# THE THEORY OF FUNCTIONS.

*A Treatise on the Theory of Functions.* By JAMES HARKNESS and FRANK MORLEY. New York, Macmillan & Co., 1893. 8vo. IX + 503 pp.

IN the vast realm of Mathematical Science, as it stands before us to-day, there exists, owing to the genius, activity, and assiduity of mathematicians, such a great variety of widely extended subdivisions that it seems to be an impossibility for one man to keep pace with the rapid development of all of them. The consequence has been that mathematicians try to confine themselves to some special line of their science, not only in research work, but also in reading up mathematical literature. This is a pity, because the cultivation of any special branch suffers considerably by disregarding the methods and results of other branches, while, on the other hand, the most beautiful results have often been obtained by the combination of apparently unconnected regions: it is to be regretted, but it is a fact, and will remain so as long as the capacity of the human intellect remains limited.

There exists, however, one branch of mathematical science whose bearing on nearly all the other parts is so evident that it appears to be, with perhaps a few exceptions, indispensable for special work of any kind—that is, the *Theory of Functions.* Even in applied mathematics it seems to gain a foothold: the theoretical astronomer as well as the mathematical physicist has frequently enough, when he wants to feel safe ground under his feet in his calculations, to fall back on the theory of functions.

While thus the high importance of function theory will be readily admitted, it has until lately been impossible to obtain a fair knowledge of this subject without consulting quite a number of different German or French treatises and original papers. There did not exist any English text-book on the subject at all.* But there is also no Continental work which deals with the theory of functions in any degree of completeness. Either Cauchy's, or Riemann's, or Weierstrass's method is given alone, whereas a combination of all three methods is needed in order to pursue the study of the subject to advantage.

An urgent demand was met therefore when there appeared, some months ago, two new, comprehensive treatises, both in English: Forsyth, "Theory of Functions of a Complex Vari-

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* I should perhaps except here some chapters in Chrystal's Algebra, which bear on subjects belonging to the theory of functions.

able;" and the work of which we shall try to give a short review in the following pages: Harkness and Morley, "A Treatise on the Theory of Functions."

The first thing which strikes the reader is the richness of material contained in the work. Besides an exhaustive presentation of the principal conceptions and deductions as they are given by the great founders of the theory, Cauchy, Riemann, and Weierstrass, we find in the book a full account of Weierstrass's theory of elliptic functions, of double Theta-functions, and of Abelian integrals and functions, including the difficult and intricate proofs of existence theorems, leading up to the most recent investigations, and foreshadowing further developments in this direction. Also some collateral branches of the theory are taken up, as, for instance, algebraic curves and (non-analytic) functions of a real variable.

And this rich material is not merely compiled in a loose manner, but it has been worked over carefully, and systematized in a way which gives the book a thoroughly scientific character. Even in those parts which treat of the most difficult problems of the theory the impression is forced upon the reader that he is guided by the hand of men who are thoroughly versed in modern mathematical thought.

An added value is conferred upon the work by the fulness of references which are given, not only throughout the whole volume, but also at the end of each chapter in a separate appendix. This should be done in every scientific treatise, for, besides its immediate purpose, it always exerts an inspiring influence upon the student, and it leads him to a principle only too often violated: "*Never study one book only.*"

As to the manner in which the subject is presented to the reader, it seems to be natural that in a work like this the language cannot be otherwise than concise, sometimes summary. The conciseness is, however, in some places carried so far as to mislead a beginner. To make up for compactness of expression, all should have been done to facilitate a clear understanding of the text. Unfortunately this is not always the case. The theorems enunciated are not sufficiently, or even not at all, separated from the context. Thus it frequently happens that the reader cannot decide where the enunciation of a theorem ends and the explanation of it begins. Definitions of vital importance are given in the middle of a paragraph without being made prominent in any way. Much of this could have been avoided very easily by the use of italics.

Furthermore it would have added considerably to perspicuity if, after long deductions, the result of the investigation were expressly stated. As it is, the reader has in many paragraphs to find out for himself what the authors have in mind. This may perhaps be less felt by a reader who studies the volume word by word, but it will prove a more serious drawback for one who wants to use the book for reference, on account of the difficulty in ascertaining at a rapid glance the bearing of the different paragraphs on the subject in question.

It is to be hoped, however, that these defects will be removed in a second edition.

