

NOT SO MANY NON-DISJOINT TRANSLATIONS

ANDRZEJ ROSŁANOWSKI AND VYACHESLAV V. RYKOV

(Communicated by Heike Mildenerger)

ABSTRACT. We show that, consistently, there is a Borel set which has uncountably many pairwise very non-disjoint translations, but does not allow a perfect set of such translations.

1. INTRODUCTION

There is some interest in the literature in Borel sets admitting many pairwise disjoint translations. For instance, Balcerzak, Rosłanowski, and Shelah [1] studied the σ -ideal of subsets of ${}^\omega 2$ generated by Borel sets with a perfect set of pairwise disjoint translations. In this article we are interested in a somewhat dual property of Borel sets: many overlapping translations.

If $B \subseteq {}^\omega 2$ is an uncountable Borel set, then it includes a perfect set P , and then for $x, y \in P$ we have

$$\mathbf{0}, x + y \in (P + x) \cap (P + y).$$

Consequently, every uncountable Borel subset of ${}^\omega 2$ has a perfect set of pairwise non-disjoint translations. However, if we demand that the intersections are more substantial, then the problem of many non-disjoint translations becomes more interesting. One should notice that if $x + b_0 = y + b_1$, then also $x + b_1 = y + b_0$, so if $x \neq y$ and $(B + x) \cap (B + y)$ is finite, then $|(B + x) \cap (B + y)|$ must be even.

Here we investigate the first non-trivial case when $(B + x) \cap (B + y)$ has at least four elements. We show that it is consistent with ZFC that there is a Σ_2^0 subset B of the Cantor space ${}^\omega 2$ such that

- for some uncountable set $H \subseteq {}^\omega 2$, $|(B + h) \cap (B + h')| \geq 4$ for all $h, h' \in H$, but
- for every perfect set $P \subseteq {}^\omega 2$ there are $x, x' \in P$ such that

$$|(B + x) \cap (B + x')| \leq 2.$$

Our proof follows the spirit of the proof of Shelah [5, Theorem 1.13], but since we cut on generality, our arguments are more straightforward. We fully utilize the algebraic properties of $({}^\omega 2, +)$, in particular the fact that all elements of ${}^\omega 2$ are self-inverse.

This line of research will be continued in Rosłanowski and Shelah [4], where we will deal with the general case of κ many pairwise non-disjoint translations (getting the full parallel of [5, Theorem 1.13]).

Received by the editors November 11, 2017, and, in revised form, November 14, 2017, May 27, 2018, and August 13, 2018.

2010 *Mathematics Subject Classification*. Primary 03E35; Secondary 03E15, 54H05.

Notation and terminology. Our notation is rather standard and compatible with that of classical textbooks (like Jech [2]). However, in forcing we keep the convention that *a stronger condition is the larger one*.

Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ, ξ . Natural numbers (finite ordinals) will be called i, j, k and ℓ, n .

For a forcing notion \mathbb{P} , all \mathbb{P} -names for objects in the extension via \mathbb{P} will be denoted with a tilde below (e.g., \tilde{h}, \tilde{T}), and $G_{\mathbb{P}}$ will stand for the canonical \mathbb{P} -name for the generic filter in \mathbb{P} .

For two sequences η, ν we write $\nu \triangleleft \eta$ whenever ν is a proper initial segment of η , and $\nu \trianglelefteq \eta$ when either $\nu \triangleleft \eta$ or $\nu = \eta$. A *tree* is a \triangleleft -downward closed set of sequences.

The set of all sequences of length n and with values in $\{0, 1\}$ is denoted by ${}^n 2$ and we let ${}^{\omega} 2 = \bigcup_{n < \omega} {}^n 2$. For $\sigma \in {}^{\omega} 2$ let $[\sigma] = \{x \in {}^{\omega} 2 : \sigma \triangleleft x\}$. The Cantor space ${}^{\omega} 2$ of all infinite 0–1 sequences is equipped with the topology generated by sets of the form $[\sigma]$ and the coordinate-wise addition $+$ modulo 2. Thus $({}^{\omega} 2, +)$ is a topological group.

For a tree $T \subseteq {}^{\omega} 2$ we set $[T] = \{x \in {}^{\omega} 2 : (\forall n < \omega)(x \upharpoonright n \in T)\}$.

For a set $A \subseteq X \times Y$ and $x \in X$ and $y \in Y$ let

$$A_x = \{z \in Y : (x, z) \in A\} \quad \text{and} \quad A^y = \{z \in X : (z, y) \in A\}.$$

2. SOME TECHNICALITIES

Definition 2.1. Let $1 < \ell < \omega$. A 4-arrangement in ${}^{\ell} 2$ is a tuple $\langle a, b, c, d \rangle \subseteq {}^{\ell} 2$ such that $a <_{\text{lex}} b <_{\text{lex}} c <_{\text{lex}} d$ and

$$\begin{aligned} \min\{k < \ell : a(k) \neq c(k)\} &= \min\{k < \ell : b(k) \neq c(k)\} \\ &= \min\{k < \ell : a(k) \neq d(k)\} = \min\{k < \ell : b(k) \neq d(k)\}. \end{aligned}$$

Lemma 2.2. Let $15 < \ell < \omega$. Suppose that $h : [{}^{\ell} 2]^2 \rightarrow 2$ is a coloring with the property that

(\otimes) if $a, b, c \in {}^{\ell} 2$ are distinct, then $h(a, b) = 1$ or $h(a, c) = 1$, or $h(b, c) = 1$.

(That is, there is no h -homogenous triangle in color 0.) Then there is a set $\mathcal{A} \subseteq {}^{\ell} 2$ such that

- (1) $|\mathcal{A}| \geq 5$, and \mathcal{A} contains a 4-arrangement, and
- (2) \mathcal{A} is h -homogeneous in color 1, i.e., $h(a, b) = 1$ for distinct $a, b \in \mathcal{A}$.

Proof. First, for $a \in {}^{\ell} 2$ let $Z_a = \{x \in {}^{\ell} 2 : x \neq a \wedge h(x, a) = 0\}$. It follows from the assumption (\otimes) that

(*) for each a , the set Z_a is h -homogenous in color 1.

If for some $a \in {}^{\ell} 2$ the set Z_a satisfies the requirements of (1), then we are done. So suppose that

(\odot) for each $a \in {}^{\ell} 2$ either $|Z_a| \leq 4$ or Z_a contains no 4-arrangement.

Let $a \in {}^{\ell} 2$ be the sequence constantly equal to 0 and let $d \in {}^{\ell} 2$ be the $<_{\text{lex}}$ -last element of ${}^{\ell} 2 \setminus Z_a$. It follows from (\odot) that $\{x \in {}^{\ell} 2 : x \upharpoonright (\ell - 3) \equiv 1\} \setminus Z_a \neq \emptyset$, and hence $d(k) = 1$ for $k < \ell - 3$.

Let $Y = \{\sigma \in {}^{\ell-3} 2 : \sigma(0) = 0 \wedge \sigma(1) = 1\}$ and for $\sigma \in Y$ let $X_{\sigma} = \{x \in {}^{\ell} 2 : \sigma \triangleleft x\}$. By (\odot), $X_{\sigma} \setminus Z_a \neq \emptyset$ (for each $\sigma \in Y$), so we may pick $x_{\sigma} \in X_{\sigma}$ such that $h(a, x_{\sigma}) = 1$. Again by (\odot), the set $\{x_{\sigma} : \sigma \in Y\}$ cannot be contained in

Z_d , so we may pick $\sigma^* \in Y$ such that $h(d, x_{\sigma^*}) = 1$. Set $b = x_{\sigma^*}$ and note that $h(a, b) = h(b, d) = 1$.

