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EVOLUTION OF CRITERIA OF CONVERGENCE.

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THE expressions convergent and divergent series were used for the first time in 1668 by James Gregory. Newton and Leibniz felt the necessity of inquiring into the convergence of infinite series, but they had no proper criteria, excepting a test advanced by Leibniz for alternating series. By Euler and his contemporaries the *formal* treatment of series was greatly extended, while the necessity for determining the convergence was generally lost sight of. To be sure, it was Euler who first observed the semi-convergence of a series. He, moreover, remarked that great care should be exercised in the summation of divergent series. But this warning was not taken so seriously by him as it would be by a modern writer, for in the very same article* in which it occurs Euler did not hesitate to write

$$\ldots + \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \ldots = 0,$$

simply because

$$n + n^2 + \ldots. = \frac{n}{1-n};\ 1 + \frac{1}{n} + \frac{1}{n^2} + \ldots = \frac{n}{n-1}.$$

The facts are that Euler reached some very pretty results in infinite series, now well known, and also some very absurd results, now quite forgotten. Protests were made by Nicolaus Bernoulli and Varignon against the prevailing reckless use of series: isolated attempts at establishing criteria of convergence are on record: but the dominating sentiment of the age frowned down any proposition which would put limitations upon operations with series. The faults of this period found their culmination in the Combinatorial School in Germany, which has now passed into deserved oblivion.

I. *Special Criteria.* In the progress of mathematics, truth

* *Comm. Petrop.*, vol. 11, p. 116.

and error cannot go together long. Doubtful or plainly absurd results obtained from infinite series stimulated profounder inquiries into the validity of operations with them. Their *actual contents* came to be the primary, *form* a secondary consideration. The first important and strictly rigorous investigation of series was made by Gauss,* when he discussed the series designated by $F(\alpha, \beta, \gamma, x)$. He investigates the ratio of two successive terms, compares series with geometric progressions, and deduces a general criterion for series of positive terms, the ratio of successive terms of which can be expressed in a certain form. The deduction of this criterion is laborious, but it settles the question of convergence in every case which it is intended to cover, and thus bears the stamp of generality, so characteristic of Gauss's writings. Owing to the strangeness of treatment and unusual rigor, Gauss's paper excited little interest among the mathematicians of that time.

More fortunate in reaching the public was Baron Cauchy whose *Analyse Algébrique* of 1821 contains a rigorous treatment of series. All series whose sum does not approach a fixed limit as the number of terms increases indefinitely, are called divergent. Like Gauss, he institutes comparisons with geometric series and finds that series with positive terms are convergent or not, according as $(u_n)^{\frac{1}{n}}$ or u_{n+1}/u_n is ultimately less or greater than unity. To reach cases where these expressions become unity and fail, Cauchy proved (1) that if in $u_0 + u_1 + \ldots$ each term is smaller than its preceding, it is always convergent when $u_0 + 2u_1 + 4u_3 + 8u_7 + \ldots$ is convergent, and only then; (2) that if $Lu_n / L\frac{1}{n}$ † converges toward a finite limit greater than one, the series is convergent, but divergent when that limit is less than one. He shows that series with partly negative terms are convergent when the absolute values of the terms converge, and then deduces Leibniz's test for alternating series.

The most outspoken critic of the old methods in series was Abel. His letter to his friend Holmboe (June, 1826) contains severe criticisms. It is very interesting reading, even to modern students. Abel also pointed out in *Crelle's Journal*, vol. 3, the error in an article by Olivier, who pretended to have found the following extremely simple, general criteria for a series with positive terms : Σu_n diverges if the limit of nu_n is not zero and converges if it equals zero. Abel showed that the second part of this is incorrect. Among the posthumous papers of Abel ‡ it is demonstrated that a series having

* "Disquisitiones generales circa seriem infinitam". . , *Werke*, vol. 3.
† L denotes characteristic of logarithm in any system.
‡ *Oeuvres*, vol. 2, p. 197.

the general term $\frac{1}{n \lg n \lg^2 n \ldots \lg^{m-1} n}$ is divergent,* but one having the general term $\frac{1}{n \lg n \lg^2 n \ldots (\lg^m n)^{1+\alpha}}$ converges if $\alpha > 0$. In the same paper he deduces logarithmic tests in anticipation of De Morgan and Bertrand.