*Chapter I., "Geometric Introduction,"* opens with the ordinary geometrical representation of a complex variable in the plane. In order to guard against useless generalizations, it is shown in § 15 that an extension to three dimensions becomes illusory if the laws of ordinary algebra shall be maintained.

A function $w$ of a complex variable $z$ being defined at first merely as a correspondence, the definition of continuity of such a function is given in § 16, and illustrated by simple examples.

In the sequel it becomes necessary to speak also of branches of a many-valued function; § 18 deals therefore with Cauchy's proof of the fundamental theorem of algebra. Now this proof, as is known, shows only that the modulus of an integral function possesses the lower limit—zero. This point should be emphasized *somewhere* in the book, if not in the introductory chapter, certainly later on, after it has been shown that a continuous function always attains its limit.

The important criterion of a monogenic function $w$ of $z$, i.e., a function whose derivative $\frac{dw}{dz}$ is independent of $dz$, is developed in § 20, and in connection therewith numerous and very instructive examples are given,—in §§ 21–30 for one-valued, in §§ 46–48 for many-valued, functions. §§ 29 and 30, which treat of conform representation by fractional linear functions, may serve as well as a preliminary exercise for the study of modular functions.

In the remaining paragraphs the authors give a very elegant exposition of several points in the theory of invariants for cubic and biquadratic binary quantics. For the benefit of those students, however, who are not yet acquainted with invariants, it might be said that these paragraphs are not absolutely required for the study of the following chapters. Perhaps it would have been a better plan to confine the subject to only the most important points, as, for instance, the deduction of $g_2$ and $g_3$, and to set forth this deduction in such a way that there would not be presupposed, on the part of the reader, familiarity with the theory of invariants.

*Chapter II., "Real functions of a real variable,"* gives a succinct account of some of the most important recent researches concerning the foundations of the infinitesimal calculus.

Since the time when Abel wrote, in his well-known letter to Hansteen, "Je consacrerai toutes mes forces à répandre de la lumière sur l'immense obscurité qui règne aujourd'hui dans l'analyse," much has been done in the direction indicated by the illustrious mathematician. The "immense obscurity" has been traced back to its source, viz., a comparatively small number of fundamental conceptions which seem simple enough as long as we are satisfied with geometrical evidence, but which turn out to be full of the most intricate difficulties as soon as they are exposed to the search-light of purely analytical methods. Such are the conceptions of: *Irrational numbers* (which may be said to be at the bottom of all difficulties of the infinitesimal calculus); that of a *limit* with its delicate distinctions between upper limit and maximum, between uniform and non-uniform convergence; the general idea of a *function,* with its wide range of possibilities. These and other difficulties have been cleared up by the combined efforts of men like *Abel, Cauchy, Bolzano, Weierstrass, du Bois-Reymond, Cantor,* and others, and the infinitesimal calculus has been put upon a basis as sound and solid as the foundation of arithmetic.

The importance of these researches to the theory of functions of a complex variable can hardly be overestimated; for the same difficulties that face us in the elements of real infinitesimal calculus will turn up over and over again, whether we consider Cauchy-Riemann's or Weierstrass's theory. As examples, I may mention Goursat's proof of Cauchy's theorem and Weierstrass's theorem on the existence of singular points on the circumference of the circle of convergence.

There is still another, more pedagogical, reason why a discussion of functions of a real variable should not be omitted in a treatise on analytic functions. We understand much better the properties of these latter functions and the consequences of the restrictions imposed upon them by their very definition when we have already obtained an insight into the many difficulties by which non-analytical functions are surrounded.

We are therefore indebted to the authors for having given to this important subject a place in their volume.

We find in §§ 50–54 an exceedingly lucid account of Cantor's definition of irrational numbers, which is followed by definitions of upper and lower limits, points of accumulation, mass of points, and Cantor's derived mass of points.

All this may be called rather a theory of the *independent* variable; but certainly this theory should always precede a theory of the *dependent* variable, i.e., a theory of functions.

In §§ 60–64 important definitions and theorems on upper and lower limits of functions and on continuity of functions are deduced. Weierstrass's famous example illustrates in § 65 the possibility of a continuous function which has nowhere a differential quotient.

*Chapter III.* "*The theory of infinite series.*"—This chapter, forming the foundation of function-theory, may be divided into two parts: (1) Infinite series and products; (2) Weierstrass's theory of analytic functions.

