

It is simple to find condition (i) of Corollary 3.1 gives the infinite strip

$$\beta \ge -\frac{1}{2}, \quad \frac{1}{2} \le \alpha < \frac{3}{2}, \quad \alpha \le \frac{1}{2}\left(\beta + \frac{3}{2}\right) \tag{21}$$

as a positivity region. Likewise, conditions (ii), (iii) correspond to the boundary lines

$$\beta \ge \frac{3}{2},\ \alpha = \frac{3}{2} \quad \text{and} \quad -1 < \beta \le -\frac{1}{2},\ \alpha = \beta + 1. \tag{22}$$

To fill out the remaining positivity region, we observe from Lemma 3.1 that if (20) is positive with some parameters $\alpha_0, \beta_0,$ then positivity continues to hold true for all parameters $\alpha_0 - \delta,\ \beta_0 + \delta,\ \delta \geq 0,$ that is, for all α, β lying on the half-line emanating from (β_0, α_0) defined by $\alpha = -\beta + \alpha_0 + \beta_0,\ \beta \geq \beta_0$. By adding all half-lines emanating from $(\beta, \alpha) \in \mathcal{P}$ constructed from (21), (22), we obtain the full stated region $\mathcal{P}$. $\square$

In the special case $\mu = 1,$ reduction (18) is unnecessary for

$$\begin{aligned} \int_0^x (x-t)^\lambda\, t^\alpha J_\beta(t) dt &= \frac{B(\alpha+\beta+1, \lambda+1) x^{\lambda+\alpha+\beta+1}}{2^\beta\, \Gamma(\beta+1)} \\ &\times\ {}_2F_3\left(\begin{array}{c} \frac{\alpha+\beta+1}{2},\ \frac{\alpha+\beta+2}{2} \\ \beta+1,\ \frac{\alpha+\beta+\lambda+2}{2},\ \frac{\alpha+\beta+\lambda+3}{2} \end{array} \Bigg| -\frac{x^2}{4} \right) \end{aligned} \tag{23}$$

in which the generalized hypergeometric functions are of type (16) with

$$a = \frac{\alpha+\beta+1}{2},\ b = \frac{\lambda+1}{2},\ c = \beta.$$

A direct application of Theorem 3.1 yields the following improved result.

Theorem 4.2. *For $\lambda > 0$ and $(\beta, \alpha) \in \mathcal{O},$ we have*

$$\int_0^x (x-t)^\lambda\, t^\alpha J_\beta(t) dt > 0 \qquad (x > 0),$$

where $\mathcal{O}$ denotes the set of parameters defined by

$$\mathcal{O} = \begin{cases} \beta > -1,\ -\beta + 1 - \lambda < \alpha \leq \min\left[\frac{1}{2}\left(\beta + \lambda + \frac{1}{2}\right), \lambda + \frac{1}{2}\right], & \text{if} \quad \lambda < 1, \\ \beta > -1,\ -\beta - 1 < \alpha \leq \min\left[\beta + 1,\ \frac{1}{2}\left(\beta + \lambda + \frac{1}{2}\right), \lambda + \frac{1}{2}\right], & \text{if} \quad \lambda \geq 1. \end{cases}$$

As the proof proceeds in the same fashion as above, we shall omit it. We refer to Gasper [7], [8] for related results and further applications.

In view of the identity

$$J_{-\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi\, t}}\, \cos t,$$

the case $\beta = -1/2$ in Theorem 2.1, Theorem 4.1 corresponds to Kuttner's result.

Corollary 4.1. *If $\mu > 0,\ \lambda \geq 0,\ -1 < \alpha \leq -\frac{1}{2}$ or $0 < \mu \leq 1 \leq \lambda,\ -1 < \alpha \leq 0,$ then*

$$\int_0^x (x^\mu - t^\mu)^\lambda\, t^\alpha \cos t\, dt > 0 \qquad (x > 0).$$

In the case $\alpha = 0,$ the problem of determining the positivity range of $\lambda = \lambda(\mu)$ is a long-standing open problem and we refer to Golubov [11] and Gneiting, Konis and Richards [10] for partial results and further references.

