PICARD'S TRAITÉ D'ANALYSE.

Traité d'Analyse. By ÉMILE PICARD. Vol. I., 1891, pp. XII. + 457; Vol. II., 1893, pp. XIV. + 512. Paris, Gauthier-Villars.

ONE of the ablest of American mathematicians said to the writer not long ago, "we have waited fifty years for this book"! While neither the speaker's age nor the state of mathematical analysis would warrant one in taking this statement literally, it nevertheless expressed a feeling which must have been experienced during recent years by every student of analysis. A great many treatises having the above title, or one differing but little from it, have appeared, particularly in France and Germany, during the last twenty-five years or thereabouts, many of them good, some of them excellent—as for example Jordan's "Cours d'Analyse," while some had perhaps no really good reason for existence. None of these treatises though, however valuable they may have been, have filled or even attempted to fill the place which will be occupied by Picard's "Traité d'Analyse."

The necessity for a treatise on analysis which should present the subject from the modern point of view has for several years been most obvious. The extraordinary developments in the theory of functions, in differential equations, and in certain purely algebraical theories, and the important applications of the results of these developments to geometrical, physical, and astronomical problems, have made such a treatise almost indispensable. The difficulties caused by the magnitude and complexity of the subject might well deter any one from undertaking to give an account of it, so that any mathematician, whatever his merit, would probably be thanked if he had made a fair attempt in that direction. When, however, such a mathematician as M. Émile Picard undertakes the task more than ordinary gratitude is due. M. Picard is one of the very first analysts of the age, both as an original investigator and in virtue of the vast range of his knowledge; joined to these claims to be considered the proper person to write the Traité d'Analyse of to-day, he possesses a most remarkably clear and elegant style of presenting a subject, whether it be in the form of a memoir embodying the results of his own personal researches or in the form of a lecture or chapter containing an account of the researches of others. The elegance and conciseness of M. Picard are not, however, at the expense of rigor, as every reader of his work is aware, and as every student who has had the pleasure of listening to his lectures can testify.

The first volume of M. Picard's work is in a measure an introductory volume, and is principally concerned with the development of comparatively elementary subjects, though some of the subjects treated and the general method of development would hardly find place in a treatise for beginners. There is in many places such a close relation between the subjects treated in vol. I. and vol. II., that it was thought better to notice these two volumes together. Vol. I. is divided into three parts: Part First, Simple and multiple integrals; Part Second, Laplace's equation and its applications; developments in series; Part Third, Geometrical applications of the infinitesimal calculus. Vol. II. contains seventeen chapters. Fifteen deal with functions of a complex variable, further developments in series, conform representation, Dirichlet's problem, roots common to two simultaneous equations (also discussed in vol. I.), integrals of non-uniform functions, algebraic functions of one variable, Riemann's surfaces, and Abelian integrals. There is one chapter on general theorems concerning differential equations, and one on the applications of these theorems. A word is necessàry as to the author's primary intention and the reasons which caused him to modify it. In the introduction to vol. I. he says: "En publiant ce *Traité d'Analyse*, j'ai pour but principal de développer la partie de mon cours de la Faculté des Sciences, relative à la théorie des équations différentielles. Cet ouvrage sera donc surtout un traité général sur la théorie des équation différentielles à une ou plusieurs variables. Je n'ai cependant pas cru devoir adopter ce dernier titre, et cela pour deux raisons." The first of these reasons concerns principally M. Picard's students, and need not be cited. "Un autre motif, d'un caractère tout scientifique, m'engageait encore à garder le titre un peu vague de *Traité d'Analyse;* c'est que la théorie des équations différentielles est intimement liée a plus d'une autre théorie qu'il nous faudra approfondir. Pour ne citer qu'un exemple, l'étude préliminaire des fonctions algébriques est indispensable, quand on veut s'occuper de certaines classes d'équations différentielles. Nous ne nous bornerons donc pas strictement à l'étude des équations différentielles; nous rayonnerons autour de ce centre." In the introduction to vol. II. we find the following statement: "J'avais annoncé dans le premier volume que je comptais m'occuper surtout dans ce Traité de la théorie des équations différentielles. On trouvera ici un seul chapitre consacré à cette théorie telle qu'on l'entend ordinairement dans les ouvrages classiques. Je pourrais prétexter que l'équation de Laplace est une équation différentielle; j'aime mieux avouer que mon plan s'est un peu élargi. Je m'occuperai particulièrement, dans le tome III., de l'étude des équations différentielles, mais je n'oserais pas affirmer cependant que je n'aurais pas encore plusieurs parenthèses à ouvrir." As an indication of a most important subject to be treated later, the writer may perhaps be permitted to quote a line from a private letter: ". . . Les fonctions fuchsiennes, hyper-fuchsiennes etc. ne seront pas oubliées dans la rédaction de mon traité." That is certainly pleasant news for the student of those functions.

Chapter I. of vol. I. is concerned with definite integrals. The opening words of the chapter may be quoted: "Le calcul intégral a pris naissance le jour où l'on s'est posé la question suivante: une fonction $f(x)$ étant donné existe-t-il une fonction qui admette $f(x)$ pour derivée, c'est-à-dire une fonction telle que l'on ait

$$\frac{dy}{dx}=f(x).$$

On a d'abord répondu à cette question par une représentation géométrique qui n'a aucune valeur par elle-même, mais qui n'en a pas moins fait faire de grands progrès à la science. On construisit la courbe $y=f(x)$, et l'on considérait l'aire comprise entre cette courbe, l'axe des x et deux parallèles à l'axe des y, l'une fixe, l'autre variable; on montrait que l'aire, considérée comme fonction de l'abscisse x de cette dernière ordonnée, est une fonction de x ayant $f(x)$ pour dérivée. Il est clair qu'à moins d'admettre que la notion d'aire est une notion première, il n'y a pas là une réponse rigoureuse au problème posé."