Now we repeat the above procedure “on d 's side” for both a and b and d . We let $Y' = \{\sigma \in {}^{\ell-3}2 : \sigma(0) = 1 \wedge \sigma(1) = 0\}$ and $Y'' = \{\rho \in {}^{\ell-6}2 : \sigma(0) = 1 \wedge \sigma(1) = 0\}$. For $\sigma \in Y'$ consider $X_\sigma = \{x \in {}^\ell 2 : \sigma \triangleleft x\}$ and note that by our assumptions we may pick $x'_\sigma \in X_\sigma$ such that $h(a, x'_\sigma) = 1$. Now, for each $\rho \in Y''$ we may choose $\sigma_\rho \in Y'$ such that $\rho \triangleleft \sigma_\rho$ and $h(b, x'_{\sigma_\rho}) = 1$. By our assumptions, for some $\rho^* \in Y''$ we also have $h(d, x'_{\sigma_{\rho^*}}) = 1$. Set $c = x'_{\sigma_{\rho^*}}$ and note that $\langle a, b, c, d \rangle$ is a 4-arrangement which is homogenous in color 1.

Repeating the above procedure again, but starting with $Y^+ = \{\sigma \in {}^{\ell-3}2 : \sigma(0) = \sigma(1) = 0 \wedge \sigma(2) = 1\}$, going through levels $\ell - 3$, $\ell - 6$, and $\ell - 9$ and dealing with a, b, c, d one may find $e \in {}^\ell 2$ such that $\mathcal{A} = \{a, b, c, d, e\}$ satisfies the demands (1)+(2). \square

Lemma 2.3. *Let $0 < \ell < \omega$ and let $\mathcal{B} \subseteq {}^\ell 2$ be a linearly independent set of vectors (in $({}^\ell 2, +, \cdot)$ over $(\mathbb{2}, +_2, \cdot_2)$).*

- (a) *If $a, b, c \in {}^\ell 2$ are pairwise distinct and $\{a, b, c\} + \{a, b, c\} \subseteq \mathcal{B} + \mathcal{B}$, then for some pairwise distinct $\eta, \nu, \rho \in \mathcal{B}$ we have*

$$a + b = \eta + \nu \quad \text{and} \quad a + c = \eta + \rho.$$

- (b) *If $\mathcal{A} \subseteq {}^\ell 2$, $|\mathcal{A}| \geq 5$, and $\mathcal{A} + \mathcal{A} \subseteq \mathcal{B} + \mathcal{B}$, then for a unique $x \in {}^\ell 2$ we have $\mathcal{A} + x \subseteq \mathcal{B}$.*

Proof. (a) Let $\nu_a, \nu_b, \eta_a, \eta_c, \rho_b, \rho_c \in \mathcal{B}$ be such that

$$a + b = \nu_a + \nu_b, \quad a + c = \eta_a + \eta_c, \quad \text{and} \quad b + c = \rho_b + \rho_c.$$

Then $\nu_a \neq \nu_b$, $\eta_a \neq \eta_c$, $\rho_b \neq \rho_c$, and

$$\rho_b + \rho_c = b + c = a + b + a + c = \nu_a + \nu_b + \eta_a + \eta_c.$$

By the linear independence of \mathcal{B} we conclude $\{\nu_a, \nu_b\} \cap \{\eta_a, \eta_c\} \neq \emptyset$.

(b) Let $\mathcal{A} = \{a_0, a_1, \dots, a_{n-1}\}$, $n = |\mathcal{A}| \geq 5$. Using clause (a) we may choose $\eta, \nu, \rho \in \mathcal{B}$ such that

$$a_0 + a_1 = \eta + \nu \quad \text{and} \quad a_0 + a_2 = \eta + \rho.$$

Note that

- (\oplus) if $2 < k < n$ and $a_0 + a_k = \eta^* + \eta^+$ with $\eta^*, \eta^+ \in \mathcal{B}$, then

$$|\{\eta^*, \eta^+\} \cap \{\eta, \nu\}| = |\{\eta^*, \eta^+\} \cap \{\eta, \rho\}| = 1.$$

(Just apply clause (a) to a_0, a_1, a_k and to a_0, a_2, a_k , remembering that \mathcal{B} is linearly independent.)

Let $x = a_0 + \eta$. We will argue that $a_i + x \in \mathcal{B}$ for all $i < n$. Clearly by our choices this holds for $i \leq 2$. Suppose $2 < i < n$ is such that $a_i + x \notin \mathcal{B}$. Let $\eta^*, \eta^+ \in \mathcal{B}$ be such that $a_0 + a_i = \eta^* + \eta^+$. If we had $\eta \in \{\eta^*, \eta^+\}$, then $a_i + (a_0 + \eta) \in \mathcal{B}$, contradicting $a_i + x \notin \mathcal{B}$. Therefore $\eta \notin \{\eta^*, \eta^+\}$ and it follows from (\oplus) that $\nu, \rho \in \{\eta^*, \eta^+\}$, so $a_0 + a_i = \nu + \rho$. Let $j < n$ be such that $j \notin \{0, 1, 2, i\}$, and let $a_0 + a_j = \nu^* + \nu^+$, $\nu^*, \nu^+ \in \mathcal{B}$. Then (using (\oplus)) we get

$$|\{\nu^*, \nu^+\} \cap \{\eta, \nu\}| = |\{\nu^*, \nu^+\} \cap \{\eta, \rho\}| = |\{\nu^*, \nu^+\} \cap \{\nu, \rho\}| = 1,$$

a contradiction. The uniqueness of x follows from the linear independence of \mathcal{B} . \square

Lemma 2.4. *Suppose that $P \subseteq {}^\omega 2$ is a perfect set and $A_n \subseteq P \times P$ are Σ_1^1 sets (for $n < \omega$) such that $P \times P = \bigcup_{n < \omega} A_n \cup \{(x, x) : x \in P\}$. Then there is a perfect set $P^* \subseteq P$ with the following property:*

- (\heartsuit) *For some increasing sequence of integers $0 = n_0 < n_1 < n_2 < n_3 < \dots$, for each $k < \omega$ and any distinct $x, y \in P^*$ we have*
- (a) *if $x \upharpoonright n_{k+1} \neq y \upharpoonright n_{k+1}$, then for all $x', y' \in P^*$*
- $$(x \upharpoonright n_{k+1} = x' \upharpoonright n_{k+1} \wedge y \upharpoonright n_{k+1} = y' \upharpoonright n_{k+1}) \Rightarrow ((x, y) \in A_k \Leftrightarrow (x', y') \in A_k),$$
- (b) *the set $\{z \upharpoonright n_{k+1} : z \in P^* \wedge z \upharpoonright n_k = x \upharpoonright n_k\}$ has exactly two elements.*

Proof. We will use the general result of Mycielski on the existence of independent sets in topological algebras. To be able to quote his theorem we have to introduce some definitions.

We say that a set $S \subseteq P^m$ is obtained by identification of variables from $R \subseteq P^{m+1}$ if for some $i, j \leq m$ we have

$$(x_1, \dots, x_m) \in S \Leftrightarrow (x_1, \dots, x_i, x_j, x_{i+1}, \dots, x_m) \in R.$$

For $n < \omega$ let J_n consist of all pairs $(\sigma, \rho) \in {}^{\omega > 2} \times {}^{\omega > 2}$ such that the set $([\sigma] \times [\rho]) \cap A_n$ is meager (in P^2) and $[\sigma] \cap [\rho] = \emptyset$, and let K_n consist of pairs $(\sigma, \rho) \in {}^{\omega > 2} \times {}^{\omega > 2}$ for which $([\sigma] \times [\rho]) \setminus A_n$ is meager and $[\sigma] \cap [\rho] = \emptyset$. For $(\sigma, \rho) \in J_n$ we may fix a Borel meager (in P) set $X_{\sigma, \rho} \subseteq P$ such that

$$\begin{aligned} &(\forall x \in [\sigma] \setminus X_{\sigma, \rho})((A_n)_x \cap [\rho] \text{ is meager}) \text{ and} \\ &(\forall y \in [\rho] \setminus X_{\sigma, \rho})((A_n)^y \cap [\sigma] \text{ is meager}). \end{aligned}$$

Similarly, if $(\sigma, \rho) \in K_n$, then a Borel meager set $X_{\sigma, \rho} \subseteq P$ is such that

$$\begin{aligned} &(\forall x \in [\sigma] \setminus X_{\sigma, \rho})([\rho] \setminus (A_n)_x \text{ is meager}) \text{ and} \\ &(\forall y \in [\rho] \setminus X_{\sigma, \rho})([\sigma] \setminus (A_n)^y \text{ is meager}). \end{aligned}$$

For $(\sigma, \rho) \in J_n \cup K_n$ let

$$R_{\sigma, \rho}^n = \{(x_1, x_2, y_1, y_2) \in P^4 : \begin{aligned} &x_1, x_2 \in [\sigma] \setminus X_{\sigma, \rho} \wedge y_1, y_2 \in [\rho] \setminus X_{\sigma, \rho} \\ &\wedge (x_1, y_1) \in A_n \wedge (x_2, y_2) \notin A_n \end{aligned}\}.$$

Clearly for every $(\sigma, \rho) \in J_n \cup K_n$ the set $R_{\sigma, \rho}^n$ is meager (in P^4), moreover if $S \subseteq P^k$ is obtained from $R_{\sigma, \rho}^n$ by repeated identification and/or permutation of variables, then S is meager in P^k as well.