The researches of Cauchy and Abel caused a considerable stir. We are told that after a scientific meeting in which Cauchy had presented his first researches on series, Laplace hastened home and remained there in seclusion until he had examined the series in his *Mécanique céleste*. Luckily, every one was found to be convergent! We must not conclude, however, that the new ideas at once displaced the old. On the contrary, the new views were generally accepted only after a severe and long struggle. As late as 1844 De Morgan began a paper on "Divergent Series" in this style: "I believe it will be generally admitted that the heading of this paper describes the only subject yet remaining, of an elementary character, on which a serious schism exists among mathematicians as to absolute correctness or incorrectness of results."† Some mathematicians, for instance Poisson, promptly rejected infinitely diverging series, but seemed to employ with confidence finitely diverging series; they appeared content to equate $\frac{1}{2}$ to $1 - 1 + 1 - \ldots$, if this series be regarded as the limit of a convergent series $1 - g + g^2 - \ldots$ Difficult questions of this nature arose in the study of certain trigonometric series, particularly that of Fourier, upon which much light was thrown by the researches of Dirichlet.

First in time, in the evolution of more delicate criteria of convergence and divergence come the researches of Raabe‡ who starts out with the following theorem previously given by Cauchy in 1827, and subsequently used by other writers: "If $\varphi(x)$ be a function which becomes zero when x increases indefinitely and which has for all values of x beween a and ∞, always finite values of the same sign, then $\Sigma^n \varphi(a+n)$ is convergent or divergent, according as $\int_a^\infty \varphi(x)\,dx$ is finite or infinite." Raabe then shows that the series with the general term $e^{-x(1+\frac{1}{2}+\cdots+\frac{1}{n})}$ is convergent when $x > 1$ and divergent when $x \overline{\lessgtr} 1$; and thence deduces the theorem that a series $\Sigma\, u_n$ is convergent when, in the expression $\lim \left(\frac{u_{n+1}}{u_n}\right)^n = e^{-x}$ or in the equivalent expression (a test bearing the

* The index is here used as in the calculus of operations, *e.g.*, $\lg^2 n \equiv \lg \lg n$.

† *Trans. Cambr. Philos. Soc.*, vol. 8, pt. ii.

‡ *Zeitschrift für Physik und Math.*, vol. 10 (1832).

name of the author) $\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=x$, $x>1$, and divergent when $x<1$, the case $x=1$ remaining undetermined.

We come now to the criteria of De Morgan and Bertrand which have been unsurpassed for practical adaptability to many series whose ratio of convergence is ultimately unity. De Morgan established the logarithmic scale of functional dimensions which makes $x^a\,(\lg x)^b$, b being positive, of a higher dimension than x^a and of a lower dimension than x^{a+k}, however small k may be and however great b may be. Between x^a and x^{a+k} may be found an infinite number of functions higher in dimension than the first and lower than the second. Built on this idea is De Morgan's test,* the series being $\frac{1}{\varphi(a)}+\frac{1}{\varphi(a+1)}+\frac{1}{\varphi(a+2)}+\ldots$ "First examine $P_0=x\,\varphi'(x)\,/\,\varphi(x)$, when x is infinite. If, then, a_0, the limit of P_0, be >1, the series is convergent; if <1, divergent. But if $a_0=1$, find a_1, the limit of P_1, or $\lambda x\,(P_0-a_0)$; then if $a_1>1$ the series is convergent, if <1, divergent. But if $a_1=1$, find a_2, the limit of P_2 or $\lambda^2 x\,(P_1-a_1)$; then if $a_2>1$, the series is convergent, if <1 divergent. But if $a_2=1$, examine P_3, etc., etc.," [$\lambda^2 x\equiv\lg(\lg x)$]. De Morgan says that "if a function could be shown for which a_0, a_1, etc. *ad inf.* are severally $=1$, this criterion does not determine whether the series is convergent or divergent." This statement left it doubtful whether such functions could be conceived or not. Ossian Bonnet † remarked that De Morgan's test never fails, except when the number of logarithms grows to infinity, "a case which is to some degree the point of junction of convergent series and of divergent series." Bonnet's view, if correct, would make De Morgan's test an absolute criterion for all series of positive terms, but Du Bois Reymond ‡ has shown that there is a region of convergence where the logarithmic criteria completely fail and has actually constructed a demonstrably convergent series for which the logarithmic criteria fail. A. Pringsheim § illustrates the same thing by the following comparatively simple series,