The author proceeds now to give a precise meaning to the notion of a definite integral in the case when the function to be integrated, $f(x)$, is continuous between the limits within which the variable x is restrained to lie. Admitting that there exists a function y satisfying the equation

$$\frac{dy}{dx}=f(x),$$

and taking the value y_0 for $x=a$ and the value Y for $x=b$; divide the interval (a, b) into n intervals given by the values $x_1, x_2 \ldots x_{n-1}$, and let $y_1, y_2 \ldots y_{n-1}$ be the corresponding values of y. If the interval x_1-a is small enough, the quotient

$$\frac{y_1-y_0}{x_1-a}$$

differs but little from $f(a)$, and we can write the approximate relations

$$y_1 - y_0 = (x_1 - a)f(a);$$

$$y_2 - y_1 = (x_2 - x_1)f(x_1);$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Y - y_{n-1} = (b - x_{n-1})f(x_{n-1}).$$

Adding these, we get

$$Y - y_0 = (x_1 - a)f(a) + (x_2 - x_1)f(x_1) + \ldots + (b - x_{n-1})f(x_{n-1});$$

an approximate equality, which will be more and more accurate as the number of intervals is increased, each of them becoming smaller and smaller. A preliminary lemma is now established where by the *oscillation* of a continuous function in an interval is meant the difference between the greatest and the least values which it takes in the interval. The lemma is as follows: Suppose an interval (a, b) where to fix the ideas $a < b$ and a function $f(x)$ continuous in this interval. Having given a positive number ϵ as small as we please, *we can always find a positive quantity δ, such that in every interval contained in (a, b) and less than δ the oscillation of the function shall be less than ϵ.*

This lemma established, the author proceeds to the demonstration of the following fundamental theorem. *The sum*

$$(x_1 - a)f(a) + \ldots + (b - x_{n-1})f(x_{n-1})$$

tends towards a limit, when all the intervals (x_{i+1}, x_i) tend towards zero according to any law whatever at the same time that their number increases indefinitely. Following the establishment of this theorem come some geometrical applications to the areas and lengths of arcs of curves, and then the notion of integration by parts is introduced. Under this head Picard gives an important formula of Kronecker's. Let $f(x)$, $g(x)$ be any two functions of x, and let as usual $f^{(n)}(x)$ denote the nth derivative of $f(x)$, and let $g^{(n)}(-x)$ denote the nth derivative of $g(x)$ when after differentiation x has been replaced by $-x$. Start from the identity

$$f^{(h)}(x)g^{(n-h)}(-x) - f^{(h-1)}(x)g^{(n-h+1)}(-x)$$
$$= \frac{d}{dx}[f^{(h-1)}(x)g^{(n-h)}(-x)].$$

Making successively $h = 1, 2, \ldots, n$, and adding, we have

$$f^{(n)}(x)g(-x) - f(x)g^{(n)}(-x)$$
$$= \sum_{h=1}^{h=n} \frac{d}{dx}[f^{(h-1)}(x)g^{(n-h)}(-x)];$$

integrate now between a and b, and we have

$$\int_a^b f^{(n)}(x)g(-x)dx - \int_a^b f(x)g^{(n)}(-x)dx$$
$$= \sum_{h=1}^{h=n}\Big[f^{(h-1)}(x)g^{(n-h)}(-x)\Big]_a^b. \qquad (\alpha)$$

The calculation of the second integral is thus conducted to that of the first. This formula of Kronecker's was given in 1884, and has not before appeared in any treatise on analysis, so far as the writer is aware. It is capable of many interesting applications. Picard gives two applications. First writing

$$f(x) = F'(x), \qquad g(x) = \frac{(x+b)^n}{1 \cdot 2 \ldots n},$$

we get

$$F(b) = F(a) + (b-a)F'(a) + \ldots$$
$$+ \frac{(b-a)^n}{1 \cdot 2 \ldots n}F^{(n)}(a) + \int_a^b F^{(n+1)}(x)\frac{(b-x)^n}{1 \cdot 2 \ldots n}dx,$$

which is Taylor's theorem. The remainder here presents itself in the form of a definite integral, but is readily changed into the ordinary form. Again, writing

$$g(x) = (x+a)^n(x+b)^n,$$

and letting $f(x)$ denote an arbitrary polynomial of degree $n-1$, we find from Kronecker's formula (α)

$$\int_a^b f(x)P_n(x)dx = 0,$$

where

$$P_n(x) = A\frac{d^n(x-a)^n(x-b)^n}{dx^n}, \qquad (\beta)$$

A denoting a constant. The relation (β) is shown to define completely the polynomial P_n to a constant factor *près*—this polynomial is, of course, Legendre's polynomial—i.e., a zonal spherical harmonic.

The case of the change of variable in the integral is now considered, and then the notion of a definite integral is extended to the case where one limit becomes infinite. Here Cauchy's rule relative to the convergence of series furnishes a natural and interesting application, which, however, need not be quoted. The conditions under which we may differentiate under the sign $\int$ are now carefully considered, and as a consequence of this method of calculating an integral the formula

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

is found. This chapter closes by an extension of the notion of a definite integral to the case of complex functions of a real variable. If $F(x)$ is a real function of x, and if $\psi(x)$ is a positive function between a and b, we know that

$$\int_a^b F(x)\psi(x)dx = F(\xi)\int_a^b \psi(x)dx,$$

where ξ is a value of x between a and b. Darboux has given a formula analogous to this in the case where the function $F(x)$ is of the form $f(x) + i\phi(x)$, x being a real variable. Write

$$I = \int_a^b F(x)\psi(x)dx,$$

where $F(x) = f(x) + i\phi(x)$ and where $\psi(x)$ is positive between a and b. Darboux's formula is

$$I = \lambda F(\xi)\int_a^b \psi(x)dx,$$

where ξ is a value of x lying between a and b, and λ is a quantity whose modulus is at most equal to unity. From this formula we derive readily the extension of Taylor's theorem to the case where $F(x)$ has the above form.

Chapter II. is devoted to indefinite integrals. It begins with a brief and elegant account of the integrals of rational fractions, and then passes on to hyperelliptic integrals, and so on to the integrals of algebraic differentials in general.

The hyperelliptic integrals are shown to reduce to the two types

$$\int \frac{f(x)dx}{\sqrt{R(x)}}, \quad \int \frac{\phi(x)dx}{(x-a)^\alpha \sqrt{R(x)}},$$

where $f(x)$, $\phi(x)$, and $R(x)$ are polynomials and α is a positive integer. The cases where $R(x)$ is of odd degree or of even degree are seen to be conducted the one to the other, so that $R(x)$ is supposed at once to be of degree $2p + 1$. It is now proved that there exist $2p$ integrals of the first type, and of the form

$$\int \frac{x^\mu}{\sqrt{R(x)}} dx, \qquad \mu = 0, 1, 2, \ldots (2p - 1),$$

and then that among these there are p integrals of the *first kind*, that is, integrals which remain finite when x increases indefinitely, the remaining integrals of this type being of the *second kind*.

Passing now to integrals of the second type,

$$\int \frac{\phi(x)dx}{(x-a)^\alpha \sqrt{R(x)}},$$

it is seen to be necessary to distinguish between the two cases when a is and is not a root of $R(x) = 0$; and it is finally shown that the integrals of the second type conduct to those of the first type and to integrals of the form

$$\int \frac{dx}{(x-a)\sqrt{R(x)}},$$

where a is not a root of $R(x) = 0$.

The consideration of the hyperelliptic integrals includes as a particular case the elliptic integrals, and the case where $R(x)$ is a polynomial of degree two.