The sets A_n have the Baire property and hence the sets $J_n \cup K_n$ are dense in ${}^{\omega > 2} \times {}^{\omega > 2}$. Let

$$X = \bigcup \{X_{\sigma, \rho} : (\sigma, \rho) \in J_n \cup K_n \wedge n < \omega\}$$

and

$$R_n = \{(x, y) \in P^2 : x \neq y \text{ and for all } (\sigma, \rho) \in J_n \cup K_n \text{ we have } x \notin [\sigma] \vee y \notin [\rho]\}.$$

Easily, X is a meager subset of P , each R_n is meager in P^2 and identification of variables in R_n leads to an empty set (so meager).

By [3, Theorem 1, p. 141] there is a perfect set $P' \subseteq P$ such that

- $(P' \times P' \times P' \times P') \cap R_{\sigma, \rho}^n = \emptyset$ for all $n < \omega$ and $(\sigma, \rho) \in J_n \cup K_n$, and
- $P' \cap X = \emptyset$ and $(P' \times P') \cap R_n = \emptyset$ for all $n < \omega$.

Clearly, if $x \neq y$ are from P' and $n < \omega$, then for some $N < \omega$ we have

$$(\forall x', y' \in P')((x \upharpoonright N = x' \upharpoonright N \wedge y \upharpoonright N = y' \upharpoonright N) \Rightarrow ((x, y) \in A_n \Leftrightarrow (x', y') \in A_n)).$$

By shrinking the perfect P' one can construct a perfect set $P^* \subseteq P'$ and an increasing sequence $0 = n_0 < n_1 < n_2 < \dots$ such that the demands in (\heartsuit) are satisfied. \square

3. THE MAIN RESULT

Theorem 3.1. *There exists a ccc forcing notion \mathbb{P} of size ω_1 such that in $\mathbf{V}^{\mathbb{P}}$, there is a Σ_2^0 set $B \subseteq {}^\omega 2$ with the properties that*

- (♠) (a) *for some sequence $\langle h_\alpha : \alpha < \omega_1 \rangle$ of pairwise distinct elements of ${}^\omega 2$ we have $|(h_\alpha + B) \cap (h_\beta + B)| \geq 4$, but*
- (b) *in each perfect set $P \subseteq {}^\omega 2$ there are $f, g \in P$ with $|(f+B) \cap (g+B)| \leq 2$.*

Proof. A condition $p \in \mathbb{P}$ is a tuple

$$p = \langle u, n, \bar{\eta}, m_*, \bar{t}, \mu, K \rangle = \langle u^p, n^p, \bar{\eta}^p, m_*^p, \bar{t}^p, \mu^p, K^p \rangle$$

satisfying the following demands:

- (1) $\emptyset \neq u \in [\omega_1]^{<\omega}$, $0 < m_*, n < \omega$, and $\bar{\eta} = \langle \eta_\alpha : \alpha \in u \rangle \subseteq {}^{n_2}$.
- (2) $\bar{t} = \langle t_m : m < m_* \rangle$, each $t_m \subseteq {}^{n \geq 2}$ is a tree with all maximal nodes of length n .
- (3) $\mu : [u]^2 \rightarrow {}^{n_2 \times m_*}$, and if $\alpha \neq \beta$ are from u , then we will write $\mu(\alpha, \beta) = \mu(\beta, \alpha) = (\rho_{\alpha, \beta}, \ell_{\alpha, \beta})$.
- (4) If $\alpha \neq \beta$ are from u , then both $\eta_\alpha + \rho_{\alpha, \beta} \in t_{\ell_{\alpha, \beta}}$ and $\eta_\beta + \rho_{\alpha, \beta} \in t_{\ell_{\alpha, \beta}}$.
- (5) $K : u \rightarrow m_* : \alpha \mapsto K_\alpha$ and $\eta_\alpha \in t_{K_\alpha}$.
- (6) If $\alpha < \beta < \gamma$ are from u , then $\{K_\alpha, K_\gamma, \ell_{\alpha, \gamma}\} \neq \{K_\beta, K_\gamma, \ell_{\beta, \gamma}\}$.
- (7) If $m < m' < m_*$, then $t_m \cap t_{m'} \cap {}^{n_2} = \emptyset$.
- (8) If $m < m_*$, then $t_m \cap {}^{n_2} \subseteq \{\eta_\alpha + \rho_{\alpha, \beta} : \alpha \neq \beta \wedge \alpha, \beta \in u\} \cup \{\eta_\alpha : \alpha \in u\}$.
- (9) $\langle \eta_\alpha : \alpha \in u \rangle \frown \langle \rho_{\alpha, \beta} : \alpha < \beta \wedge \alpha, \beta \in u \rangle$ is a list of linearly independent vectors (in $({}^{n_2}, +, \cdot)$ over $(2, +_2, \cdot_2)$); in particular they are pairwise distinct.

The order $\leq_{\mathbb{P}} = \leq$ of \mathbb{P} is defined by:

$p \leq q$ if and only if the following conditions are satisfied:

- (i) $u^p \subseteq u^q$, $n^p \leq n^q$, and $m_*^p \leq m_*^q$.
- (ii) If $\alpha \in u^p$, then $\eta_\alpha^q \upharpoonright n^p = \eta_\alpha^p$.
- (iii) If $m < m_*^q$, then $t_m^q \cap {}^{n^p} = t_m^p \cap {}^{n^p}$.
- (iv) If $\alpha \in u^p$, then $K_\alpha^p = K_\alpha^q$ and if $\alpha \neq \beta$ are from u^p , then $\ell_{\alpha, \beta}^p = \ell_{\alpha, \beta}^q$ and $\rho_{\alpha, \beta}^p \trianglelefteq \rho_{\alpha, \beta}^q$.

Claim 3.1.1. (\mathbb{P}, \leq) is a partial order of size ω_1 .

Claim 3.1.2.

- (1) If $p \in \mathbb{P}$ and $b_0, c_0, b_1, c_1 \in \bigcup_{m < m_*^p} (t_m^p \cap {}^{n^p})$ are pairwise distinct and satisfy $b_0 + c_0 = b_1 + c_1$, then for some $\alpha \neq \beta$ from u^p we have

$$\{b_0, c_0, b_1, c_1\} = \{\eta_\alpha^p, \eta_\beta^p, \eta_\alpha^p + \rho_{\alpha, \beta}^p, \eta_\beta^p + \rho_{\alpha, \beta}^p\}.$$

- (2) Also, for some $i < 2$, $\{b_i, c_i\} = \{\eta_\alpha^p, \eta_\beta^p\}$ or $\{b_i, c_i\} = \{\eta_\alpha^p + \rho_{\alpha, \beta}^p, \eta_\beta^p\}$ or $\{b_i, c_i\} = \{\eta_\alpha^p + \rho_{\alpha, \beta}^p, \eta_\alpha^p\}$.