$$S=\Sigma u_n=\sum_1^{\infty}{}_{\nu}\left\{\frac{\varepsilon_0^{(\nu)}}{\nu^r a^{\nu^{r+s}}}+\frac{\varepsilon_1^{(\nu)}}{\nu^r a^{\nu^{r+s}}}+\ldots+\frac{\varepsilon_\mu^{(\nu)}}{\nu^r a^{\nu^{r+s}}}\right\},$$

in which $a>1$, $r>1$, $s>0$, $\mu=$ the largest integral number contained in $a^{\nu^{r+s}}$. Suppose, moreover, that either

* De Morgan's Calculus, p. 326.
† *Liouville's Journal*, vol. 8 (1843).
‡ *Crelle's Journal*, vol. 76.
§ *Mathematische Annalen*, vol. 35 (1889).

$\varepsilon_0^{(\nu)} = \varepsilon_1^{(\nu)} = \ldots = \varepsilon_\mu^{(\nu)} = 1$, or $1 + h_\nu = \varepsilon_0^{(\nu)} > \varepsilon_1^{(\nu)} > \ldots > \varepsilon_\mu^{(\nu)} = 1$, h_ν being selected positive and so as to make the last term in each group of terms of the series larger than the first term in the next following group. This series is convergent, for

$$S < \sum_1^\infty \frac{(1+h)(\mu+1)}{\nu^r a^{\nu^{r+s}}} < (1+h)\left\{\sum_1^\infty{}_\nu \frac{1}{\nu^r} + \sum_1^\infty{}_\nu \frac{1}{\nu^r a^{\nu^{r+s}}}\right\},$$

the two series in the brace being evidently convergent. Let us now apply Bonnet's form of the logarithmic criteria, which says that a series is convergent if $L_\kappa(n)\lg_\kappa^\rho n \, . \, a_{n+p}$, where $L^\kappa(n) \equiv \lg_0 n \, . \lg_1 n \, . \, \ldots \, . \lg^\kappa n$, approaches a limit less than some positive finite quantity.* Let $u_n = \frac{\varepsilon_\mu^{(\nu)}}{\nu^r a^{\nu^{r+s}}}$ be the last term in any group (except the first). Then $n > \mu + 1 > a^{\nu^{r+s}}$. Hence $n \lg n \, . \, u_n > a^{\nu^{r+s}} . \lg a^{\nu^{r+s}} . \frac{1}{\nu^r a^{\nu^{r+s}}}$ *i.e.* $> \lg a \, . \, \nu^s$, which increases indefinitely with ν. Thus, even though the terms u_n never increase, we have among the values of lim $(n \lg n . u_n)$ the value ∞; all the more will lim $\{L_\kappa(n)\lg_\kappa^\rho n \, . \, u_n\}$ have ∞ values, and the expression does *not* approach a limit less than some finite quantity. Thus the logarithmic tests *may* fail to indicate convergence even where the convergence can be easily established by other means. If in the above series we make $r = 1$, the resulting series is easily seen to be divergent. But the logarithmic expression lim $\{L_\kappa(n) \, . \, u_n\}$, which indicates divergence if it is greater than some positive quantity, becomes here zero if the first term of a group be taken for u_n. Thus we have divergent series to which logarithmic criteria are inapplicable.

Some of De Morgan's results were reached independently by Bertrand. De Morgan's criteria were expressed in more convenient form by him and by Bonnet. His memoir † contains a discussion of various forms of criteria and a proof of the equivalence of his tests with De Morgan's, and that of Raabe as generalized by himself.