We pass on now to the case of the integrals of algebraic differentials in general, *i.e.*, to the case of the Abelian integrals. These are all comprised in the form

$$\int F(x, y)dx,$$

where F is a rational function of x and y, and where x and y are connected by the irreducible algebraic equation $f(x, y) = 0$. The curve $f = 0$ is supposed of degree m, and, further, the aggregate of all the homogeneous terms of degree m is supposed made up of m distinct linear factors, none of which reduce to either x or y, which comes to saying geometrically that the m asymptotic directions of the curve f are distinct, and none of them are parallel to the axes of x or y. This is, of course, a permissible supposition, as such a condition can always be realized by a homographic transformation.

The above integral is now conducted by a known and purely algebraic reduction to the two types

$$\int \frac{P(x,y)dx}{f'_y}, \int \frac{Q(x, y)dx}{(x-a)^\alpha f'_y},$$

where P and Q are polynomials in x and y and α is a positive integer; and, finally, it is here shown that in the integrals of the first type the degree of the polynomial $P(x,y)$ can be lowered to $2m-4$.

In the case of the integrals of the second type two cases have to be considered, according as the straight line $x-a=0$ meets the curve $f=0$ in m distinct points, or is tangent to the curve. Again, the curve $f=0$ may or may not have singular points; if, however, the curve have only double points with distinct tangents, it is shown that under both of the above hypotheses as to the line $x-a=0$ the integral

$$\int \frac{Q(x,y)dx}{(x-a)^\alpha f'_y}, \quad \alpha > 1,$$

can be conducted to an analogous integral, in which the exponent of $x-a$ is equal to unity, say

$$\int \frac{R(x, y)dx}{(x-a)f'_y},$$

when R is a polynomial in (x, y). This can finally be reduced by aid of the equation $f(x, y)=0$ to the form

$$\int \frac{R(y)dx}{(x-a)f'_y},$$

where R is a polynomial in y alone of the degree $m-1$ at most. This chapter closes by a brief account of Hermite's researches on the integration of rational functions of sin x and cos x.

The subject of the Abelian integrals is resumed in chapter XIV., of vol. II., after having devoted one chapter, XIII., to the general properties of algebraic functions of one variable, a theorem of Nöther's, and Riemann's surfaces. It will perhaps be as well to notice these chapters in this place rather than wait until they occur in the course of going over the two volumes chapter by chapter. In chapter XIII. the author begins by defining an algebraic function, and obtains its development in the region of a point. These familiar results are presented in an exceedingly simple and elegant form. Let $f(u, z) = 0$ be the irreducible equation defining the function u, and suppose it to be of the degree m in u. Introducing now the notion of *critical points*, and making use of the known theorem (established in vol. I.) that if this equation has for $z = \alpha$ n roots equal to β, it will have for z near α n roots and n only near β, the following fundamental theorem of algebraic functions is established: *The roots which for $z = \alpha$ become equal to β form one or several circular systems, and the roots of any one circular system are, in the region of α, represented by a development of the form*

$$u = \beta + A(z - \alpha)^{\frac{1}{n'}} + B(z - \alpha)^{\frac{2}{n'}} + \dots$$

To the n' determinations of $(z - \alpha)^{\frac{1}{n'}}$ correspond n' values of u, and these are the values which permute circularly around the point α.

This result is sufficient for the general theory of algebraic functions and their integrals, but practically we need to know how to obtain the different circular systems and the corresponding numbers n'. A very brief, but sufficient, account is here given of Puiseux's researches in this direction.

Following these classical investigations on circular systems comes a most welcome account of the important theorem of Nöther's, which shows that the generality of the theory of algebraic functions is in no wise lessened if we confine ourselves to curves having no other singular points than *multiple points with distinct tangents*. The importance of this theorem, which was given by Nöther in vol. IX. of the *Mathematische Annalen*, is at once seen by any one who has studied even a very little of the theory of algebraic functions. The demonstration given of this theorem is due to M. Simart, and is of such interest that it deserves to be reproduced here.

Let $f(x, y) = 0$ be the equation of an algebraic curve of degree m, and let the origin be a multiple point of order n on the curve, that is, having n tangents. This equation can be put in the form

$$\phi_n(x, y) + \phi_{n+\lambda}(x, y) + \dots + \phi_m(x, y) = 0, \qquad (E)$$

and we shall admit that aside from the origin the axes of x and y only meet the curve in simple points, that the tangents at the origin do not coincide with the axes, and, finally, that the asymptotic directions are distinct and different from the directions of the axes. These hypotheses granted, make the substitution $y = \frac{x}{Y}$. To the curve (E) will correspond point by point the curve (E') of degree $2m - n$, viz.,

$$F(Y, x) = Y^{m-n}\phi_n(Y, 1) + x^{\lambda} Y^{m-n-\lambda}\phi_{n+\lambda}(Y, 1) + \dots + x^{m-n}\phi_m(Y, 1) = 0. \qquad (E')$$

To a multiple point of order n' of the curve (E), not on the axis of x or on that of y, corresponds evidently a multiple point of order n' of the curve (E'). To the value $x = 0$ on the first curve correspond n values of y which are zero, and $m - n$ other distinct values. To these points there correspond in the second curve first a multiple point of order $m - n$ $(x = 0,\ Y = 0)$ with $m - n$ distinct tangents, since for $x = 0$ the $m - n$ corresponding values of y are different from each other and different from zero; and again, n simple or multiple points of the axis of y determined by

$$\phi_n(Y, 1) = 0.$$

To the value $x = \infty$ correspond m distinct values of Y; further, for $Y = \infty$ the $m - n$ values of x are distinct.

We have then substituted for the curve (E) a curve (E'), which, outside of the axis of x, has the multiple points of the first; but which has at the point $x = 0,\ Y = 0$ a multiple point of order $m - n$ with distinct tangents, and upon the axis of Y $(x = 0)$, n points determined by $\phi_n(Y, 1) = 0$, which may be distinct or coincident.

Let us suppose that the equation

$$\phi_n(Y, 1) = 0$$

has a multiple root $Y = Y_1$, to which corresponds a multiple point of order $n' \leqq n$. By an arbitrary homographic transformation

$$x = \frac{ax' + by'}{a''x' + b''y' + c''}, \qquad Y - Y_1 = \frac{a'x' + b'y'}{a''x' + b''y' + c''},$$

we can substitute for the curve (E') the curve (E_1) of degree $2m - n$, viz.:

$$\phi'_{n'}(x', y') + \phi'_{n'+\lambda'}(x', y') + \dots + \phi'_{2m-n}(x', y') = 0. \quad (E_1)$$

This curve satisfies the same conditions as the curve (E), and its asymptotic directions are distinct and different from zero. It has, aside from the points determined by $\phi_n(Y, 1) = 0, x = 0$, multiple points corresponding to the multiple points of (E), plus: *one* multiple point of order $m - n$ with distinct tangents arising from the point $x = 0$, $Y = 0$; *one* multiple point of order $m - n$ with distinct tangents arising from the point $Y = \infty$; and, finally, *one* multiple point of order m with distinct tangents arising from the point $x = \infty$.