Proof of the claim. It follows from the definition of \mathbb{P} (clause (8)) that $b_0, c_0, b_1, c_1 \in \{\eta_\alpha, \eta_\alpha + \rho_{\alpha, \beta} : \alpha \neq \beta \text{ are from } u^p\}$. Since, by clause (9), $\langle \eta_\alpha^p : \alpha \in u \rangle \frown \langle \rho_{\alpha, \beta} : \alpha < \beta \wedge \alpha, \beta \in u \rangle$ are linearly independent we easily get our conclusion. \square

Claim 3.1.3. For every $N, M < \omega$ and $\delta < \omega_1$ the set

$$Z_\delta^{N, M} = \{p \in \mathbb{P} : n^p \geq N \wedge m_*^p \geq M \wedge \delta \in u^p\}$$

is open dense in \mathbb{P} .

Proof of the claim. Suppose that $p \in \mathbb{P}$ and let $\alpha \in \omega_1 \setminus u^p$.

Let $\langle \alpha_0, \dots, \alpha_k \rangle$ be the increasing enumeration of u^p . Set $u = u^p \cup \{\alpha\}$, $n = n^p + k + 2$, and $m_* = m_*^p + k + 2$. For $i \leq k$ let

$$\eta_{\alpha_i} = \eta_{\alpha_i}^p \frown \underbrace{\langle 0, \dots, 0 \rangle}_{k+2} \quad \text{and} \quad \rho_{\alpha_i, \alpha} = \rho_{\alpha, \alpha_i} = \underbrace{\langle 0, \dots, 0 \rangle}_{n^p+i+1} \frown \langle 1 \rangle \frown \underbrace{\langle 0, \dots, 0 \rangle}_{k-i}.$$

We also let $\eta_\alpha = \underbrace{\langle 0, \dots, 0 \rangle}_{n^p} \frown \langle 1 \rangle \frown \underbrace{\langle 0, \dots, 0 \rangle}_{k+1}$ and we put $\ell_{\alpha_i, \alpha} = \ell_{\alpha, \alpha_i} = m_*^p + i$ and

$K_\alpha = m_*^p + k + 1$. Next, for $i \leq k$ we define $K_{\alpha_i} = K_{\alpha_i}^p$ and for $i < j \leq k$ we let $\rho_{\alpha_j, \alpha_i} = \rho_{\alpha_i, \alpha_j} = \rho_{\alpha_i, \alpha_j}^p \frown \underbrace{\langle 0, \dots, 0 \rangle}_{k+2}$, and $\ell_{\alpha_j, \alpha_i} = \ell_{\alpha_i, \alpha_j} = \ell_{\alpha_i, \alpha_j}^p$. (So a function

$\mu : [u]^2 \longrightarrow n^2 \times m_*$ is defined now too.) For $m < m_*^p$ let

$$t_m = t_m^p \cup \{\sigma \frown \underbrace{\langle 0, \dots, 0 \rangle}_j : \sigma \in t_m^p \cap n^p \cdot 2 \wedge j < k + 3\}$$

and for $m = m_*^p + i < m_* - 1$ let $t_m = \{(\eta_{\alpha_i} + \rho_{\alpha_i, \alpha}) \upharpoonright j, (\eta_\alpha + \rho_{\alpha, \alpha}) \upharpoonright j : j \leq n\}$ and $t_{m_*-1} = \{\eta_\alpha \upharpoonright j : j \leq n\}$. Finally, let $\bar{t} = \langle t_m : m < m_* \rangle$.

It is straightforward to verify that $q = \langle u, n, \bar{\eta}, m_*, \bar{t}, \mu, K \rangle$ satisfies the demands of the definition of a condition in \mathbb{P} . Moreover, q is a condition stronger than p , and $\alpha \in u^q$, $m_*^q \geq m_*^p + 2$ and $n^q \geq n^p + 2$.

Now the claim readily follows. \square

Claim 3.1.4. The forcing notion \mathbb{P} has the Knaster property.

Proof of the claim. Suppose that $\langle p_\xi : \xi < \omega_1 \rangle$ is a sequence of conditions from \mathbb{P} . Applying the standard Δ -lemma based cleaning procedure we may find an uncountable set $A \subseteq \omega_1$ such that $\{u^{p_\xi} : \xi \in A\}$ forms a Δ -system of finite sets and for $\xi < \zeta$ from A we have:

- (*)₁ $n^{p_\xi} = n^{p_\zeta}$, $m_*^{p_\xi} = m_*^{p_\zeta}$, $\bar{t}^{p_\xi} = \bar{t}^{p_\zeta}$,
- (*)₂ $|u^{p_\xi}| = |u^{p_\zeta}|$, $u^{p_\xi} \cap u^{p_\zeta}$ is an initial segment of both u^{p_ξ} and u^{p_ζ} and $\max(u^{p_\xi} \setminus u^{p_\zeta}) < \min(u^{p_\zeta} \setminus u^{p_\xi})$,
- (*)₃ if $\pi : u^{p_\xi} \longrightarrow u^{p_\zeta}$ is the order preserving bijection, then for every $\alpha \in u^{p_\xi}$ we have

$$K_\alpha^{p_\xi} = K_{\pi(\alpha)}^{p_\zeta} \quad \text{and} \quad \eta_\alpha^{p_\xi} = \eta_{\pi(\alpha)}^{p_\zeta}$$

and $\mu^{p_\xi}(\alpha, \beta) = \mu^{p_\zeta}(\pi(\alpha), \pi(\beta))$ for all $\alpha < \beta$ from u^{p_ξ} .

We may assume that $u^{p_\xi} \cap u^{p_\zeta} \neq \emptyset \neq u^{p_\xi} \setminus u^{p_\zeta}$ for distinct $\xi, \zeta \in A$.

We will argue that if $\xi < \zeta$ are from A , then the conditions p_ξ, p_ζ are compatible.

Let

- (*)₄ $\langle \gamma_0, \dots, \gamma_{k_0} \rangle$ be the increasing enumeration of $u^{p_\varepsilon} \cap u^{p_\zeta}$, let $\langle \alpha_0, \dots, \alpha_{k_1} \rangle$ be the increasing enumeration of $u^{p_\varepsilon} \setminus u^{p_\zeta}$, and let $\langle \beta_0, \dots, \beta_{k_1} \rangle$ be the increasing enumeration of $u^{p_\zeta} \setminus u^{p_\varepsilon}$ (so $2 + k_0 + k_1 = |u^{p_\varepsilon}|$) and let
- (*)₅ $k^* = (k_1 + 1)(k_0 + k_1 + 3) + \frac{(k_1 - 1)(k_1 + 2)}{2} + 1$, $n = n^{p_\varepsilon} + k^*$, and $m_* = m_*^{p_\varepsilon} + (k_1 + 1)^2$, and
- (*)₆ $\nu_i = \underbrace{\langle 0, \dots, 0 \rangle}_i \frown \langle 1 \rangle \frown \underbrace{\langle 0, \dots, 0 \rangle}_{k^* - i - 1} \in k^* 2$ for $i < k^*$.

To define a common upper bound to p_ε and p_ζ we put

- (*)₇ $u = u^{p_\varepsilon} \cup u^{p_\zeta} = \{\alpha_i : i \leq k_1\} \cup \{\beta_i : i \leq k_1\} \cup \{\gamma_i : i \leq k_0\}$, and
- (*)₈ for $i \leq k_1$ we set $\eta_{\alpha_i} = \eta_{\alpha_i}^{p_\varepsilon} \frown \underbrace{\langle 0, \dots, 0 \rangle}_{n - n^{p_\varepsilon}}$, $\eta_{\beta_i} = \eta_{\beta_i}^{p_\zeta} \frown \nu_i$ and for $i \leq k_0$ we let

$$\eta_{\gamma_i} = \eta_{\gamma_i}^{p_\varepsilon} \frown \underbrace{\langle 0, \dots, 0 \rangle}_{n - n^{p_\varepsilon}}, \text{ and}$$

- (*)₉ $K_{\alpha_i} = K_{\alpha_i}^{p_\varepsilon}$, $K_{\beta_i} = K_{\beta_i}^{p_\zeta}$ (for $i \leq k_1$) and $K_{\gamma_i} = K_{\gamma_i}^{p_\varepsilon}$ (for $i \leq k_0$).