II. *General Criteria.* The treatment of the question of convergence from a still wider point of view, culminating in a regular mathematical theory, was begun by Kummer and continued by Dini, Du Bois Reymond, Kohn, and Pringsheim. The tests thus far given are called by Pringsheim *special*

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* The inferior index here indicates the order of the functional operation, while the superior index is the power exponent, *e. g.* $\lg_2^3(n) \equiv (\lg\lg n)^3$.

† *Liouville's Journal*, vol. 7.

criteria, because they all depend upon a comparison of the n^{th} term of the series with special functions, a^n, n^κ, $n(\lg n)^\kappa$, etc. Kummer's article * antedates the papers of De Morgan and Bertrand. He establishes the theorem that the series Σu_p is convergent if a function $\varphi(p)$ can be found such that $\lim \varphi(p) \cdot u_p = 0$ and $\lim \left\{ \varphi(p) \frac{u_p}{u_{p+1}} - \varphi(p+1) \right\}$ is not equal to zero. Raabe's test can be deduced from this. Du Bois Reymond divides all criteria into two classes: criteria of the *first kind* and criteria of the *second kind*, according as the general term u_p or the ratio of u_{p+1} and u_p is made the basis of research. Kummer's looks like a mixed criterion, but it is really of the second kind. The true significance of the second part of that criterion was at first overlooked. Bertrand says in his *Calcul différentiel* (vol. 1, p. 244) that the great indeterminateness in the mode of applying it is an advantage to those who know how to profit thereby. But Du Bois Reymond † points out that there is nothing indeterminate, that in the selection of the function φ we have practically the choice only between the quantities $L_\kappa(n)$ arising out of the criteria of the second kind, and if criteria of the second kind were discovered, affording more delicate tests than do the logarithmic ones, such new criteria would be embraced by the second part of Kummer's criterion. $\operatorname{Lim} \left\{ \varphi(p) \frac{u_p}{u_{p+1}} - \varphi(p+1) \right\}$ is therefore a necessary general form of all criteria of the second kind which renders an investigation of the first part of Kummer's criterion, viz. $\lim u_p \cdot \varphi(p)$, superfluous. The above criterion, bereft of the first part, was invented anew, over half a century after its first publication, by Jensen‡, who was unaware of Kummer's researches.

Dini generalized Kummer's result, but his paper § (known to the writer only through the remarks upon it made by Pringsheim) remained unnoticed, and the same ground was traversed independently six years later by Du Bois Reymond, though with somewhat greater thoroughness. The investigations of both rest upon a comparison of the terms u_n of a series with the expression $\psi(n) - \psi(n+1)$, where $\psi(n)$ invariably either increases or decreases with n. Du Bois Reymond makes the following general statement, from which the useful criteria of the first kind can be deduced as special cases: If the terms u_n of any series of positive terms be brought to the form

$$\frac{1}{\lambda n} \left\{ \psi(n) - \psi(n+1) \right\},$$

* *Crelle*, vol. 13 (1835).

† *Crelle*, vol. 76 (1873).

‡ *Comptes Rendus*, vol. 106 (1888).

§ *Annali dell' Univ. Tosc.*, vol. 9.

then the given series converges whenever limit $\psi(n)$ remains finite, and limit λn does not become zero; the series diverges whenever limit $\psi(n)$ becomes infinite and limit λn does not become infinite. This formula is outwardly the same for divergence as for convergence, but in reality it differs considerably, since $\psi(n)$ invariably increases for divergence and decreases for convergence. Criteria for continually weaker degrees of convergence or divergence are obtained by substituting for $\psi(n)$ a succession of functions continually decreasing or increasing more and more slowly. By selecting for such a succession

$$e^{\mp \alpha n},\ n^{\mp \alpha},\ (\lg n)^{\mp \alpha},\ \ldots\ (\lg_r n)^{\mp \alpha},$$

where α is a positive and finite number, the logarithmic criteria are obtained. By putting $\psi(n) = u_n \varphi(n)$, where $\varphi(n)$ is positive, Du Bois Reymond's general criteria of the second kind are obtained: if

$$\lim_{n=\infty} \left\{ \varphi(n) - \frac{u_{n+1}}{u_n} \varphi(n+1) \right\} > 0,$$

then Σu_n is convergent. But it is divergent if this limit is negative and $\varphi(n)$ is selected so as to render $\Sigma \frac{1}{\varphi(n)}$ divergent. The researches of Du Bois Reymond were continued by G. Kohn,* who showed how from any convergent (divergent) series a new series can be obtained which converges (diverges) more rapidly or less rapidly than the original series, according to the nature of the function introduced in the process. He thus arrives at a new criterion.

