ON LISTS OF COVARIANTS.

BY DR. EMORY MCCLINTOCK.

I USE the term covariants to include invariants, and I write particularly concerning lists of covariants (groundforms) of the binary quintic and sextic, those of quantics of lower degree being few and well known. When the weight of a covariant is spoken of in this article, it must be understood to mean the weight of its first term or "source." The symbol 5_n will denote that covariant of the quintic whose weight is n, and 6_m that covariant of the sextic whose weight is m. Thus, for example, 5_2 and 6_2 represent the hessians (weight 2) of the quintic and sextic respectively. The only case of ambiguity is 6_{15}, for which weight there are two covariants: one of these may be denoted by 6_{15a}, the other by 6_{15b}.

The table printed on the next page exhibits the terminology of different writers. Professor Cayley's* superb collection of the covariants of the quintic, in which each is designated by a letter of the alphabet, is arranged, as will be observed, first according to the degree in the coefficients, and secondly according to the order in the variables. Thus 5_0, of the first degree, is called A, and 5_2 and 5_4, of the second degree, come next; but 5_4 being of order 2 while 5_2 is of order 6, the letter B is assigned to 5_4, and so on. The small italics contained in the column headed by the name of Dr. Salmon † are the symbols used in a table at the end of his work, illustrative of transvection, and denote the seminvariants which form the sources of covariants of the quintic and of higher quantics as well. The letters a, g, h, i, j, k, are therefore used by him also for the sextic, together with l, m, n, q, representing respectively 6_8, 6_{14}, 6_{20}, 6_{10}. Clebsch ‡ and Gordan § differ but slightly in their nomenclature. Faà de Bruno ‖ designates invariants by the letter I, with subscripts indicating degree, and other covariants by the letter C, with subscripts indicating order and degree. The column headed by the name of Professor Sylvester contains his table ¶ of *germs* for the quintic, each source having its distinguishing germ, *i. e.*, the coefficient in it of the highest power of the final coefficient of the quintic. Thus, the quintic being

---

* *Mathematical Papers*, II, 273–309; Cambridge, 1889.

† *Modern Higher Algebra*, 4th Edition.

‡ *Theorie der Binären Algebraischen Formen*, Leipzig, 1872.

§ *Invariantentheorie*, herausgegeben von Kerschensteiner, Leipzig, 1887.

‖ *Théorie des Formes Binaires*, Turin, 1876.

¶ *American Journal of Mathematics*, V, 89.

COVARIANTS OF THE QUINTIC.

By Weight.	Deg-Order.	Cayley.	Salmon.	Clebsch.	Gordan.	Faà de Bruno.	Sylvester.
5_0	1–5	A	U, a	f	f	$C_{5,1}$	a
5_2	2–6	C	H, h	H	φ	$C_{6,2}$	(c)
5_3	3–9	F	g	T	t	$C_{9,3}$	(d)
5_4	2–2	B	S, i	i	i	$C_{2,2}$	(e)
5_5	3–5	E	k			$C_{5,3}$	a^2f
5_6	3–3	D	T, j	j	j	$C_{3,3}$	(e')
5_7	4–6	I				$C_{6,4}$	$a(c)f$
5_8	4–4	H	e			$C_{4,4}$	$(d)f$
5_9	5–7	L				$C_{7,5}$	$(c)^2f$
5_{10}	4–0	G	J	A	A	I_4	a^2f^2
5_{11}	5–3	K				$C_{3,5}$	$3a(e')f-2(c)(e)f$
5_{12}	5–1	J	α	α	α	$C_{1,5}$	$a(c)f^2$
5_{13}	6–4	N				$C_{4,6}$	$a(d)f^2$
5_{14}	6–2	M	τ	τ	τ	$C_{2,6}$	$(c)^2f^2$
5_{15}	7–5	P				$C_{5,7}$	$(c)(d)f^2$
5_{17}	7–1	O		β	β	$C_{1,7}$	$a^2(c)f^3$
5_{19}	8–2	R	ϑ	ϑ	ϑ	$C_{2,8}$	$a(c)^2f^3$
5_{20}	8–0	Q	K	B	B	I_8	$(c)(d)f^3$
5_{21}	9–3	S				$C_{3,9}$	$(c)^3f^3$
5_{27}	11–1	T		γ	γ	$C_{1,11}$	$(c)^2(d)f^4$
5_{30}	12–0	U	L	C	C	I_{12}	$(c)^2\Delta f^4$
5_{32}	13–1	V		δ	δ	$C_{1,13}$	$(c)^3(e')f^4$
5_{45}	18–0	W	R, I	R	R	I_{18}	$a(c)^5f^7$