We define also a function $\mu : [u]^2 \longrightarrow n 2 \times m_* : \{\delta, \varepsilon\} \mapsto (\rho_{\delta, \varepsilon}, \ell_{\delta, \varepsilon})$ as follows:

- (*)₁₀ if $\delta < \varepsilon$ are from u^{p_ε} , then $\rho_{\delta, \varepsilon} = \rho_{\delta, \varepsilon}^{p_\varepsilon} \frown \underbrace{\langle 0, \dots, 0 \rangle}_{n - n^{p_\varepsilon}}$ and $\ell_{\delta, \varepsilon} = \ell_{\delta, \varepsilon}^{p_\varepsilon}$;
- (*)₁₁ if $i \leq k_0$ and $j \leq k_1$, then $\rho_{\gamma_i, \beta_j} = \rho_{\gamma_i, \beta_j}^{p_\zeta} \frown \nu_k$, where $k = (k_1 + 1) + i(k_1 + 1) + j$, and $\ell_{\gamma_i, \beta_j} = \ell_{\gamma_i, \beta_j}^{p_\zeta}$;
- (*)₁₂ if $i, j \leq k_1$, then $\rho_{\alpha_i, \beta_j} = \underbrace{\langle 0, \dots, 0 \rangle}_{n^{p_\zeta}} \frown \nu_k$, where $k = (k_0 + 2)(k_1 + 1) + i(k_1 + 1) + j$, and $\ell_{\alpha_i, \beta_j} = m_*^{p_\zeta} + i(k_1 + 1) + j$;
- (*)₁₃ if $i < j \leq k_1$, then $\rho_{\beta_i, \beta_j} = \rho_{\beta_i, \beta_j}^{p_\zeta} \frown \nu_k$, where $k = (k_1 + 1)(k_0 + k_1 + 3) + \frac{i(2k_1 - i + 1)}{2} + (j - i - 1)$, and $\ell_{\beta_i, \beta_j} = \ell_{\beta_i, \beta_j}^{p_\zeta}$.

Finally, let $\bar{t} = \langle t_m : m < m_* \rangle$, where

- (*)₁₄ if $m < m_*^{p_\zeta}$, then

$$\begin{aligned} t_m = & t_m^{p_\zeta} \cup \{\sigma \frown \underbrace{\langle 0, \dots, 0 \rangle}_k : \sigma \in t_m^{p_\zeta} \cap n^{p_\zeta} 2 \wedge k \leq k^*\} \\ & \cup \{\eta_{\beta_i} \upharpoonright k : i \leq k_1 \wedge K_{\beta_i} = m \wedge k \leq n\} \\ & \cup \{(\eta_{\gamma_i} + \rho_{\gamma_i, \beta_j}) \upharpoonright k : i \leq k_0 \wedge j \leq k_1 \wedge \ell_{\gamma_i, \beta_j} = m \wedge k \leq n\} \\ & \cup \{(\eta_{\beta_j} + \rho_{\gamma_i, \beta_j}) \upharpoonright k : i \leq k_0 \wedge j \leq k_1 \wedge \ell_{\gamma_i, \beta_j} = m \wedge k \leq n\} \\ & \cup \{(\eta_{\beta_i} + \rho_{\beta_i, \beta_j}) \upharpoonright k : i < j \leq k_1 \wedge \ell_{\beta_i, \beta_j} = m \wedge k \leq n\} \\ & \cup \{(\eta_{\beta_j} + \rho_{\beta_i, \beta_j}) \upharpoonright k : i < j \leq k_1 \wedge \ell_{\beta_i, \beta_j} = m \wedge k \leq n\} \end{aligned}$$

and

- (*)₁₅ for $m = m_*^{p_\zeta} + i(k_1 + 1) + j < m_*$, $i, j \leq k_1$, we let

$$t_m = \{(\eta_{\alpha_i} + \rho_{\alpha_i, \beta_j}) \upharpoonright k, (\eta_{\beta_j} + \rho_{\alpha_i, \beta_j}) \upharpoonright k : k \leq n\}.$$

One easily verifies that $q = \langle u, n, \bar{\eta}, m_*, \bar{t}, \mu, K \rangle$ satisfies the demands of the definition of a condition in \mathbb{P} and that this condition is a common upper bound of p_ζ and p_ε . \square

We define \mathbb{P} -names \underline{h}_α (for $\alpha < \omega_1$), \underline{T}_m (for $m < \omega$) and $\underline{r}_{\alpha,\beta}$ (for $\alpha < \beta < \omega_1$) by

- $\Vdash_{\mathbb{P}} \underline{h}_\alpha = \bigcup \{ \eta_\alpha^p : p \in \mathcal{G}_{\mathbb{P}} \wedge \alpha \in u^p \}$,
- $\Vdash_{\mathbb{P}} \underline{T}_m = \bigcup \{ t_m^p : p \in \mathcal{G}_{\mathbb{P}} \wedge m < m_*^p \}$,
- $\Vdash_{\mathbb{P}} \underline{r}_{\alpha,\beta} = \bigcup \{ \rho_{\alpha,\beta}^p : p \in \mathcal{G}_{\mathbb{P}} \wedge \alpha, \beta \in u^p \}$.

Claim 3.1.5. For $\alpha < \beta < \omega_1$ and $m < \omega$ we have

- (1) $\Vdash_{\mathbb{P}} \underline{h}_\alpha, \underline{r}_{\alpha,\beta} \in {}^\omega 2$,
- (2) $\Vdash_{\mathbb{P}} \underline{T}_m \subseteq {}^{>\omega} 2$ is a tree with no maximal nodes,
- (3) $\Vdash_{\mathbb{P}}$ “if $m < m' < \omega$, then $[\underline{T}_m] \cap [\underline{T}_{m'}] = \emptyset$ ”.

Proof of the claim. By Claim 3.1.3 and the definition of the order of \mathbb{P} . \square

Let \underline{B} be the \mathbb{P} -name for the Σ_2^0 subset of ${}^\omega 2$ given by $\Vdash_{\mathbb{P}} \underline{B} = \bigcup_{m < \omega} [\underline{T}_m]$.

Claim 3.1.6. For each $\alpha < \beta < \omega_1$ we have

$$\Vdash_{\mathbb{P}} \left| (\underline{h}_\alpha + \underline{B}) \cap (\underline{h}_\beta + \underline{B}) \right| \geq 4.$$

Proof of the claim. It should be clear that $\underline{h}_\alpha, \underline{h}_\beta, \underline{h}_\alpha + \underline{r}_{\alpha,\beta}$, and $\underline{h}_\beta + \underline{r}_{\alpha,\beta}$ are forced to belong to \underline{B} and they all are pairwise distinct. Therefore, $\mathbf{0}, \underline{r}_{\alpha,\beta}, \underline{h}_\alpha + \underline{h}_\beta$ and $\underline{h}_\alpha + \underline{h}_\beta + \underline{r}_{\alpha,\beta}$ are distinct elements of the intersection $(\underline{h}_\alpha + \underline{B}) \cap (\underline{h}_\beta + \underline{B})$. \square

Claim 3.1.7.

$\Vdash_{\mathbb{P}}$ “for every perfect set $P \subseteq {}^\omega 2$ there are $f, g \in P$ with $|(f + \underline{B}) \cap (g + \underline{B})| < 4$ ”.