A remarkable advance in the general theory of convergence and divergence was made by A. Pringsheim † in an article of 100 pages which establishes a simple, more coherent, and more complete general theory. He criticises Du Bois Reymond's theory because the convergent criteria in it are heterogeneous in nature to the divergent criteria; because it does not disclose the existence of general disjunctive criteria in which the decision as to convergence or divergence can be reached from the examination of one and the same expression; because the general criteria of the second kind do not flow naturally from those of the first kind. Pringsheim endeavors to steer clear of these objections in his own theory, of which what follows is a very meagre outline: Let Σa_ν be a series of positive terms, then the simplest types of criteria of the first kind may be

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* *Grunert's Archiv*, vol. 67 (1882).
† *Mathematische Annalen*, vol. 35 (1889).

expressed in several equivalent forms of which the following are two.

$$\Sigma a_\nu \begin{cases} \text{diverges if } a_{\nu+p} \geqq h \,.\, d_\nu, \text{ or if } \lim \dfrac{a_{n+p}}{d_n} \geqq g > h \\ \text{converges if } a_{\nu+p} \leqq H \,.\, c_\nu, \text{ or if } \lim \dfrac{a_{n+p}}{c_n} \leqq G < H \end{cases}$$

where $n = \infty$, h and H are finite positive quantities, $d_\nu = D_\nu^{-1}$ is the general term of a divergent series, $c_\nu = C_\nu^{-1}$ is the general term of a convergent series and p represents any constant positive integer. After proving that the general term of any divergent series Σa^ν, can be expressed in the form $\dfrac{M_{\nu+1} - M_\nu}{M_\nu}$, and that the term of any convergent series Σa_ν can be expressed in the form $\dfrac{M_{\nu+1} - M_\nu}{M_{\nu+1} \,.\, M^\nu}$, and where M_ν is positive and finite for finite values of ν, and invariably increases with ν, being ∞ when ν is ∞, he writes the above general criterion thus,

$$\Sigma a_\nu \begin{cases} \text{diverges if } \lim \dfrac{M_n}{M_{n+1} - M_n} \,.\, a_{n+p} \geqq g > 0 \\ \text{converges if } \lim \dfrac{M_{n+1}\, M_n}{M_{n+1} - M_n} \,.\, a_{n+p} \leqq G < \infty \end{cases}$$

Putting $M_\nu = \nu$, and replacing ν successively by $\lg \nu$, $\lg_2 \nu, .. \lg^\kappa \nu$, he deduces Bertrand's and Bonnet's forms of logarithmic criteria and then reaches a remarkable generalization constituting a new general criterion of the first kind, analogous to Kummer's criterion of the second kind, viz., the series Σa_ν is always convergent if a positive number $\varphi(\nu)$ exists for which

$$\lim \frac{\lg \dfrac{1}{\varphi(n)\, a_{n+r}}}{\sum\limits_0^n \nu \; \dfrac{1}{\varphi(\nu)}} > 0$$

The other special criteria are also deduced from the general form. The general type of criteria of the second kind is

$$\Sigma a_\nu \begin{cases} \text{diverges if } \lim P_n \left(D_n a_{n+p} - D_{n+1} a_{n+p+1}\right) < 0 \\ \text{converges if } \lim P_n \left(C_n a_{n+p} - C_{n+1} a_{n+p+1}\right) > 0 \end{cases}$$

where P_n may be any positive factor. If we take

$P_n = \frac{1}{a_{n+p+1}}$, the criteria become

$$\Sigma a_\nu \begin{cases} \text{diverges if } \lim \left(D_n \dfrac{a_{n+p}}{a_{n+p+1}} - D_{n+1} \right) < 0 \\ \text{converges if } \lim \left(C_n \dfrac{a_{n+p}}{a_{n+p+1}} - C_{n+1} \right) > 0 \end{cases}$$

From this Kummer's and the special criteria of the second kind are deduced, including those of Gauss.