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5,$$

the germ of any covariant is the coefficient of the highest power of f appearing in its source. In the column in question,

$$\begin{aligned}(c) &= ac - b^2,\\ (d) &= a^2d - 3abc + 2b^3,\\ (e) &= ae - 4bd + 3c^2,\\ (e') &= ace - ad^2 + 2bcd - c^3 - b^2e,^*\\ \Delta &= a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd.\end{aligned}$$

This germ-theory of Professor Sylvester will doubtless lead in future to important results. We may even now make some practical use of it as an aid in reducing covariants to their simplest forms.

The collection of covariants of the quintic lately made by Professor Cayley from his past publications is not likely to be superseded for many years. It appears in that great series of volumes, not yet complete, which will endure as the noblest monument of their illustrious author. It gives each covariant in the fullest detail, with all the terms arranged in the most complete order, and with the numerical coefficients verified, in every instance, as perfectly as that mode of verification can accomplish it, by calculations printed at the foot of the columns. The covariants as published are free from any inaccuracy which I have been able to discover,† with the single exception of the one (5_6) called I. In this the third and fifth columns should each be multiplied throughout by 5, and in the second column $abcf$—10 should read ac^2e—10. Yet, perfect as this collection is, it does not profess to give, and in fact does not always give, each covariant in its simplest form. An instance in point may be seen as the result of an examination of the germs. The germ of V as printed is the coefficient of f^5, namely, in Professor Sylvester's notation, $a(c)^2(d)$. If we suppose that note has been taken, as in our column headed "Sylvester," of the germs of the preceding covariants tabulated by Professor Cayley, we see that the germ of V is the product of the respective germs of J and Q. In fact, the addition of $2\,JQ$ to V as printed would simplify it materially, both by eliminating from the "source" all terms in f^5 and otherwise. The germ of V, thus modified, is the coefficient of f^4, $(c)^3(e')$, as in the "Sylvester" column. Yet it does not follow necessarily that the simplest ground-

* Printed erroneously ad^2 in the paper cited. The germ of 5_{30} is also printed incorrectly.

† As regards the quintic. The last column of 4_2, No. 9 of the quartic, is incorrect.

form may not have a compound germ. The case of 6_{14} is an instance to the contrary.

The synoptical tables of Clebsch and Gordan do not give the covariants, but merely symbolic expressions indicating how the covariants may be computed. Let no one, however, undertake to compute covariants as directed by the symbolic analysis. The expressions resulting from the application of the Clebsch-Gordan formulæ are often highly complicated. For instance, their formula for the covariant of weight 15 gives the complicated function $86BK + 7EG - 252\phi$, and that for weight 21 gives $252\psi + 29GK - 59BO$, where ϕ means a form of 5_{15} which I think simpler than P as tabulated, and ψ means a form of 5_{21} in some respects simpler than S. Yet of course these complicated expressions are true covariants, of the right weights, degrees, and orders. I mention them merely to illustrate the necessity, for those engaged in computing and tabulating covariants, of a simple method.

I am unable to prove that the method which I prefer will in every case produce the simplest form of covariant, and it will not apply to all covariants, but I have not yet known it to fail when applied, and so I give it for what it may be worth. If we call by the name of "simple transvection" that form of transvection (*Ueberschiebung*) in which one of the two covariants concerned is the quantic itself, my plan is to produce any desired covariant, when possible, by simple transvection from the nearest available covariant of lower weight. Simple transvection increases the degree by 1 and the weight (in the case of the quintic) by from 1 to 5, and it cannot be performed when the desired increase of weight exceeds the order of the covariant operated upon. Observing these limitations, it is not difficult to pick out a succession of available operations, for instance for the quintic, by referring to the table of weights, degrees, and orders of possible independent covariants. Representing by $[n]$ the operation of simple transvection which is to increase the weight by n, we shall have, successively,