5. Buhmann's radial basis functions

While studying scattered data approximations, Buhmann [4] introduced a 4-parameter family of compactly supported *radial basis functions* on $\mathbb{R}^n$ defined as follows.

Definition 5.1 (Buhmann's radial basis functions)**.** For $\delta > 0$, $\rho \geq 0$, $\lambda > -1$ and $\alpha > -n/2 - 1$, define

$$W(\mathbf{x}) = \int_0^\infty \left(1 - \frac{\|\mathbf{x}\|^2}{t}\right)_+^\lambda t^\alpha \left(1 - t^\delta\right)_+^\rho dt \qquad (\mathbf{x} \in \mathbb{R}^n), \tag{24}$$

where $\|\mathbf{x}\|$ stands for the Euclidean norm $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$.

For this family of compactly supported radial functions, Buhmann proved the following *positive definiteness* (see also [20]).

Theorem C (Buhmann, [4])**.** *Let $\mathcal{B}_n$ be the set of parameters (λ, α) defined by*

$$\begin{aligned}
\mathcal{B}_1 &= \left\{\lambda > -\frac{1}{2},\, -1 < \alpha \leq \min\left(\lambda - \frac{1}{2}, \frac{\lambda}{2}\right)\right\},\\
\mathcal{B}_2 &= \left\{\lambda > -\frac{1}{2},\, -1 < \alpha \leq \min\left[\frac{1}{2}\left(\lambda - \frac{1}{2}\right), \lambda - \frac{1}{2}\right]\right\},\\
\mathcal{B}_3 &= \left\{\lambda \geq 0,\, -1 < \alpha \leq \frac{1}{2}(\lambda - 1)\right\},\\
\mathcal{B}_n &= \left\{\lambda > \frac{n-5}{2},\, -1 < \alpha \leq \frac{1}{2}\left(\lambda - \frac{n-1}{2}\right)\right\} \quad \textit{if} \quad n \geq 4.
\end{aligned}$$

For $0 < \delta \leq \frac{1}{2}$, $\rho \geq 1$, if $(\lambda, \alpha) \in \mathcal{B}_n$, then W has a strictly positive Fourier transform and hence induces positive definite matrices on $\mathbb{R}^n$.

As Buhmann calculated, the Fourier transform of W is $\widehat{W}(\xi) = \omega(\|\xi\|)$, $\xi \in \mathbb{R}^n$, where

$$\omega(x) = \frac{(2\pi)^{\frac{n}{2}} 2^{\lambda+1} \Gamma(\lambda+1)}{x^{n+2+2\delta\rho+2\alpha}} \int_0^x \left(x^{2\delta} - t^{2\delta}\right)^\rho t^{2\alpha+1-\lambda+\frac{n}{2}} J_{\lambda+\frac{n}{2}}(t)\, dt \tag{25}$$

for $x > 0$ and Buhmann exploited Theorem A to establish the above result.

Our purpose here is to extend $\mathcal{B}_n$ in several directions.

We begin with extending Theorem C with the aid of Theorem 4.1.

Theorem 5.1. *Let $\mathcal{P}_n$ be the set of parameters (λ, α) defined by*

$$\mathcal{P}_n = \left\{\lambda > -1,\, -\frac{n+2}{2} < \alpha \leq \min\left[\frac{1}{4}\left(3\lambda - \frac{n+1}{2}\right), \frac{1}{2}\left(\lambda - \frac{n-1}{2}\right)\right]\right\}.$$

For $0 < \delta \leq \frac{1}{2}$, $\rho \geq 1$, if $(\lambda, \alpha) \in \mathcal{P}_n$, then W has a strictly positive Fourier transform and hence induces positive definite matrices on $\mathbb{R}^n$.