As regards the scope of criteria of the first and of the second kind, we find that the first will always be decisive whenever the second are, but the second may fail ever so often when the first do not. Though more limited, the criteria of the second kind are nevertheless of value, for they often yield results more easily and quickly. This narrower range of application is due to the fact first clearly pointed out by Pringsheim that the terms a_ν of a series may lie always above or always below the corresponding terms b_ν of another series and yet there may be no fixed relation whatever between the ratios $\frac{a_{\nu+1}}{a_\nu}$ and $\frac{b_{\nu+1}}{b_\nu}$. Since in a series of positive terms the order in which the terms come has nothing to do with the convergence or divergence of the series, it is clear that $\frac{a_{n+1}}{a_n}$ does not usually approach a limit. Thus the case $\frac{a_{\nu+1}}{a_\nu} >$ or $< \frac{b_{\nu+1}}{b_\nu}$ is only a very special one, and the probability that the series Σa_ν, yielding to the tests of the first kind, yield also to those of the second kind, is very small.

The range of special tests of the first kind has been partly considered in connection with logarithmic criteria. Pringsheim points out that the logarithmic criteria have been much overvalued, that they are applicable only to series whose terms are *essentially* decreasing, so that the increase or decrease in the values of terms does not exceed certain limits. The terms must be always above or always below the corresponding terms in $\Sigma \frac{g}{L_\kappa(\nu)}$ or $\Sigma \frac{G}{L_\kappa(\nu) \lg_\kappa^\rho(\nu)}$. But this property depends again upon the order of the terms. Suppose a certain order is favorable for the use of these tests, then a promiscuous displacement of terms may render the logarithmic scale or any other scale wholly inapplicable, and all this without altering the convergence or divergence, or even the sum of the series. This failure is in no way due to the *form* of the *general* criteria, for Pringsheim shows that for every Σa_ν, *practical* criteria of the first and second kind do exist; but his proof of this fact yields no method of finding them, except when he knows beforehand the very thing to be determined, namely, whether Σa_ν be convergent or not!

In addition to the criteria of the first kind and second kind, Pringsheim establishes an entirely new criterion of a *third kind* and also *generalized criteria of the second kind*, which apply, however, only to series with never increasing terms. Those of the third kind rest mainly upon the consideration of the limit of the difference, either of consecutive terms or of their reciprocals. In the generalized criteria of the second kind he does not consider the ratio $\frac{a_{\nu+1}}{a_\nu}$ of two consecutive terms, but the ratio of any two terms, however far apart, and deduces, among others, two criteria previously given by Kohn * and Ermakoff,† respectively.

COLORADO COLLEGE, *April* 12, 1892.

NOTE ON AN ERROR IN BALL'S HISTORY OF MATHEMATICS.

BY DR. ARTEMAS MARTIN.

I DESIRE to call attention to what seems to me to be an error in Ball's "Short History of Mathematics," page 102, concluding clause of last paragraph, where the author ascribes to Diophantus the statement "that the sum of three square integers can never be expressed as the sum of two squares."

That the above statement is not in accordance with the facts is evident, since

$$(q^2+r^2-s^2-u^2)^2+(2qu)^2+(2ru)^2=(q^2+r^2-s^2+u^2)^2+(2su)^2$$

identically, no matter what values be assigned to q, r, s, u.

If we take $q = 1$, $r = 2$, $s = 3$, $u = 4$, then, after dividing all the numbers by 4^2, we have

$$2^2 + 4^2 + 5^2 = 3^2 + 6^2 = 45.$$

Let $q = 1$, $r = 2$, $s = 4$, $u = 3$, and we find, after dividing by 2^2,

$$3^2 + 6^2 + 10^2 = 1^2 + 12^2 = 145, = 8^2 + 9^2.$$

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* *Grunert's Archiv*, vol. 67, pp. 63–95.

† *Darboux's Bulletin*, vol. 2, p. 250; vol. 17, p. 142.