We can now apply to the curve (E_1) the same transformations as to the curve (E), and continue until we arrive at an equation such as $\phi_n(Y, 1) = 0$, all of whose roots are distinct. *This series of operations with certainty come to an end;* for, suppose that we have constantly $n' = n$. The curve (E_1) of degree $m_1 = 2m - n$ has multiple points corresponding to those of the curve (E) plus *two* multiple points of order $m - n$ with distinct tangents equivalent to $(m - n)(m - n - 1)$ double points, and *one* multiple point of order m with distinct tangents equivalent to $\frac{m(m-1)}{2}$ double points. The curve (E_2) of degree $m_2 = 2m_1 - n$ differs from the curve (E_1) in the same way that (E_1) differs from (E).

Finally, denoting by d_k the number of double points of the curve E_k corresponding to the multiple points with distinct tangents which have been successively introduced, we shall have

$$d_k = \sum_{i=0}^{i=k-1} \left[(m_i - n)(m_i - n - 1) + \frac{m_i(m_i - 1)}{2} \right]$$

Now we have

$$m_i = 2^i(m - n) + n;$$

hence

$$d_k = \frac{3}{2}(m - n)^2\left(\frac{2^{2k} - 1}{2^2 - 1}\right)$$

$$+ (m - n)\frac{2n - 3}{2}(2^k - 1) + k\frac{n(n-1)}{2}.$$

The difference between this number and the maximu number of double points of an irreducible curve of order m_k, viz.,

$$\frac{(m_k - 1)(m_k - 2)}{2},$$

is

$$-\frac{(m-n)^2}{2} - (m-n)\frac{2n-3}{2} + k\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2}$$

This difference would become positive if k were sufficiently large; but this is impossible: it is therefore necessary that n diminish until it becomes equal to unity. A time will come when the multiple point at the origin of the primitive curve will have been replaced in the transformed curve by multiple points of a lower order, and during this transformation we shall only have introduced multiple points with distinct tangents. Proceeding thus step by step, *we shall finally arrive at a curve which has only multiple points with distinct tangents.*

After a few words on birational transformations Picard enunciates Nöther's theorem in the following form: *We can always, by a Cremona transformation, transform any algebraic curve whatever into another having only multiple points with distinct tangents.*

It is next shown that (leaving aside a certain difficulty which need not be mentioned here) we can by certain transformations always arrive at a curve having only double points with distinct tangents. Finally, if α_i denote the number of multiple points of order i with distinct tangents, and w denote the number of points of ramification (that is, values of x for which *two* values of y from the equation $f(x, y) = 0$ permute), we arrive at the formula

$$w = m(m-1) - \Sigma\alpha_i i(i-1),$$

the sum Σ being relative to the different values of $i (i \geqq 2)$.

This chapter concludes with a rather brief but very satisfactory section devoted to Riemann's surfaces. Very few words are necessary in speaking of this section of the Traité, but these may be preceded by another quotation from the Introduction: "Un chapitre traite des surfaces de Riemann, dont l'étude a été laissée un peu trop de côté en France; on peut, par une représentation géometrique convenable, rendre intuitifs les principaux résultats de cette théorie. Cette vue claire de la surface de Riemann une fois obtenue, toutes les applications se déroulent avec la même facilité que dans la théorie classique de Cauchy relative au plan simple. Mais il importe de juger à sa véritable valeur la belle conception de

Riemann. Ce serait une vue incomplète que de la regarder seulement comme une méthode simplificative pour présenter la théorie des fonctions algébriques. Si importante que soit la simplification apportée dans cette étude par la considération de la surface à plusieurs feuillets, ce n'est pas là ce qui fait le grand intérêt des idées de Riemann. Le point essentiel de sa théorie est dans la conception *a priori* de la surface connexe formée d'un nombre limité de feuillets plans, et dans le fait qu'à une telle surface conçue dans toute sa généralité correspond une classe de courbes algébriques. Nous n'avons donc pas voulu mutiler la pensée profonde de Riemann, et nous avons consacré un chapitre à la question difficile et capitale de l'existence des fonctions analytiques sur une surface de Riemann arbitrairement donnée; le problème même est susceptible de se généraliser si l'on prend une surface fermée arbitraire dans l'espace et qu'on considère l'équation de Beltrami qui lui correspond."

It was undoubtedly well to thus enunciate anu emphasize the essential point in Riemann's theory, as many students of the subject, particularly those who work it up privately, are prone to regard the Riemann surface as simply an ingenious way of representing an algebraic function.

The well-known theorems of Clebsch and Lüroth are given in a very simple form in chapter XIII. In a foot-note to page 371, Picard says *à propos* of Riemann's fundamental memoir: "On trouvera dans le premier chapitre de la Thèse de M. Simart (Paris, 1882) un exposé très complet et très rigoureux des théorèmes relatifs à la connexité de ces surfaces, qui pour la plupart ne sont qu'énoncés par Riemann." The writer can cordially recommend this thesis to the student of Riemann's original memoir. Another foot-note on page 375 will be of interest to English readers: "C'est le géomètre anglais Clifford qui paraît avoir, le premier, remplacé la surface de Riemann sur le plan par une surface à p trous dans l'espace. Pour faire cette transformation, nous nous sommes servi de la méthode qu'il a employée dans un petit mémoire, d'une remarquable simplicité, consacré à cette théorie" [On the canonical form and dissection of a Riemann's surface, *Proceedings of the London Mathematical Society*, vol. VIII.]. The memoir of Clifford's is well known to all English readers, but it may be a little surprise to some to learn that Clifford was the first to replace the Riemann surface on the plane by a surface in three dimensional space having p holes.

The section on Riemann's surfaces closes with the application of Cauchy's theorems to functions of a complex variable on a Riemann's surface, and a proof is given of the theorem that *every function which is uniform over a Riemann's sur-*

face and which has over the surface no other singular points than poles is a rational function of x and y.

Chapter XIV. begins with a study of the periodicity of the Abelian integrals. The student of the theory of the Abelian integrals is well aware of their different determinations between given limits, these differences being linear functions of the periods and depending on the path followed from the lower limit of the integral to the upper limit. It is therefore scarcely worth while to do more than mention Picard's concise and elegant account of these different determinations of the integrals of the first category (that is, where the poles of the function to be integrated give rise to no logarithmic terms) and also of the integrals of the second category where *polar periods* arise from the fact of one at least of the poles of the function to be integrated giving rise to a logarithmic term.