It was felt in many researches that the old definition of a function as a dependence of one variable on the other was too vague to make functions accessible to analytic investigation. In order to do the latter, it proved to be necessary to impose some restrictions upon this very general idea of a function, to limit its wide range in such a way as to attain, if possible, concrete mathematical expressions. This has been done, as is well known, by Cauchy, who, *confining* himself to mongenic functions, arrived at the expansion of such a function into a power-series.

Weierstrass's method is just the opposite. He starts from given functions, well defined by simple analytic expressions, and, never deviating for a moment from the safe ground of a concrete mathematical representation, *amplifies* and generalizes these functions in such a way as to cover as much as possible the ample ground occupied by the general notion of a function. Thus he reaches, from precisely the opposite end, not only connection with Cauchy's theory, but he also comes, by his analytic functions, as near as one pleases—in a strictly mathematical sense—to every continuous non-analytical function.*

Turning now to the first part of this chapter, we find in §§ 67–69 a statement of the principal definitions and theorems on series with real terms. As to the proofs, reference is made to the usual text-books.

Series with complex terms are considered in §§ 70–73, including in particular the definition of uniform convergence. Examples are given also for *non*-uniform convergence. This is the more instructive case, since the student will be inclined to think, at first, that convergence always implies uniform convergence.

In the following sections on multiple series there is of special interest the example taken from the theory of elliptic functions:

$$\Sigma\Sigma' \frac{1}{(m_1\omega_1 + m_2\omega_2)^\lambda}.$$

Two proofs are given for the convergence of this series for $\lambda > 2$, the second being an extension to double series of "Cauchy's integral test for the convergence of simple series."

* See page 48 of the "Evanston Colloquium," by Felix Klein. New York, 1894.

It would be very desirable if this test for simple series * were given *somewhere* in the book—which is not the case,—the more so as the account of its extension to double series is extremely succinct. The same extension recurs in chapter VIII., § 223, where the condition for the convergence of double Theta-series is deduced.

The second part of this chapter, §§ 82–107, affords a comprehensive and lucid introduction into Weierstrass's theory of analytic functions. Instructive illustrations are given of the continuation of a power-series, the influence of singular points on this process, and especially of the case where the region of the analytic function is limited by a closed line (§§ 103 and 104).

*Chapter IV., " Algebraic functions,"* deals with the expansions of algebraic functions in the vicinity of branch-points and singular points, and may be regarded as an essential preparation for the following chapters on Riemann's theory. The discussion here presented to the reader, which is generally omitted in other treatises on function-theory, confers a special value on this chapter.

It is important for the student to learn the different points of view from which an equation $f(x, y) = 0$ is to be looked at in geometry and in the theory of function. If this equation is of the $m^{\text{th}}$ degree in $x$ and of the $n^{\text{th}}$ degree in $y$, it will represent geometrically, in general, a curve of the $m + n^{\text{th}}$ order, so that the line $x = 0$ for instance will have $m + n$ points of intersection with the curve; whereas in the light of function-theory only $n$ values of $y$ correspond to any given value of $x$.

With regard to singularities, there is also a difference between the views taken in geometry and in function-theory. While the branch-points of the function $y$, defined as above by the equation $f(x, y) = 0$ as a function of $x$, turn out to be of paramount importance in the theory of functions,† they are, geometrically, nothing else than the points of contact of tangents parallel to the $Y$-axis, and therefore of no higher interest than any other system of points of intersection between the curve and the first polar with regard to any point in the plane.

Expansions in the vicinity of singular points, such as nodes, cusps, etc., bear, of course, immediately on geometry, since they determine the geometrical character of the singularity. Interesting examples of this kind are given in § 122.

As to the exposition of the subject, the authors follow, at first, Noether's method of resolving by Cremona-transformations — speaking geometrically — higher singularities with

* Cf. Picard's Traité d'Analyse, vol. I., chapter 1, § 17.
† This is not so much the case in Weierstrass's theory.

partly coinciding tangents into those whose tangents are all distinct. Also Puiseux's method, emanating from Newton, is described and illustrated by some examples.

The remainder of the chapter is devoted to Clebsch-Lüroth's interesting theory of loops. This method of investigation, frequently used by French mathematicians, has, at least for some lines of work, decided advantages, the independent variable moving only in a singly-covered plane. Besides, it can be used for simplifying the study of Riemann's surfaces, as pointed out in chapter VI.