Remark 5.1. The proof follows trivially upon renaming parameters, that is,

$$2\delta \to \mu,\ \rho \to \lambda,\ 2\alpha + 1 - \lambda + n/2 \to \alpha,\ \lambda + n/2 \to \beta.$$

It is easy to observe $\mathcal{B}_n \subset \mathcal{P}_n$ in a proper way, as shown in Figure 4 in the one-dimensional case. In addition, an inspection on the two boundary lines of $\mathcal{P}_n$ reveals

$$\mathcal{P}_n = \left\{\lambda > -1,\, -\frac{n+2}{2} < \alpha \leq \frac{1}{2}\left(\lambda - \frac{n-1}{2}\right)\right\} \quad \text{for} \quad n \geq 5.$$

In the case $\delta = 1/2$, Theorem 4.2 gives an improvement.

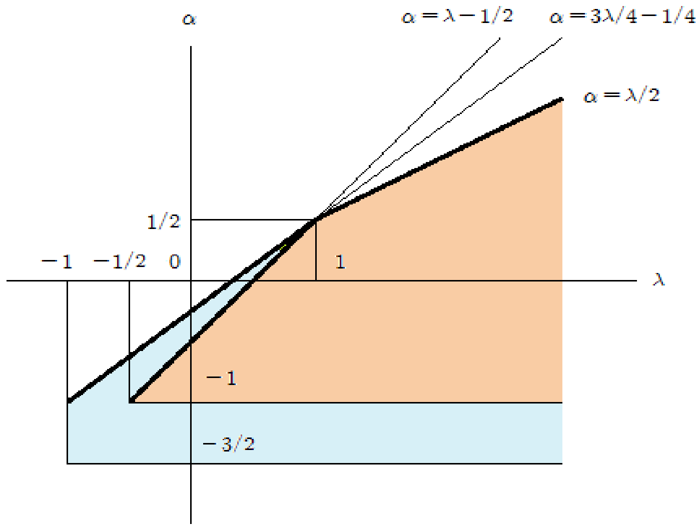

FIGURE 4. The regions of positive definiteness $\mathcal{B}_1$ (yellow) and $\mathcal{P}_1$.

Theorem 5.2. *For* $\delta = \frac{1}{2}$, $\rho \geq 1$, *Theorem* 5.1 *continues to hold true if* $\mathcal{P}_n$ *is replaced by the set* $\mathcal{O}_n$ *of parameter pairs* (λ, α) *defined as*

$$\mathcal{O}_n = \left\{\lambda > -1,\, -\frac{n+2}{2} < \alpha \leq \min\left[\lambda,\, \frac{1}{4}\left(3\lambda - \frac{n+3}{2} + \rho\right),\, \frac{1}{2}\left(\lambda - \frac{n+1}{2} + \rho\right)\right]\right\}.$$

Remark 5.2. In the special occasion $\delta = \frac{1}{2}$, $\rho = \frac{n+1}{2} + \sigma$, $\lambda = \alpha$ with $\sigma \geq 0$, Buhmann's radial basis functions take the form

$$W(\mathbf{x}) = 2\int_{\|\mathbf{x}\|}^{1} \left(t^2 - \|\mathbf{x}\|^2\right)^{\alpha} t\,(1-t)^{\frac{n+1}{2}+\sigma}\, dt, \tag{26}$$

known as Wendland's functions (see [15], [17], [18]), which are are easily seen to be positive definite for $-1 < \alpha \leq \sigma - 1$ according to the above theorem. Obviously, Buhmann's original theorem, Theorem C, is not applicable in this case.

In the unrestricted case $\delta > 0$, $\rho \geq 0$, Theorem 2.1 yields the following.

Theorem 5.3. *Let* $\mathcal{R}_n$ *be the set of parameters* (λ, α) *defined by*

$$\mathcal{R}_1 = \left\{\lambda > -1,\, -\frac{3}{2} < \alpha \leq \frac{1}{2}\left(\lambda - \frac{3}{2}\right)\right\} \cup \left\{\lambda > -\frac{1}{2},\, \frac{1}{2}\left(\lambda - \frac{3}{2}\right) < \alpha \leq \min\left[\lambda - \frac{1}{2},\, \frac{1}{2}(\lambda - 1)\right]\right\},$$

$$\mathcal{R}_n = \left\{\lambda > -1,\, -\frac{n+2}{2} < \alpha \leq \min\left[\lambda - \frac{1}{2},\, \frac{1}{2}\left(\lambda - \frac{n+1}{2}\right)\right]\right\}, \quad n \geq 2.$$

For $\delta > 0$, $\rho \geq 0$, *if* $(\lambda, \alpha) \in \mathcal{R}_n$, *then each* W *has a nonnegative non-vanishing Fourier transform and hence induces positive definite matrices on* $\mathbb{R}^n$.