We pass now to Abel's theorem, which Picard first gives in the form in which it was stated by Abel. He says: "Sous cette forme, le théorème paraît tout à fait élémentaire, et il n'y a peut-être pas, dans l'histoire de la Science, de proposition aussi importante obtenue à l'aide de considérations aussi simple."

Starting from the algebraic relation

$$(1) \qquad f(x, y) = 0,$$

consider a family of algebraic curves

$$(2) \qquad \lambda(x, y, a_1, a_2, \ldots a_r) = 0$$

depending on r arbitrary parameters $a_1, a_2, \ldots a_r$. We will suppose that λ contains these parameters rationally The curves (1) and (2) have a certain number, say μ, of points in common,

$$(x_1, y_1), (x_2, y_2), \ldots (x_\mu, y_\mu),$$

which vary with the parameters a. The abscissas $x_1, x_2, \ldots x_\mu$ are roots of a certain equation of degree μ, say

$$(3) \qquad \theta(x, a_1, a_2, \ldots a_r) = 0,$$

whose coefficients are rational in the a's. If the axes do not occupy any particular position relative to the two curves, we can always admit that the corresponding value of y is given by

$$y = \psi(x, a_1, a_2, \ldots a_r),$$

where ψ is rational in $x, a_1, a_2, \ldots a_r$. This granted, consider any Abelian integral whatever,

$$\int_{(x_0, y_0)}^{(x, y)} R(x, y)dx,$$

where $R(x, y)$ is a rational function of x and y, and form the sum

$$S = \sum_{n=1}^{n=\mu} \int_{(x_0, y_0)}^{(x_n, y_n)} R(x, y)dx.$$

This sum is determinate to a sum *près* of multiples of the polar and cyclic periods of the integral, periods which are independent of the a's. The object of Abel's theorem is to determine the nature of this sum considered as a function of the parameters a. Denoting by δ a total differential with respect to these parameters, we have

$$\delta S = R(x_1, y_1)\delta x_1 + \ldots + R(x_\mu, y_\mu)\delta x_\mu.$$

Now by differentiating (3) we can calculate successively

$$\delta x_1, \delta x_2, \ldots \delta x_\mu;$$

substituting these values in δS and replacing the y's by their values ψ, we shall have for the coefficient of δa_1 a rational function of $x_1, x_2, \ldots, x_\mu$ and of the a's. Further, it will evidently be symmetrical with respect to $x_1, x_2, \ldots x_\mu$, and consequently the coefficient of δa_1 will be a rational function of $a_1, a_2, \ldots a_r$, and the same is of course true of the other coefficients. We have therefore

$$\delta S = P_1(a_1, a_2, \ldots a_r)\delta a_1 + \ldots + P_r(a_1, a_2, \ldots a_r)\delta a_r,$$

where the P's are rational functions of $a_1, a_2, \ldots a_r$.

This equality constitutes Abel's theorem in its primordial form; it expresses the fact that S is an algebraico-logarithmic function of the parameters a. In fact, integration of the total differential in the second member leads necessarily to an expression of the form

$$\phi + \Sigma A \log \Phi,$$

the A's denoting constants and ϕ and Φ rational functions of the a's.

Next follows the well-known form of Abel's therorem when applied to Abelian integrals of the first kind. The derivation of this most important form of the theorem is so brief and simple that it may be given here.

We consider the Abelian integrals of the first kind,

$$\int_{(x_0,\, y_0)}^{(x,\, y)} R(x,\, y)dx,$$

that is the integrals which remain finite for every value of x and y on the Riemann's surface. Apply Abel's theorem to such an integral. In the first place, the algebraico-logarithmic function of the a's must remain finite for every finite or infinite value of the a's, since S itself remains always finite. Now a function of the form

$$\phi + \Sigma A \log \Phi,$$

where ϕ and Φ represent rational functions of the a's and where the A's are constants, cannot remain finite for every value of the parameters a; it must therefore reduce to a constant It follows then that the sum S is independent of the parameters a. For integrals of the first kind we can therefore enunciate Abel's theorem as follows: *The sum*

$$\sum_{n=1}^{n=\mu} \int_{(x_0,\, y_0)}^{(x_n,\, y_n)} R(x,\, y)dx,$$

where $(x_n,\, y_n)$ *denote the points of intersection of the curves* (1) *and* (2), *points which vary with the parameters* a, *does not depend on these parameters.*

The sum has a constant value, leaving aside, of course, linear functions of certain fixed periods which can always be introduced by varying the path between $(x_0,\, y_0)$ and $(x_n,\, y_n)$. The theorem for this case can also be put in the form

$$R(x_1,\, y_1)dx_1 + R(x_2,\, y_2)dx_2 + \ldots + R(x_\mu,\, y_\mu)dx_\mu = 0,$$

the d's denoting total differentials with respect to $a_1, a_2, \ldots, a_r$.

The great importance of this special form of Abel's theorem in analysis and in the theory of algebraic curves seems to the writer a sufficient justification for reproducing here Picard's very concise and simple statement of it.

The consideration of integrals of the first kind is now taken up and it is first shown that these integrals are necessarily of the form

$$\int \frac{Q(x,\, y)dx}{f_y'},$$

where $Q(x,\, y)$ is a polynomial, and then the necessary and sufficient condition is found in order that an integral of this form may really be an integral of the first kind; viz., that $Q(x, y)$ must be a polynomial of degree $m - 3$ in x and y and the curve

$$Q(x, y) = 0$$

must have as multiple points of order $i - 1$ the multiple points of order i of the curve $f(x, y) = 0$.

An exceedingly interesting investigation now follows concerning the number of linearly independent integrals of the first kind, but it cannot be entered into here. Certain fundamental theorems concerning integrals of the first kind are now given, and incidentally a new demonstrution is given of the theorem concerning the number of linearly independent integrals of the first kind. In finishing the subject of integrals of the first kind it must suffice to merely state the following theorem established on page 409: *We can form an integral of the first kind for which the real parts of the 2p periods have arbitrary given values.* This chapter closes with a brief but most admirable account of the integrals of the second and third kinds.