*Chapter V. "Integration."*—Two ways lie open before the student for entering into the theory of functions. One of them has been described in chapter III. The other is afforded by Cauchy's beautiful method of integration along a closed line in the plane of complex numbers.

It cannot be denied that this second way is a shorter one, but only as long as certain fundamental points, which seem at a first glance to be evident, are taken for granted. As soon, however, as absolute rigor is required, the number of those fundamental difficulties bearing on the foundation of the infinitesimal calculus, which have been pointed out in the review of chapter II., will prove to be much greater in Cauchy's than in Weierstrass's theory. On the other hand, Cauchy's methods, important and attractive in themselves, are indispensable in Riemann's theory of Abelian integrals.

In the present chapter Cauchy's method is developed and carried on until connection with Weierstrass's theory is reached.

Among the different proofs of Cauchy's fundamental theorem Goursat's method has been chosen as the one which makes immediate use of the existence of a differential quotient and which does not apply double integration. The presentation of this proof being exceedingly concise, the student should consult, besides, the original paper of Goursat * and one of the other proofs referred to. Riemann's classical proof is given in chapter VI., and may also be consulted, as it takes only a singly-sheeted plane instead of the general Riemann surface.

In §§ 135–141 the important applications of Cauchy's theorem to integration about singular points are discussed, leading to Taylor's and Laurent's theorems.

The connection with Weierstrass's theory being thus established, we find in the following paragraphs those fundamental theorems which refer mostly to analytic functions whose singularities are known,—in particular a comprehensive account of

* Cf. also *Franklin:* "Two proofs of Cauchy's theorem." *American Journal of Math.*, vol. IX., p. 389.

Mittag-Leffler's theorem, and Weierstrass's factor-representation of a function with assigned zeros.

Applications to the calculation of definite integrals and a succinct account of Cauchy's proof of the existence of integrals of differential equations close the chapter.

*Chapter VI., "Riemann surfaces."*—The first half of this chapter, being devoted to the theory of Riemann surfaces proper, §§ 158–173, can be highly recommended as a lucid and pedagogical introduction into the subject.

At first the principal ideas of Riemann surfaces are developed, leaving aside all generalities. A number (9) of very instructive, well-chosen, and elaborated examples showing how to construct a Riemann surface will prove to be of extremely good service to the student.

Theorems on the connection of surfaces in general and of Riemann surfaces in particular are given in §§ 168–170, leading to the definition of the deficiency $p$.

It must be put down as, in a certain sense, a drawback of Riemann's theory that these geometrical considerations on the connection of surfaces are, in the general case, and if any degree of reliability in the conclusions is to be reached, of a highly complicated character, and lead into abstruse difficulties. Now, certainly, no mathematician should object to mathematical difficulties; but these difficulties, as they are presented here, bearing mostly on "analysis situs," lie in a region which is entirely incongruous with pure analysis.

If, however, the geometrical theorems in question have been proved or are taken for granted, the beauty of Riemann's theory lies open before us. No simpler and clearer insight into the real meaning and nature of the deficiency $p$, for instance, can be gained than the perception that $p$ constitutes the one and only invariant of the surface with regard to continuous deformation.

The canonical dissection of a Riemann surface is based in § 171 upon the theory of loops discussed in chapter IV. Also, Klein's normal surface and its dissection, which exhibits so very clearly the true nature of the problem, is presented to the reader, as well as Riemann's original method.

The subject of the second part of this chapter is, at first, the study of functions on a Riemann surface. Integrals on the surface are then discussed at some length, but only as far as to the demonstration of periods, a further development being the chief object of chapter X.

There follows now in §§ 183–190 an investigation which is of a special theoretical value. The theory of algebraic functions, as it has been developed by Kronecker and Dedekind-Weber, is sketched briefly in order to show that a curve $f(w, z) = 0$ can be transformed rationally in such a way that the final equation does not contain any higher singularities than simple nodes. The presentation of this subject is very succinct, and can hardly be regarded as more than a guide through Kronecker's and Dedekind-Weber's papers.

The chapter concludes with the explanation of Klein's interesting transformation of a curve of class 3, $p = 1$, into the normal surface of deficiency 1, that is, into an anchor-ring.

*Chapter VII. "Elliptic functions."*—The subject presented in this chapter is, in essence, Weierstrass's theory of elliptic functions.