Remark 5.3. The Fourier transform of W is indeed strictly positive unless

$$\rho = 0,\ \alpha = -\frac{n}{2}\,,\ \lambda = -\frac{n-1}{2}\,,\ n = 1, 2. \tag{27}$$

In such an exceptional case of (27), it is simple to evaluate

$$W(\mathbf{x}) = \begin{cases} 2\,(1 - \|\mathbf{x}\|) & \text{if} \quad n = 1, \\ 2\ln\left(\dfrac{1 + \sqrt{1 - \|\mathbf{x}\|^2}}{\|\mathbf{x}\|}\right) & \text{if} \quad n = 2, \end{cases} \tag{28}$$

for $\|\mathbf{x}\| \le 1$ and zero otherwise. Moreover, its Fourier transform is given by

$$\widehat{W}(\xi) = 2\pi^{\frac{n-1}{2}}\Gamma\left(\frac{3-n}{2}\right)\left[\frac{\sin(\|\xi\|/2)}{\|\xi\|/2}\right]^2 \ge 0 \qquad (\xi \in \mathbb{R}^n). \tag{29}$$

An improvement in the case $\delta = 1$ owes to Theorem 2.2.

Theorem 5.4. *For $\delta = 1$, $\rho \ge 0$, Theorem 5.3 continues to hold true if $\mathcal{R}_n$ is replaced by the set $\mathcal{S}_n$ of parameter pairs (λ, α) defined as*

$$\begin{aligned} \mathcal{S}_1 &= \left\{\lambda > -1,\ -\frac{3}{2} < \alpha \le \frac{1}{2}\left(\lambda - \frac{3}{2}\right)\right\} \\ &\quad \cup \left\{\lambda > -\frac{1}{2},\ \frac{1}{2}\left(\lambda - \frac{3}{2}\right) < \alpha \le \min\left[\lambda - \frac{1}{2},\ \frac{1}{2}\,(\lambda - 1 + \rho)\right]\right\}, \\ \mathcal{S}_n &= \left\{\lambda > -1,\ -\frac{n+2}{2} < \alpha \le \min\left[\lambda - \frac{1}{2},\ \frac{1}{2}\left(\lambda - \frac{n+1}{2} + \rho\right)\right]\right\}, \quad n \ge 2. \end{aligned}$$

Remark 5.4. The Fourier transform of W is strictly positive unless

$$\alpha = \rho - \frac{n}{2}\,,\ \lambda = \rho - \frac{n-1}{2}\,,\ 1 \le n \le [2\rho + 3]\,. \tag{30}$$

In such an exceptional case, $W, \widehat{W}$ are explicitly given by

$$\begin{cases} W(\mathbf{x}) = 2\displaystyle\int_{\|\mathbf{x}\|}^{1} \left(t^2 - \|\mathbf{x}\|^2\right)^{\rho + \frac{1}{2} - \frac{n}{2}} \left(1 - t^2\right)^{\rho}\, dt, \\ \widehat{W}(\xi) = \dfrac{\pi^{\frac{n+1}{2}}\Gamma(\rho+1)\Gamma\left(\frac{2\rho+3-n}{2}\right)}{(\|\xi\|/2)^{2\rho+1}} J^2_{\rho+\frac{1}{2}}\left(\dfrac{\|\xi\|}{2}\right) \end{cases} \tag{31}$$

for $\|\mathbf{x}\| \le 1$ and $\xi \in \mathbb{R}^n$. In the special case $\rho = \frac{n-1}{2}$, radial basis function W is often referred to as Euclid's hat function (see Gneiting [9]).