Chapter XV. of vol. II. is entitled *Des fonctions uniformes sur une surface de Riemann.* It is impossible to give here any adequate account of this most interesting and important chapter; it will not submit to condensation—any condensation would merely be mutilation. The different sections of the chapter are as follows : I. Decomposition of rational functions of x and y into simple elements. II. The Riemann-Roch theorem. Special functions. This section contains Brill and Nöther's law of reciprocity. III. Birational transformations of curves into themselves. This section contains a demonstration by Picard of an important theorem of Schwarz, viz., *the curves of genus zero and of genus one are the only ones which can be transformed into themselves by a birational substitution involving an arbitrary parameter.* This demonstration is peculiarly interesting, as Picard obtains it by following the same path which led him to an analogous theorem for algebraic surfaces in his celebrated "mémoire couronné" of 1888: "*Mémoire sur la théorie des fonctions algébriques de deux variables indépendantes,*" chap. III. (*Journal de Mathématiques,* 1889). IV. Classes of algebraic curves. Normal curves. V. Curves of genus two. In section IV. there are so many interesting theorems that it is difficult to select any one or two as illustrations of the section. From section V., how ever, we may quote one theorem which is established, viz., *every curve of genus two corresponds point to point to a curve of the fourth order having one double point.*

Chapter XVI. is entitled *Théorèmes généraux relatifs à l'existence des fonctions sur une surface de Riemann.*

I. Statement of the question; preliminary theorems. II. Existence of harmonic functions on an open Riemann surface. III. Existence of harmonic functions on a closed Riemann surface. IV. Functions of a complex variable on a Riemann surface. In this section a demonstration of the fundamental theorem of the chapter is given, viz., *to an arbitrarily given Riemann surface corresponds a class of algebraic curves.* The importance of this theorem, as placing in its true light the conception of the Riemann surface, can hardly be exaggerated. V. Moduli of a class of algebraic curves. This is a most important section, but no account of it can be given here. VI. Existence theorems for Beltrami's equation corresponding to any surface whatever. This section deals with Beltrami's generalization of Laplace's equation to any surface whatever. This generalization will be referred to later. In the present connection it will be sufficient to quote the fundamental theorem concerning this generalization: *To the given surface S in space having p holes corresponds uniformly an algebraic curve of genus p.* The footnote to page 493 gives an interesting history of this theorem. A limiting case of the surface considered gives the following theorem due to Schottky. To every plane disk with p holes there corresponds a class of algebraic curves.

Chapter XVII., which closes vol. II., has for title *Courbes des genres zero et un.* It is divided into three sections: I. Unicursal curves. II. Curves of genus one. III. Generalities on doubly periodic functions. The theorems in this short chapter are well known and need not be recapitulated. It is sufficient to say that they are presented in the graceful way peculiar to Picard. An historical remark in a footnote to page 498 is interesting. Picard says: "Les mots *courbes unicursales* ont été employés pour la première fois par M. Cayley."*

We return now to vol. I. Chapters III., IV., and V. are respectively entitled *Intégrales curvilignes, Des intégrales doubles,* and *Des Intégrales multiples.* These chapters, interesting and important as they are, must be passed over with a mere mention; they will be of great interest to the physicist as well as to the mathematician. The conditions that a line integral shall depend only on the limits and not on the path and that a surface integral shall depend only on the limiting contour of the surface are given in an exceedingly elegant form ; the fundamental notion of the calculus of variations is introduced here. The sections dealing with the question of the roots

* See *Comptes rendus*, vol. LXII.

common to two equations (chap. III.), and the roots common to three equations (chap. IV.) are of particular interest, as Picard has himself probably said the last word on the subject of the number of roots common to n simultaneous equations in his memoir in the *Journal de Mathématiques* (1892). The reader may recall a brief discussion between Picard and Kronecker, just before the latter's death, on this subject. It is not necessary to enter into that discussion here; it will suffice to say that Kronecker gave a formula for the *difference* between the number of roots contained in a given region for which a certain determinant is positive and those for which it is negative. The *exact* number of roots in the region is thus not given by Kronecker's formula. Picard has, however, shown how the difficulty which presents itself in Kronecker's formula can be overcome, and proves that the number of roots common to two equations and contained in a certain contour can be represented by a double integral; and, as already mentioned, he gives the analogous theorem for the case of n simultaneous equations. These latter results are contained in chapter VII. of vol. II. The final theorem of this chapter may be quoted: "*On peut donc par suite trouver, par un calcul algébrique régulier, le nombre des racines des deux équations,*

$$f(x, y) = 0,$$

$$\phi(x, y) = 0,$$

(f et ϕ étant deux polynômes) contenues dans un contour défini par les n inégalités,

$$\lambda_i(x, y) < 0, \qquad (i = 1, 2, \ldots n),$$

les λ étant des polynômes."

Part second of volume I. treats of Laplace's equation and its applications and of developments in series. This second part is also of much interest to physicists, dealing as it does with Laplace's equation, Dirichlet's principle, the theory of attraction and of the potential, and particularly with trigonometric series. These subjects are all classical and need not be further spoken of, though it may be mentioned that the theorems of Cantor and Schwarz probably appear here for the first time in any treatise on analysis.

In taking up the subject of multiple series Picard first gives a generalization of Cauchy's rule for convergence in the case of simple series with positive terms. This rule is readily applied to series whose general term is of the form

$$\frac{1}{[f(m_1, m_2, \ldots m_p)]^a},$$

f denoting a definite and positive quadratic form. An interesting example of a double series is given by the absolutely and uniformly convergent development in a double trigonometric series of a function $f(x, y)$, which, together with its first four partial derivatives, is continuous and possesses the property defined by the equations

$$f(x + 2\pi, y) = f(x, y), \qquad f(x, y + 2\pi) = f(x, y).$$

Another illustration of double series is taken from the theory of the elliptic functions, and then the author considers a quadruply periodic function of two variables investigated by himself in a paper contained in the *Bulletin de la Société Mathématique*, 1889. This function is the sum of the double series

$$\sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \frac{e^{x+m\alpha+n\beta}}{(1+e^{x+m\alpha+n\beta})^2} \cdot \frac{e^{y+m\alpha'+n\beta'}}{(1+e^{y+m\alpha'+n\beta'})^2},$$

where α, β, α', β' are real numbers such that the determinant $\alpha\beta' - \alpha'\beta$ is different from zero. This series is by very simple transformations changed into a trigonometric series.

The last section of this chapter is of special interest to the student of the more recent developments in the theory of functions. It treats of series where the indices are not arbitrary, that is, do not make up all possible systems of integers, but are confined to the substitutions of a certain group. Picard then shows how it is possible to form functions of two variables which are entirely analogous to Poincaré's thetafuchsian functions. These hyperfuchsian functions are not studied here, but it is probable that the author will return to them in another volume.

Part III. of vol. I. is devoted to geometrical applications of the infinitesimal calculus. The subjects first treated are the theories of envelopes, developable surfaces, ruled surfaces, with a special study of the most general ruled surface of the third order, congruences, and complexes of lines. The essential properties of the focal surface of a congruence are investigated and a proof is given of Dupin's theorem that *the necessary and sufficient condition that the straight lines of a congruence shall be normal to a surface is that the tangent planes to the two nappes of the focal surface corresponding to any generatrix whatever shall be rectangular.*

Here follow some brief considerations of complexes, especially the linear complex, and chapter XI. closes with the theorem that *the tangents of a skew cubic belong to a linear complex.*

In chapter XII. the theory of contact of plane and skew curves is discussed, and among other things are derived Cayley's formulæ for skew algebraic curves, which are analogous to Plücker's formulæ for plane algebraic curves. In this chapter is also contained an investigation of curves whose tangents belong to a linear complex.