Proof of the claim. Suppose $G \subseteq \mathbb{P}$ is generic over \mathbf{V} and let us work in $\mathbf{V}[G]$ for awhile. Assume towards contradiction that $P \subseteq {}^\omega 2$ is a perfect set such that

$$(\forall f, g \in P) (|(f + \underline{B}^G) \cap (g + \underline{B}^G)| \geq 4).$$

Then for distinct $f, g \in P$ there are pairwise distinct $b_0, c_0, b_1, c_1 \in B$ such that $f + g = b_0 + c_0 = b_1 + c_1$. Now, for $(\ell_0, m_0, \ell_1, m_1, N) \in {}^5 \omega$ let

$$A_{\ell_0, m_0, \ell_1, m_1}^N = \{(f, g) \in P^2 : \text{for some } b_i \in [T_{\ell_i}^G], c_i \in [T_{m_i}^G] \text{ (for } i < 2) \text{ we have } \\ b_0 \upharpoonright N \neq c_0 \upharpoonright N \text{ and } \{b_0 \upharpoonright N, c_0 \upharpoonright N\} \cap \{b_1 \upharpoonright N, c_1 \upharpoonright N\} = \emptyset \\ \text{and } f + g = b_0 + c_0 = b_1 + c_1\}.$$

By our assumption on P we know that

$$(\boxtimes)_1 \text{ for each distinct } x, y \in P \text{ there are } \ell_0, m_0, \ell_1, m_1, N < \omega \text{ such that } (x, y) \in A_{\ell_0, m_0, \ell_1, m_1}^N.$$

Since the sets $A_{\ell_0, m_0, \ell_1, m_1}^N$ are Σ_1^1 , we may use Lemma 2.4 to choose a perfect set $P^* \subseteq P$ and an increasing sequence $0 = n_0 < n_1 < n_2 < n_3 < \dots$ such that

$(\boxtimes)_2$ for each $k < \omega$ and any distinct $x, y \in P^*$ we have:

- (a) if $x \upharpoonright n_{k+1} \neq y \upharpoonright n_{k+1}$, $\ell_0, m_0, \ell_1, m_1, N \leq k$, then for all $x', y' \in P^*$ satisfying $x \upharpoonright n_{k+1} = x' \upharpoonright n_{k+1}$ and $y \upharpoonright n_{k+1} = y' \upharpoonright n_{k+1}$ we have

$$(x, y) \in A_{\ell_0, m_0, \ell_1, m_1}^N \Leftrightarrow (x', y') \in A_{\ell_0, m_0, \ell_1, m_1}^N,$$

- (b) the set $\{z \upharpoonright n_{k+1} : z \in P^* \wedge z \upharpoonright n_k = x \upharpoonright n_k\}$ has exactly two elements.

Now, by induction on $j \leq 21$ we may choose $0 = k_0 < k_1 < k_2 < \dots < k_j < \dots < k_{20} < k_{21}$ and $A \subseteq P^*$ such that

$$(\boxtimes)_3 \quad |A| = 2^{20}, \text{ say } A = \{x_0, \dots, x_{2^{20}-1}\},$$

$$(\boxtimes)_4 \text{ if } j \leq 20, x, y \in A \text{ and } x \upharpoonright n_{k_j} \neq y \upharpoonright n_{k_j}, \text{ then } (x, y) \in A_{\ell_0, m_0, \ell_1, m_1}^N \text{ for some } \ell_0, m_0, \ell_1, m_1, N < k_j,$$

(\boxtimes)₅ if $j < 20$ and $x \in A$, then there is $y \in A$ such that $x \upharpoonright n_{k_j} = y \upharpoonright n_{k_j}$ but $x \upharpoonright n_{k_{j+1}} \neq y \upharpoonright n_{k_{j+1}}$.

Let \underline{P}^* , \bar{n} , $\underline{A} = \{x_0, \dots, x_{2^{20}-1}\}$, and \bar{k} be \mathbb{P} -names for the objects appearing in (\boxtimes)₁–(\boxtimes)₅ and let a condition $p \in G$ force that they have the properties listed there.

Passing to a stronger condition we may also demand that

- (\boxtimes)₆ p decides the values of k_0, k_1, \dots, k_{21} , say $p \Vdash k_j = k_j$ for $j \leq 21$,
- (\boxtimes)₇ p decides the values of $n_0, n_1, \dots, n_{k_{21}}$, say $p \Vdash n_i = n_i$ for $i \leq k_{21}$,
- (\boxtimes)₈ p decides the values of $x_0 \upharpoonright n_{k_{21}}, \dots, x_{2^{20}-1} \upharpoonright n_{k_{21}}$, say $p \Vdash x_i \upharpoonright n_{k_{21}} = \sigma_i^*$ for $i < 2^{20}$,
- (\boxtimes)₉ $n^p > n_{k_{21}}$ and $m_*^p > k_{21}$.

Note that it follows from (\boxtimes)₃ + (\boxtimes)₅ that

- (\boxtimes)₁₀ if $i < j < 2^{20}$, then $\sigma_i^* \neq \sigma_j^*$.

Since p forces that x_i 's have the properties listed in (\boxtimes)₁ and (\boxtimes)₃–(\boxtimes)₅, there are $\sigma_i \in n^{n^p 2}$ (for $i < 2^{20}$) such that

- (\boxtimes)₁₁ $\sigma_i^* \triangleleft \sigma_i$ for each $i < 2^{20}$, and
- (\boxtimes)₁₂ if $i, j < 2^{20}$ are distinct, then for some $\ell_0(i, j), m_0(i, j), \ell_1(i, j), m_1(i, j) < m_*^p$ and $b_0(i, j) \in t_{\ell_0(i, j)}^{n^p 2}, c_0(i, j) \in t_{m_0(i, j)}^{n^p 2}, b_1(i, j) \in t_{\ell_1(i, j)}^{n^p 2}, c_1(i, j) \in t_{m_1(i, j)}^{n^p 2}$ we have
 - (a) $\sigma_i + \sigma_j = b_0(i, j) + c_0(i, j) = b_1(i, j) + c_1(i, j)$, and
 - (b) $\{b_0(i, j), c_0(i, j)\} \cap \{b_1(i, j), c_1(i, j)\} = \emptyset$,
- (\boxtimes)₁₃ if $i, i', j, j' < 2^{20}$ and $k < n^p$ are such that $\sigma_i \upharpoonright k = \sigma_{i'} \upharpoonright k \neq \sigma_j \upharpoonright k = \sigma_{j'} \upharpoonright k$, then

$$\langle \ell_0(i, j), m_0(i, j), \ell_1(i, j), m_1(i, j) \rangle = \langle \ell_0(i', j'), m_0(i', j'), \ell_1(i', j'), m_1(i', j') \rangle.$$

It follows from (\boxtimes)₁₀–(\boxtimes)₁₂ that there are no repetitions in the list $b_0(i, j), c_0(i, j), b_1(i, j), c_1(i, j)$.

By Claim 3.1.2(2), for distinct $i, j < 2^{20}$ we can find $\ell < 2$ and distinct α, β from u^p such that

- either $\{b_\ell(i, j), c_\ell(i, j)\} = \{\eta_\alpha^p, \eta_\beta^p\}$ (in which case we set $h(i, j) = 1$),
- or $\{b_\ell(i, j), c_\ell(i, j)\} = \{\eta_\alpha^p + \rho_{\alpha, \beta}^p, \eta_\beta^p\}$ (and then we set $h(i, j) = 0$),
- or $\{b_\ell(i, j), c_\ell(i, j)\} = \{\eta_\alpha^p + \rho_{\alpha, \beta}^p, \eta_\alpha^p\}$ (and then we also set $h(i, j) = 0$).

Note that

- (\boxtimes)₁₄ if $i, j, k < 2^{20}$ are pairwise distinct, then $h(i, j) = 1$, or $h(j, k) = 1$, or $h(i, k) = 1$.

Why? First suppose that for some $\alpha < \beta, \gamma < \delta$ and $\varepsilon < \zeta$ from u^p we have

$$\begin{aligned} \sigma_i + \sigma_j &= b_0(i, j) + c_0(i, j) = \eta_\alpha^p + \eta_\beta^p + \rho_{\alpha, \beta}^p, \\ \sigma_j + \sigma_k &= b_0(j, k) + c_0(j, k) = \eta_\gamma^p + \eta_\delta^p + \rho_{\gamma, \delta}^p, \\ \sigma_i + \sigma_k &= b_0(i, k) + c_0(i, k) = \eta_\varepsilon^p + \eta_\zeta^p + \rho_{\varepsilon, \zeta}^p. \end{aligned}$$

Then

$$0 = (\eta_\alpha^p + \eta_\beta^p + \rho_{\alpha, \beta}^p) + (\eta_\gamma^p + \eta_\delta^p + \rho_{\gamma, \delta}^p) + (\eta_\varepsilon^p + \eta_\zeta^p + \rho_{\varepsilon, \zeta}^p).$$

However, by the linear independence, this is not possible (the ρ 's cannot be cancelled).