6. Appendix: Newton diagram of positivity

We shall assume $x > 0$ in what follows and write

$$\mathbb{J}_\nu(x) = {}_0F_1\left(\nu + 1\,; -\frac{x^2}{4}\right) \qquad (\nu > -1). \tag{32}$$

In view of (19), it is evident that the $\mathbb{J}_\nu$ share positive zeros in common with Bessel functions J_ν. A basic principle of positivity is the following analogue of Lemma 3.1 for ${}_1F_2$ hypergeometric functions which can be proved in the same manner.

Lemma 6.1. *For* $a > 0,\ b > 0,\ c > 0,$ *suppose that*

$$ {}_1F_2\left(a\,;b,c\,;-\frac{x^2}{4}\right) \ge 0. $$

Then for any $0 \le \gamma < a,\ \delta \ge 0,\ \epsilon \ge 0,$ *not simultaneously zero,*

$$ {}_1F_2\left(a-\gamma\,;b+\delta,\ c+\epsilon;-\frac{x^2}{4}\right) > 0. $$

Proof of Lemma 2.1. For part (i), if $\phi \ge 0$ and $0 < b \le a,$ then Lemma 6.1 implies

$$ {}_1F_2\left(b\,;\,b,c\,;-\frac{x^2}{4}\right) = \mathbb{J}_{c-1}(x) > 0, $$

which contradicts the fact $\mathbb{J}_{c-1}$ has infinitely many positive zeros. Thus $b > a$ and $c > a$ by symmetry. In view of the asymptotic behavior ([13])

$$ \begin{aligned} \phi(x) &= \frac{\Gamma(b)\Gamma(c)}{\Gamma(b-a)\Gamma(c-a)}\left(\frac{x}{2}\right)^{-2a}\left[1+O\left(x^{-2}\right)\right] \\ &+ \frac{\Gamma(b)\Gamma(c)}{\sqrt{\pi}\,\Gamma(a)}\left(\frac{x}{2}\right)^{-\sigma}\left[\cos\left(x-\frac{\pi\sigma}{2}\right)+O\left(x^{-1}\right)\right] \end{aligned} \tag{33} $$

as $x \to \infty,$ where $\sigma = b + c - a - 1/2,$ it is immediate to observe the condition $\sigma \ge 2a$ is necessary, that is, $b + c \ge 3a + 1/2$.

Regarding part (ii), observe first ([16], Chapter 5) that

$$ {}_1F_2\left(a\,;a+\frac{1}{2},\ 2a\,;-\frac{x^2}{4}\right) = \mathbb{J}^2_{a-\frac{1}{2}}\left(\frac{x}{2}\right) \ge 0. $$

For $a = 1/2,$ the two points of Λ coincide and the positivity of ϕ with parameter pair $(b,c) \in P_{1/2}$ follows from Lemma 6.1. For $a \ne 1/2,$ if (b,c) lies on the boundary line of P_a, that is, $c = 3a + 1/2 - b,$ then it is not hard to compute the coefficients of Gasper's sums of squares series expansion by Saalschütz's formula to deduce

$$ \begin{aligned} {}_1F_2\left(a\,;b,\ 3a+\frac{1}{2}-b\,;-\frac{x^2}{4}\right) &= \Gamma^2\left(a+\frac{1}{2}\right)\left(\frac{x}{4}\right)^{-2a-1} \\ &\times \sum_{n=0}^{\infty}\frac{2n+2a-1}{n+2a-1}\frac{(2a)_n}{n!}\frac{(2a-b)_n\,(b-a-1/2)_n}{(b)_n\,(3a+1/2-b)_n}J^2_{n+a-\frac{1}{2}}\left(\frac{x}{2}\right), \end{aligned} \tag{34} $$

which is easily seen to be positive when b lies strictly between $a + 1/2$ and $2a$. The positivity of ϕ for $(b,c) \in P_a$ now follows from this boundary case and Lemma 6.1. □

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