Chapter XIII. deals with the curvature and torsion of skew curves, and chapter XIV. with curves traced on surfaces. Here the theorems of Euler and Meunier are first obtained, and then the subject of lines of curvature is taken up and the theorems of Joachimsthal and Dupin are established. Surfaces which are the envelopes of spheres, and in particular Dupin's cyclide, generalities on asymptotic lines and asymptotic lines on certain ruled surfaces form the subjects of the closing sections of this rather long and very interesting chapter. Chapter XV. is entitled *Surfaces applicables. Representation conforme. Cartes géographiques.* The expression for the square of the element of arc of a curve on a surface is first found, and the subject of surfaces applicable to one another, and in particular to the plane, is developed. In the second section the conform representation of a plane upon a plane is studied, and the third section gives some examples of conform representation and a little introduction to the linear substitutions

$$\left(z, \frac{az+b}{cz+d}\right),$$

for which $ad - bc = 1$.

After obtaining some of the familiar properties of such a substitution he defines a *group* and then a *discontinuous group*, viz. Poincaré's Fuchsian group. Following this he considers with more detail the group formed by the substitutions

$$\left(z, \frac{az+b}{cz+d}\right),$$

where a, b, c, d are four real integers, satisfying the relation

$$ad - bc = 1.$$

This group is first proved to be discontinuous, and then it is shown that it leads to a division of the half-plane into an infinite number of triangles. The proof which Picard gives of this result is based upon the arithmetical theory of the reduction of definite quadratic forms, which makes it necessary to give a very brief account of these forms—or, more exactly, of the notion of a reduced quadratic form (*forme quadratique réduite*). Picard then shows very briefly how the notion of connecting the theory of the substitution

$$\left(z \frac{az+b}{cz+d}\right)$$

with the theory of quadratic forms can be extended to obtain the substitutions for a half-space analogous to those found for the half-plane. This investigation is contained in a paper by Picard in the *Bulletin de la Société Mathématique* for 1884. The title is *Sur un groupe de transformations des points de l'espace situés du même côté d'un plan.* The final section of this chapter and volume is a very brief one on map projections. It is a matter of regret to the writer that this beautiful part III. of vol. I. has to be noticed so briefly, but he has thought it better to touch lightly on these applications of analysis to geometry in order to leave more space for the analysis itself.

The closing chapters of vol. II. have already been noticed and need not be particularly referred to again. The principal part of vol. II. is devoted to harmonic and analytical functions. Picard says in the introduction: "Sans négliger le point de vue de Cauchy dans la théorie de ces dernières fonctions, je me suis surtout attaché à une étude approfondie des fonctions harmoniques, c'est-à-dire de l'équation de Laplace; une grande partie de ce volume est consacrée à cette équation célèbre, dont dépend toute la théorie des fonctions analytiques. Je me suis arrêté longuement sur le principe de Dirichlet, qui joue un si grand rôle dans les travaux de Riemann, et qui est aussi important pour la physique mathématique que pour l'analyse."

The first chapter begins with the definition of a function of a complex variable and the familiar conditions of Cauchy that it shall be monogenic—that is, if the function be denoted by $u + iv$ we must have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Following this is given the interesting generalization due to Beltrami of these equations to the case of any surface Σ on which the element of length is given by the equation

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2.$$

Beltrami's generalized equations are

$$\frac{\partial v}{\partial p} = \frac{F\dfrac{\partial u}{\partial p} - E\dfrac{\partial u}{\partial q}}{\sqrt{EG - F^2}};$$

$$\frac{\partial v}{\partial q} = \frac{G\dfrac{\partial u}{\partial p} - F\dfrac{\partial u}{\partial q}}{\sqrt{EG - F^2}}.$$

The combination $u + iv$ can now be called a complex function of the point (p, q) on the surface Σ. There follows now a careful study of Laplace's equation and then a most important extension to the general linear partial differential equation of the second order, with two independent variables, of some of the results obtained for Laplace's equation. The author's important contributions to the theory of these equations are well known to all students of differential equations. The last section of chapter I. is principally concerned with Neumann's method for the solution of Dirichlet's problem.

Chapter II. is a very important chapter in the theory of functions: it is entitled *Développments en series et prolongement analytique des fonctions harmoniques et des fonctions d'une variable complexe.*

In concluding certain generalities concerning harmonic functions and developments in series the author gives a proof of the following theorem of Harnack's: *Let there be given a series*

$$u_0 + u_1 + u_2 + \dots + u_n + \dots$$

of harmonic functions which are all positive inside an area limited by a contour C. If this series is convergent at a point O in the interior of the area, it will be convergent at every point in the interior of the area and will represent a harmonic function.

The pages devoted to the subject of the extension of an analytical function, to the examples of functions which cannot be extended (Freedholm's, for example), and to the recent theorems given by Hadamard, particularly his theorem relative to the region of convergence of a power series (Taylor's series), are full of most interesting and valuable results, but can only be alluded to here. The next two chapters, III. and IV., are devoted wholly to Dirichlet's problem, and contain an account of the methods of Schwarz and Poincaré—the latter method being developed in a memoir contained in vol. IX. of the *American Journal of Mathematics.*

Chapter V. contains a direct study of functions of a complex variable and begins by establishing certain well-known and general theorems of Cauchy. The second section, dealing with poles and essential singularities, gives further theorems of Cauchy's and introduces Weierstrass's notion of essential singular points and a mere mention of a most important theorem of the author's on entire functions. Section III. contains familiar elementary examples of functions of a complex variable. Section IV. deals very briefly with convergent products, and section V. is concerned with the decomposition of uniform functions into primary factors. A very elegant and rigorous demonstration of Weierstrass's results is given first and then a most interesting generalization is given of these results (which are too well known to need statement).

Picard shows that a formula similar to that of Weierstrass can be found for uniform functions which are continuous for all points of the plane except such as are situated upon a circumference C of radius R and having the origin as centre.

Let $A_1, A_2, \ldots A_n, \ldots$ be a series of quantities such that on writing $A_n = \rho_n e^{i\alpha_n}$ we have

$$| \rho_n - R | \geqq | \rho_{n+1} - R |$$

and further $\lim_{n=\infty} \rho_n = R$. It is now shown that we can form an expression depending on z which shall be uniform and continuous in all points of the plane, points of the circumference excepted, which shall represent an analytical function of z inside and outside the circumference and shall vanish for the values $A_1, A_2, \ldots A_n, \ldots$ of z.