Second, suppose

$$\begin{aligned}\sigma_i + \sigma_j &= b_0(i, j) + c_0(i, j) = \eta_\alpha^p + \eta_\alpha^p + \rho_{\alpha, \beta}^p = \rho_{\alpha, \beta}^p, \\ \sigma_j + \sigma_k &= b_0(j, k) + c_0(j, k) = \eta_\gamma^p + \eta_\delta^p + \rho_{\gamma, \delta}^p, \\ \sigma_i + \sigma_k &= b_0(i, k) + c_0(i, k) = \eta_\varepsilon^p + \eta_\zeta^p + \rho_{\varepsilon, \zeta}^p.\end{aligned}$$

Then $\rho_{\alpha, \beta}^p = (\eta_\gamma^p + \eta_\delta^p + \rho_{\gamma, \delta}^p) + (\eta_\varepsilon^p + \eta_\zeta^p + \rho_{\varepsilon, \zeta}^p)$, and this is again not possible by the linear independence.

Thirdly, the assumption that $\sigma_i + \sigma_j = \rho_{\alpha, \beta}^p$, $\sigma_j + \sigma_k = \rho_{\gamma, \delta}^p$ and $\sigma_i + \sigma_k = \eta_\varepsilon^p + \eta_\zeta^p + \rho_{\varepsilon, \zeta}^p$ leads to

$$\rho_{\alpha, \beta}^p + \rho_{\gamma, \delta}^p = \eta_\varepsilon^p + \eta_\zeta^p + \rho_{\varepsilon, \zeta}^p,$$

again a clear contradiction.

Finally, the configuration $\sigma_i + \sigma_j = \rho_{\alpha, \beta}^p$, $\sigma_j + \sigma_k = \rho_{\gamma, \delta}^p$ and $\sigma_i + \sigma_k = \rho_{\varepsilon, \zeta}^p$ is also impossible.

Using Lemma 2.2 we may find $A \subseteq \{i : i < 2^{20}\}$ such that

(\boxtimes)₁₅ (a) $|A| \geq 5$, and $\mathcal{A} = \{\sigma_i : i \in A\}$ contains a 4-arrangement (see Definition 2.1), and

(b) A is h -homogeneous in color 1, i.e., $h(i, j) = 1$ for $i < j$ from A

(remember (\boxtimes)₃ + (\boxtimes)₅). Now, (\boxtimes)₁₅(b) implies that

$$\mathcal{A} + \mathcal{A} \subseteq \{\eta_\alpha^p : \alpha \in u^p\} + \{\eta_\alpha^p : \alpha \in u^p\}.$$

Hence, by Lemma 2.3(b), there is $x \in {}^n 2$ such that $\mathcal{A} + x \subseteq \{\eta_\alpha^p : \alpha \in u^p\}$. Since $\mathcal{A} + x$ contains a 4-arrangement we may find $\alpha < \beta < \gamma$ such that $\eta_\alpha^p, \eta_\beta^p, \eta_\gamma^p \in \mathcal{A} + x$ and

$$\begin{aligned}\min\{k < n^p : \eta_\alpha^p(k) \neq \eta_\gamma^p(k)\} &= \min\{k < n^p : \eta_\beta^p(k) \neq \eta_\gamma^p(k)\} \\ &< \min\{k < n^p : \eta_\alpha^p(k) \neq \eta_\beta^p(k)\}.\end{aligned}$$

Now, $(\eta_\alpha^p + x), (\eta_\beta^p + x), (\eta_\gamma^p + x) \in \mathcal{A}$ so let $i, j, k < 2^{20}$ be such that $\eta_\alpha^p + x = \sigma_i, \eta_\beta^p + x = \sigma_j$ and $\eta_\gamma^p + x = \sigma_k$. Then

$$\eta_\alpha^p + \eta_\gamma^p = \sigma_i + \sigma_k = b_0(i, k) + c_0(i, k) = b_1(i, k) + c_1(i, k).$$

By (\boxtimes)₁₅(b) we know that $h(i, k) = 1$, so for some $x < 2$ there are distinct $\varepsilon, \zeta \in u^p$ such that $\{b_x(i, k), c_x(i, k)\} = \{\eta_\varepsilon^p, \eta_\zeta^p\}$. Then $\eta_\alpha^p + \eta_\gamma^p = \eta_\varepsilon^p + \eta_\zeta^p$ and by the linear independence of $\langle \eta_\delta^p : \delta \in u \rangle \setminus \langle \rho_{\delta, \xi}^p : \delta < \xi \wedge \delta, \xi \in u \rangle$ necessarily $\{\eta_\varepsilon^p, \eta_\zeta^p\} = \{\eta_\alpha^p, \eta_\gamma^p\}$. But now Claim 3.1.2(1) gives that $\{b_{1-x}(i, k), c_{1-x}(i, k)\} = \{\eta_\alpha^p + \rho_{\alpha, \gamma}^p, \eta_\beta^p + \rho_{\alpha, \gamma}^p\}$. Consequently, using (\boxtimes)₁₂ + (7) we get

$$\{K_\alpha^p, K_\gamma^p\} = \{\ell_x(i, k), m_x(i, k)\} \quad \text{and} \quad \{\ell_{\alpha, \gamma}^p\} = \{\ell_{1-x}(i, k), m_{1-x}(i, k)\}.$$

Therefore,

$$\{K_\alpha^p, K_\gamma^p, \ell_{\alpha, \gamma}^p\} = \{\ell_0(i, k), m_0(i, k), \ell_1(i, k), m_1(i, k)\}.$$

Similarly, considering (j, k, β, γ) instead of (i, k, α, γ) , we show that

$$\{K_\beta^p, K_\gamma^p, \ell_{\beta, \gamma}^p\} = \{\ell_0(j, k), m_0(j, k), \ell_1(j, k), m_1(j, k)\}.$$

By (\boxtimes)₁₃ we have

$$\{\ell_0(i, k), m_0(i, k), \ell_1(i, k), m_1(i, k)\} = \{\ell_0(j, k), m_0(j, k), \ell_1(j, k), m_1(j, k)\}$$

and this implies that $\{K_\alpha^p, K_\gamma^p, \ell_{\alpha, \gamma}^p\} = \{K_\beta^p, K_\gamma^p, \ell_{\beta, \gamma}^p\}$, contradicting clause (6) of the definition of \mathbb{P} . \square

\square

Theorem 3.2. (1) Assume that a Σ_2^0 set $B \subseteq {}^\omega 2$ satisfies (\spadesuit) of Theorem 3.1. Let \mathbb{Q} be a ccc forcing notion. Then $\Vdash_{\mathbb{Q}}$ “ B satisfies (\spadesuit) ”.

(2) Suppose κ is an infinite cardinal satisfying $\kappa^{\omega_0} = \kappa$. Then for some ccc forcing notion \mathbb{Q}_0 we have

$\Vdash_{\mathbb{Q}_0}$ “ $\mathfrak{c} = \kappa$ and there is a Σ_2^0 set $B \subseteq {}^\omega 2$ satisfying (\spadesuit) ”.