$[2]\ 5_0 = 5_2,$	$[3]\ 5_8 = 5_{11},$
$[1]\ 5_2 = 5_3,$	$[4]\ 5_8 = 5_{12},$
$[4]\ 5_0 = 5_4,$	$[1]\ 5_{12} = 5_{13},$
$[1]\ 5_4 = 5_5,$	$[3]\ 5_{11} = 5_{14},$
$[4]\ 5_2 = 5_6,$	$[1]\ 5_{14} = 5_{15},$
$[1]\ 5_6 = 5_7,$	$[4]\ 5_{13} = 5_{17},$
$[2]\ 5_6 = 5_8,$	$[4]\ 5_{15} = 5_{19},$
$[1]\ 5_8 = 5_9,$	$[5]\ 5_{15} = 5_{20},$
$[5]\ 5_5 = 5_{10},$	$[2]\ 5_{19} = 5_{21}.$

Although, as I have said, I cannot prove that simple transvection, applied to the nearest, will always produce the simplest possible result, it seems not unreasonable that this should be the case, since the operand is presumably in the simplest form, and the operation is of the simplest character. The operation just indicated for producing 5_8 yields an expression simpler than H, that for 5_{15} an expression simpler than P, that for 5_{21}, an expression simpler than S* ; the others produce the corresponding covariants tabulated by Professor Cayley, which are therefore, in all probability, the simplest attainable forms.

Another principle appears to be even more important than that of simple transvection from the nearest. It is that if for any quantic a groundform is wanted for any degree-weight for which one exists for a lower quantic, the same "source" should be employed. This principle enables us to use for 6_2, 6_3, 6_4, 6_5, 6_6, 6_7, 6_{10}, and 6_{11}, the sources of corresponding degree-weight for the quintic. Yet for some reason unknown to me this principle appears uniformly to be disregarded in the formation of 6_{10}. Even the "germ-table for the sextic" of Professor Sylvester assigns for 6_{10} a less simple form than 5_{10}. That it is less simple may be seen from an examination of the numerical coefficients:

6_{10} by Faà de Bruno's table,† ± 186, ± 330, ± 549, ± 330, ± 186
6_{10} from 5_{10}, ... ± 142, ± 168, ± 263, ± 168, ± 142
In fact, Faà de Bruno's 6_{10} is really $2 \cdot 6_6 \cdot 6_4 - 3 \cdot 6_{10}$.

Tables for the sextic are needed, as complete, correct, and well printed as those of Professor Cayley for the quintic. If any member of the Society, undeterred by the great labor which the task will involve, will undertake to compute such a set of tables for the sextic, to be published, say, in the *American Journal of Mathematics*, I shall be glad to contribute towards it my own computations of the first seventeen of the twenty-six groundforms, complete, with those of the simple forms of 5_8, 5_{15}, and 5_{21} already mentioned, which might usefully be published with the sextic tables. The utility of such printed tables consists largely in their availability for reference in case of need, and for this purpose they should be published, not singly or in small numbers as computed from time to time, but in masses. It is for this reason that I have not thought of publishing the computations just mentioned. I have made them, indeed, not intending publication, but in order to verify to the greatest extent my idea that the easiest way to find the simplest forms is, wherever practicable, to apply simple transvection as already explained.

---

* $H = 5_8 + B^2$; $P = 5_{15} - BK$; $S = \frac{1}{3}(GK - BO) - 5_{21}$.

† Corrected. As Professor Sylvester points out (*loc. cit.*), the tables printed by Faà de Bruno, useful as they are, contain many errors. The last column of this 6_{10} table is nearly all wrong, and only one column of the five is quite right.

Confining my attention to this question of simplicity, I have not even made search among mathematical journals to see what has already been done towards the computation of the more difficult covariants of the sextic, but will do so at the instance of any member of the Society willing to undertake the work of completing the series, who may not himself have access to a large library.