The definition of primitive periods is given in §§ 191–194 in a very clear and exhaustive manner.* Next the function $\wp(u)$ is introduced immediately, defined by the series

$$\frac{1}{u^2} + \Sigma'\left(\frac{1}{(u-w)^2} - \frac{1}{w^2}\right),$$

and the double periodicity of this function demonstrated. Liouville's theorems on doubly periodic functions follow in §§ 196–200. In § 201 the $\wp$-function is taken up again, the $\sigma$-function deduced from it and then the theory is carried through, essentially according to Halphen and Schwarz's Formelsammlung.

We regret that the authors did not follow Weierstrass's construction of the elliptic functions as the most general one-valued functions which possess an algebraic addition-theorem, although they refer to it in several places. A few steps would have been sufficient, since nearly all the auxiliary theorems are at hand (see chapter v., § 151), in order to deduce this beautiful theory, one of the masterpieces of Weierstrass's creations. No better illustration of the high practical usefulness of those general theorems on the theory of functions developed in chapter v., which may seem to the student somewhat abstract in their generality, could have been given.

Some remarks may be made with regard to notation. Weierstrass denotes his fundamental periods by $2\omega$, $2\omega'$, using afterwards, where the sum $\omega + \omega'$ comes in, the following notations:

$$\omega_1 = \omega, \quad \omega_2 = \omega + \omega', \quad \omega_3 = \omega'.$$

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* It may only be mentioned that it would perhaps have been better to define the parallelogram of periods *ab initio* as the totality of all the points within and on the rim of the parallelogram *OABC, except the points on AB and BC.*

Instead of this, Harkness and Morley introduce this notation:

$$\omega_1 = \omega,\ \omega_2 = -\omega - \omega',\ \omega_3 = \omega'.$$

Hence the relation holds:

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

The symmetry of the formulæ obtained thereby will sufficiently justify this deviation from the original notation.*

*Chapter VIII. "Double Theta-functions."*—One who wishes to make his first acquaintance with double $\theta$-functions cannot do better than to study first this chapter before taking up some of the more detailed treatises or original papers on the subject. For the benefit of some readers it might be added that there is not presupposed any knowledge of the preceding parts of the volume.

Besides the definitions and fundamental properties of double $\theta$-functions, and a very valuable digest and explanation of the different notations, by marks, current indices, and duads, the theory is developed so far as to embody the most remarkable linear, quadratic, and bi-quadratic relations between $\theta$-functions.

Some original work has been added to the subject by the authors. In the so-called $\theta$-formula of Riemann, extended by Prym-Krazer, there occurs a constant factor whose numerical value (= 4) is generally computed by expansion into series.† The authors have determined this factor by a different elegant method, which proves to be effective also for the deduction of quite a number of $\theta$-relations.

In the concluding paragraph a brief definition of $p$-tuple $\theta$-functions is added. It seems somewhat strange that the analogous exponential representation of ordinary elliptic $\theta$-functions is nowhere to be found in the volume. These functions are mentioned only briefly in chapter VII., where they are derived in trigonometrical form from the $\sigma$-functions.

*Chapter IX. "Dirichlet's problem."*—The most characteristic feature of Riemann's magnificent theory of Abelian functions is that he starts, not from an algebraic equation, but

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* *Study* in his "Sphaerische Trigonometrie" (Leipzig, 1893), has been led to the same change of notation by similar reasons (see l. c., page 189).

† Cf. *Krause:* Die Transformation der hyperelliptischen Functionen, p. 46.

from a Riemann surface. On this surface—made simply connected by a system of cross-cuts—he defines and proves the existence of "functions $w$" of $z = x + iy$, having given algebraic or logarithmic discontinuities and constant moduli of periodicity. With the help of these functions $w$ he is able to build up "functions $s$" which are single-valued in the original surface $T$, and have no other than algebraic discontinuities. The functions $s$ are proved to be algebraic functions of $z$, while the functions $w$ turn out to be integrals of rational functions of $s$ and $z$, so-called Abelian integrals.