Another series of quantities

$$B_1, B_2, \ldots B_n, \ldots$$

are taken on the circumference C and such that

$$\lim_{n=\infty} (A_n - B_n) = 0.$$

The product

$$\prod_{n=1}^{n=\infty} \frac{z - A_n}{z - B_n} e^{\phi_n(z)},$$

where

$$\phi_n(z) = \frac{A_n - B_n}{z - B_n} + \frac{1}{2}\left(\frac{A_n - A_n}{z - B_n}\right)^2 + \ldots + \frac{1}{n-1}\left(\frac{A_n - B_n}{z - B_n}\right)^{n-1},$$

is shown to be convergent and to represent an analytical function $G(z)$ having the properties mentioned. Two interesting examples of this theorem are given. The chapter closes with a theorem due to Painlevé and which is derived from the fundamental Cauchy formula

$$F(x) = \frac{1}{2\pi i}\int_C \frac{F(z)}{z - x} dz.$$

The theorem is: *Every function which is holomorphic in an area limited by a convex contour can be developed in this area in a series of polynomials.*

It must suffice simply to mention the matters treated in chapter VI., which is devoted to applications of Cauchy's theorems: Investigations of certain definite integrals; Developments in series of rational fractions; (here some important remarks are made concerniug the Weierstrass decomposition of a function into primary factors;) Cauchy's method for obtaining Fourier's and analogous series; Number of roots of an equation contained in a contour; and the Theory of indices.

Chapter VII., on the number of roots common to two simultaneous equations, has already been mentioned.

Chapter VIII. has for title *Intégrales des fonctions non-uniformes*. The first section shows the different determinations which an hyperelliptic integral of the type $\int_{x_0}^{x} \frac{f(x)dx}{\sqrt{R(x)}}$ ($f(x)$ a polynomial) can have when the path between the limits is varied. The second section considers the integrals of the first kind and the reduction of the number of periods; here a proof is given that there must be at least two distinct periods. The elementary properties of the periods of the elliptic integral of the first kind are given now, including a proof of the theorem that the ratio of the two periods is imaginary. Section III. contains an example of a non-uniform function represented by integrals, an application to the hypergeometric series, and finally some important properties of the ratio of the periods of an elliptic integral regarded as a function of the modulus. The employment of this function now enables the author to give proofs of his two celebrated theorems on uniform functions. The first is: *An entire function $G(z)$ which can never become equal to two values a and b is necessarily a constant.* For the second theorem we consider a uniform function $f(z)$ having throughout the plane only poles as singular points; Picard then shows that *there cannot be more than two finite values a and b which this function cannot take for a finite value of the variable; if there are more than two, the function reduces to a constant.*

Chapter IX. is a most admirable introduction to the study of functions of several independent variables. This chapter is rendered particularly interesting and valuable by containing the author's presentation of Poincaré's extension of Cauchy's fundamental theorem for one complex variable to the case of two such variables. As it is impossible to give any adequate account of this beautiful piece of analysis here it seems better to pass it over with a mere mention. The chapter closes with a study of Lagrange's formula for one and two equations.

Chapter X. resumes the theory of conform representation, the elementary propositions in which were given in vol. I. In this chapter the author is particularly occupied with the question of the representation of a given area upon another area equally given. It is necessary first to give a definition of an *arc of an analytic curve.* Suppose a curve such that the coordinates x and y of an arbitrary point are analytical functions of a parameter t,

$$x = f(t), \qquad y = \phi(t),$$

f and ϕ being supposed holomorphic functions of t in the region of the real value $t = t_0$, the coefficients of the developments in series according to powers of $t - t_0$ being of course real. We say, then, that this arc is analytic; furthermore, this analytic arc will be *regular* at the point corresponding to $t = t_0$ if we can choose the parameter t on which x and y depend analytically in such a way that $f'(t_0)$ and $\phi'(t_0)$ are not both zero at the same time. A determinate arc $\alpha\beta$ is said to be regular if it is regular at all its points. Suppose we have a closed contour C, and let us admit that a portion $\alpha\beta$ of this contour is a regular arc of an analytical line. Assign now a succession of values along the contour, and suppose that the ensemble of values along the arc $\alpha\beta$ forms an analytical function of the parameter t. Under these conditions the following theorem due to Schwarz is demonstrated: The harmonic function taking the given values along the contour can be prolonged analytically beyond the arc $\alpha\beta$.

The conform representation of a simple area on a circle is next taken up, and then a presentation is given of Schwarz's method for Dirichlet's principle. The chapter concludes with the consideration of a simple case of two areas limited by several contours, and it is shown how a function which is holomorphic in the interior of an ellipse can be developed in a series of polynomials.

Chapter XI. contains general theorems in differential equations; the theorems are classical, but their mode of presentation is modern, and it need hardly be said that it is both elegant and rigorous. The theorems of the existence of an integral for a differential equation, or of a system of integrals for a system of ordinary or of partial differential equations, occupy the entire chapter. Cauchy's methods are given, and also the author's own method by successive approximations, which is familiar to the readers of this BULLETIN by Dr. Fiske's translation. The *unique* determination of a system of integrals for given initial values is emphasized strongly by the author for reasons which need not be gone into.* The chapter is a

* See page 317.

thoroughly satisfactory one on this fundamental question in the theory of differential equations.

Chapter XII. contains some applications of the general theorems, and begins with some of the well-known theorems of Briot and Bouquet and then follows a most important theorem due to Painlevé and the latter's notion of fixed and movable critical points. The writer has given a brief explanation of what Painlevé means by these terms in another number of this BULLETIN, and it need not be repeated here. Riccati's equation is next studied, and the chapter closes with an account of the inversion of the elliptic integral and of certain entire functions associated with the elliptic functions. The fuller study of the subject of differential equations and of the functions defined by them is reserved for another volume.

The writer is quite conscious of the inadequacy of the preceding notice to give a satisfactory idea of this most important work of M. Picard's. The attempt has been made to show how in each theory or its application M. Picard goes at once to what is essential and in particular in the applications how he has selected really important problems in analysis, geometry, and mathematical physics. No applications are given simply because they afford pretty exercises in analysis or give rise to very symmetrical sets of formulæ. It is customary to say something about the typography of a book reviewed and concerning errata. As for the former, it is hardly necessary to comment on Gauthier-Villars' manner of getting up a book. As for the latter, they are too few and trifling to mention: still one might mention one which the reader will not find out is an error until he has read nearly a page further. In the third line from the bottom of page 45, vol. I., we find the words: "Je me place d'abord dans le premier cas." Instead of the "first case" it should be the "*second case.*"

T. CRAIG.

BALTIMORE, Oct. 10, 1893.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, October 7, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Mr. John M. Colaw, Monterey, Va.; Mr. David Lyman Pettegrew, Worcester, Mass.; Dr. Isaac J. Schwatt, University of Pennsylvania; Professor David Eugene Smith, Michigan State