(3) Assume **CH**. Let \mathbb{C}_{ω_2} be the forcing notion adding ω_2 Cohen reals. Then in $\mathbf{V}^{\mathbb{C}_{\omega_2}}$ the following holds:

If $B \subseteq {}^\omega 2$ is Borel, $\langle \eta_\alpha : \alpha < \omega_2 \rangle \subseteq {}^\omega 2$ and

$$(\forall \alpha < \beta < \omega_2) (|(B + \eta_\alpha) \cap (B + \eta_\beta)| \geq 4),$$

then there is a perfect set $P \subseteq {}^\omega 2$ such that

$$(\forall x, y \in P) (|(B + x) \cap (B + y)| \geq 4).$$

(4) Assume **CH**. Then for some ccc forcing notion \mathbb{Q}_1 we have $\Vdash_{\mathbb{Q}_1}$ “ $\mathfrak{c} = \omega_2$ and there is a Σ_2^0 set $B \subseteq {}^\omega 2$ satisfying

$(\otimes)_1$ for some sequence $\langle h_\alpha : \alpha < \omega_1 \rangle$ of pairwise distinct elements of ${}^\omega 2$ we have $|(h_\alpha + B) \cap (h_\beta + B)| \geq 4$ for all $\alpha, \beta < \omega_1$, but

$(\otimes)_2$ there is **no** sequence $\langle g_\alpha : \alpha < \omega_2 \rangle$ of pairwise distinct elements of ${}^\omega 2$ such that $|(g_\alpha + B) \cap (g_\beta + B)| \geq 4$ for all $\alpha, \beta < \omega_2$ ”.

Proof. (1) For a Σ_2^0 set $B \subseteq {}^\omega 2$, the set

$$\mathcal{Z}_B = \{(x, y) \in {}^\omega 2 \times {}^\omega 2 : |(B + x) \cap (B + y)| > 2\}$$

is Σ_2^0 . One can easily see this by noting that $(x, y) \in \mathcal{Z}_B$ if and only if

$$(\exists t_0, t_1, t_2 \in B)(x + y + t_0 \in B \wedge x + y + t_1 \in B \wedge x + y + t_2 \in B \wedge |\{t_0, t_1, t_2\}| = 3)$$

and remembering that continuous images of Σ_2^0 subsets of ${}^5({}^\omega 2)$ are Σ_2^0 . Consequently, for a Σ_2^0 set $B \subseteq {}^\omega 2$, the formula “there is a perfect set $P \subseteq {}^\omega 2$ such that $P \times P \subseteq \mathcal{Z}_B$ ” is Σ_2^1 , and thus absolute for forcing extensions.

Since $A \times A \subseteq \mathcal{Z}_B$ if and only if $|(B + x) \cap (B + y)| > 2$ for $x, y \in A$, the assertion should be clear.

(2) Let \mathbb{P} be the forcing notion given by Theorem 3.1 and let \mathbb{C}_κ be the forcing notion adding κ Cohen reals (with finite conditions). It follows from (1) above (and Theorem 3.1) that the composition $\mathbb{Q}_0 = \mathbb{P} * \mathbb{C}_\kappa$ has the required property.

(3) First note that for a Borel set $B \subseteq {}^\omega 2$, the set $\mathcal{Z}_B = \{(x, y) \in {}^\omega 2 \times {}^\omega 2 : |(B + x) \cap (B + y)| > 2\}$ is Σ_1^1 . Now, by Shelah [5, Fact 1.16], after adding ω_2 Cohen reals over a model of **CH**, every analytic subset of ${}^\omega 2 \times {}^\omega 2$ (so sets \mathcal{Z}_B in particular) which contains a square of side length ω_2 contains a perfect square.

See more in Rosłanowski and Shelah [4, Section 3].

(4) Let \mathbb{P} be the forcing notion given by Theorem 3.1, let \mathbb{C}_{ω_2} be the forcing notion adding ω_2 Cohen reals, and let $\mathbb{Q}_1 = \mathbb{P} * \mathbb{C}_{\omega_2}$. In $\mathbf{V}^{\mathbb{Q}_1}$ consider the Σ_2^0 set B added by \mathbb{P} . As in (2) above, $\Vdash_{\mathbb{Q}_1}$ “ B satisfies (\spadesuit) ”, so in particular $\Vdash_{\mathbb{Q}_1}$ “ B satisfies $(\otimes)_1$ ”. Since $\mathbf{V}^{\mathbb{P}} \models \mathbf{CH}$, we may also use (3) to argue that $\Vdash_{\mathbb{Q}_1}$ “ B satisfies $(\otimes)_2$ ”. \square

Theorem 3.3. Assume **MA** + $\neg \mathbf{CH}$. Then there is a Σ_2^0 set $B \subseteq {}^\omega 2$ satisfying (\spadesuit) of Theorem 3.1.

Proof. Standard consequence of the proof of Theorem 3.1. With notation as there, let \mathcal{M} be a transitive model of a large part of ZFC and such that $\omega_1^{\mathcal{M}} = \omega_1$ and $|\mathcal{M}| = \omega_1$. (For instance, take the transitive collapse of $\mathcal{N} \prec \mathcal{H}(\chi)$, $|\mathcal{N}| = \omega_1$, $\omega_1 \subseteq \mathcal{N}$.) Then $\mathbb{P} \in \mathcal{M}$. By **MA** + \neg **CH** we may find an \mathcal{M} -generic $G \subseteq \mathbb{P}$. Consider $\mathcal{M}[G]$ which is a model of a large part of ZFC. By the proof of Theorem 3.1 there are $\langle T_m : m < \omega \rangle$ and $\langle \eta_\alpha : \alpha < \omega_1 \rangle$ belonging to $\mathcal{M}[G]$ such that

(♠) $^{\mathcal{M}[G]}$ in $\mathcal{M}[G]$,

- (a) $T_m \subseteq {}^{\omega}2$ is a tree with no terminal nodes (for every $m < \omega$) and $\eta_\alpha \in {}^{\omega}2$ (for each $\alpha < \omega_1$),
- (b) $|(\eta_\alpha + \bigcup_{m < \omega} [T_m]) \cap (h_\beta + \bigcup_{m < \omega} [T_m])| \geq 4$,
- (c) for every perfect set $P \subseteq {}^{\omega}2$ there are $f, g \in P$ with $|(f + \bigcup_{m < \omega} [T_m]) \cap (g + \bigcup_{m < \omega} [T_m])| \leq 2$.

The properties stated in (♠) $^{\mathcal{M}[G]}$ (a),(b) are clearly absolute between \mathcal{M} and \mathbf{V} . By an argument as in Theorem 3.2(1), the property stated in (♠) $^{\mathcal{M}[G]}$ (c) is Π_2^1 , so also absolute between \mathcal{M} and \mathbf{V} . \square

For the completeness of the picture we would like to know the answer to the following problem (compare this with Theorem 3.2(3)).

Problem 3.4. Is it consistent that for every Borel set $B \subseteq {}^{\omega}2$,

if there is a sequence $\langle \eta_\alpha : \alpha < \omega_1 \rangle$ of distinct elements of ${}^{\omega}2$ such that

$$(\forall \alpha < \beta < \omega_1) (|(B + \eta_\alpha) \cap (B + \eta_\beta)| \geq 4),$$

then there is a perfect set $P \subseteq {}^{\omega}2$ such that

$$(\forall x, y \in P) (|(B + x) \cap (B + y)| \geq 4) ?$$

ACKNOWLEDGMENT

We would like to thank the anonymous referee for very valuable comments and corrections.

REFERENCES

- [1] Marek Balcerzak, Andrzej Rosłanowski, and Saharon Shelah, *Ideals without ccc*, J. Symbolic Logic **63** (1998), no. 1, 128–148, DOI 10.2307/2586592. MR1610790
- [2] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded. MR1940513
- [3] Jan Mycielski, *Independent sets in topological algebras*, Fund. Math. **55** (1964), 139–147, DOI 10.4064/fm-55-2-139-147. MR0173645
- [4] Andrzej Rosłanowski and Saharon Shelah, *Borel sets without perfectly many overlapping translations*. Reports on Mathematical Logic, submitted, arxiv:1806.06283.
- [5] Saharon Shelah, *Borel sets with large squares*, Fund. Math. **159** (1999), no. 1, 1–50. MR1669643

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA AT OMAHA, OMAHA, NEBRASKA 68182-0243

Email address: arosłanowski@unomaha.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA AT OMAHA, OMAHA, NEBRASKA 68182-0243

Email address: vrykov@unomaha.edu