The class of cases to which I have referred as unsuitable for the application of simple transvection are those in which there is no groundform near enough upon which to operate. For instance, to produce by simple transvection the invariant 6_{18}, of the sixth degree in the coefficients, we should need as a basis of operation a covariant of degree 5 whose weight should not be less than 12, and whose weight and order combined should exceed 17. The only groundforms of degree 5, are, however, 6_{13} and 6_{14}, the former of order 4, the latter of order 2, and neither of them can be used to produce 6_{18}. It is of course possible in such cases to apply simple transvection to a complex covariant—as, for instance, to $6_4 . 6_8$, of order 6, for producing 6_{18}—but that will not usually produce the best results, and it is doubtless preferable to employ transvection (no longer "simple") of groundforms other than the quantic itself, in accordance with the recommendations of the text-books. Of the four text-books already cited which supply formulæ for computing the groundforms of the quintic and sextic, the formulæ collected by Salmon are apparently the best. So far as my observation has gone, the application of Salmon's formulæ has given simple results in most cases. Among the exceptions to this remark are 5_{21}, 6_{10}, 6_{14}.

After once applying simple transvection to produce 6_{15}, for which weight there are two groundforms of the same degree in the coefficients and order in the variables, we cannot again employ satisfactorily the rule of the nearest for producing the other form. Thus, [1] 6_{14} gives 6_{15a}, and we cannot again use [1] 6_{14} for producing 6_{15b}; nor can we profitably use [2] 6_{13}, perhaps because it is not only not so "near" as [1] 6_{14}, but even not so "near" as any combination of [2] 6_{13} and [1] 6_{14}. In this case the usual symbolic formula—Jacobian of 6_6 and 6_8—is the best for practical application.*

The nine groundforms of the sextic which remain to be computed or collected (if in simplest form) from other publications—6_{18}, for instance, is well known—are, as to weight, degree, and order, as follows, the weight being denoted by the subscript: 6_{18}, 6, 0; 6_{19}, 7, 4; 6_{20}, 7, 2; 6_{23}, 8, 2; 6_{25},

* I have not tested [4] 6_{11}.

9, 4; 6_{29}, 10, 2; 6_{30}, 10, 0; 6_{35}, 12, 2; 6_{45}, 15, 0. Of these nine, five may be derived by simple transvection, viz., [4] $6_{15} = 6_{19}$, [5] $6_{15} = 6_{20}$, [3] $6_{20} = 6_{23}$, [2] $6_{23} = 6_{25}$, [4] $6_{25} = 6_{29}$. I have written in two places 6_{15} for 6_{15a} or 6_{15b}, not known which is to be preferred, a matter to be settled in either case most easily by computing a few terms upon each basis. The three of higher weights, to which simple transvection will not apply, may probably be derived most simply by means of the formulæ given by Salmon.

To illustrate the process of simple transvection, which, although sufficiently implied, is not usually illustrated in the books, I give [4] $6_4 = 6_8$ in detail in the form of a table:

OPERAND.	MULTIPLIERS.		
6_4.	For (1).	For (2).	For (3).
	$\div 2$		$\div 2$
$ae - 4bd + 3c^2$	e	f	g
$2af - 6be + 4cd$	$-d$	$-e$	$-f$
$ag - 9ce + 8d^2$	c	d	e
$2bg - 6cf + 4de$	$-b$	$-c$	$-d$
$cg - 4df + 3e^2$	a	b	c

$$\left.\begin{array}{l}(1) = acg - 3adf + 2ae^2 - b^2g + 3bcf - bde - 3c^2e + 2cd^2 \\ (2) = -bcg - 8bdf + 9be^2 + 9c^2f - 17cde + adg + 8d^3 - aef \\ (3) = aeg - 3bdg + 2c^2g - af^2 + 3bef - cdf - 3ce^2 + 2d^2e\end{array}\right\} 6_8$$

The multipliers in this instance are extremely simple. The coefficient of a is always 1, as in this case, but in general those of the other multipliers are other integers. The rule which I find best for determining the integers forming the coefficients of the multipliers for simple transvection is given in another paper, as a special case of a broader rule for transvection in general. The paper in question, "On the Computation of Covariants by Transvection," to be read before the Society on January 2, 1892, will be printed elsewhere, the pages of the *Bulletin* being intended rather for critical and historical notes than for original investigations.