The theorem of the existence of the functions $w$ is therefore of paramount importance for Riemann's theory. Riemann's own proof is based on Dirichlet's principle, viz., the conclusion that among all functions $u$ of $x$ and $y$ satisfying certain boundary- and continuity-conditions, there exists one for which the integral

$$\int_{(T)} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] dT$$

takes a minimum value. This being once granted, it follows at once from the elements of the calculus of variations that the minimizing function satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Dirichlet's conclusion lies, however, open to serious objections, since the possibility is not taken into account that the integral might be made to tend towards zero by functions $u$ which approach functions no longer satisfying the conditions imposed upon $u$. The conclusion is, in fact, one of the many examples where the distinction between lower limit and minimum is neglected, and it is entirely analogous to the conclusion, known to be wrong, that, the sum of the angles of a spherical triangle always being greater than $\pi$, there must exist a triangle for which the sum of the angles is a minimum.*

This gap in Riemann's theory has been filled by Neumann, Schwarz, and others, who have given rigorous proofs of Riemann's existence theorems independent of Dirichlet's conclusion. These proofs form the subject of chapter IX., whose length is entirely in proportion with the importance of the subject, though some readers may perhaps wish it were shorter.

* Weierstrass gave this example in one of his lectures.

The problem in its simplest form is stated as follows: "To find a function $u(x, y)$ which, together with its differential quotients of the first two orders, shall be one-valued and continuous in a region $T$, which shall satisfy Laplace's equation and shall take assigned values upon the boundary of the region."

After the demonstration of two theorems of Painlevé's on rim-values and Green's theorem for two variables, with those consequences which bear on the subject, Schwarz's solution of the problem is deduced for the simple case where the given region $T$ consists of a circle and the function on its circumference is supposed to be continuous.

It is shown next—leaving aside some allied investigations—how these restrictions can be removed. Singularities of a certain kind are admitted on the circumference, and—by the help of conform representation and Schwarz's alternating process—the solution is extended to regions bounded by any analytic lines.

The following sections are devoted to Harnack's investigations, which lead, on the basis of Schwarz's important results and by the help of Green's function $g$ to the solution of the problem for a simply connected region whose rim consists of an integrable curve, and has only a finite number of changes of direction.

The last section of the chapter extends the solution to the case where interior discontinuities are given of the kind met with in the discussion of Abelian integrals.

The reader who wishes to become familiar with the subject will have some difficulty in working his way through this chapter, but the subject offers so many intricate difficulties in itself, that he will find it necessary to fall back on the study of at least some of the original papers.

From the solution of Dirichlet's problem Riemann's existence-theorems can be derived without further difficulties. They are given at the end of the following chapter, in §§ 306 and 307. Klein's methods of proof, as exhibited in Klein-Fricke's Modulfunctionen, are adopted.

*Chapter X.*, "*Abelian integrals*," gives a detailed exposition of Riemann's theory, with a supplementary sketch of Clebsch-Gordan's work, so that this chapter may be regarded as a continuation of chapter VI. as well as of chapter IV.

Following strictly Riemann's ideas, the surface $T$ is made the starting-point, and accordingly the theory of the integrals precedes that of rational functions on the surface. The properties of the integrals of the first kind, their periods and their derivatives "the functions $\Phi$," are studied in detail in §§ 277–281.

Next follow the integrals of the second kind, with their important application to the construction of rational functions with arbitrarily assigned zeros and poles, leading to Riemann-Roch's theorem.

In this connection a short account of Riemann's theorems on the class-moduli is added. The elementary integrals of the third kind, their periods, and the theorem on the interchange of limits and parameters are treated in § 285, while § 286 deals with the most general Abelian integral.

After a digression on Clebsch and Gordan's method of homogeneous variables in §§ 287–290, Abel's theorem is taken up in § 291, and two proofs are given—Neumann's and Riemann's. Then follow Riemann's theorems on zeros of the function $\theta(u_1 - e_1, \ldots u_p - e_p)$. The determination of the constants $k$ is discussed at length, and Riemann's as well as Clebsch and Gordan's choice of the lower limits, by means of the contact-curves of order $n - 2$, is explained (§§ 294–300).

A short sketch of Jacobi's inversion-problem and Clebsch-Gordan's solution of it close the general part of the chapter, while the remaining paragraphs contain a very interesting and instructive application of the preceding general theories to the case $p = 2$. As normal curve a nodal quartic is taken, which lends itself more readily than the hyperelliptic curve to Clebsch's method. The determination of the constants $k$ leads to the important correspondence between the "transcendental" and "algebraic" characteristic of a theta-function which throws a new light upon the duad-notation explained in chapter VIII.

Finally, Klein's expression of the even and odd double $\theta$-functions in terms of algebraic functions and of a single integral of the third kind is derived, and the reader thus finds himself at the end of the chapter landed in the midst of the most important recent discoveries on Abelian functions.

The whole chapter is not only of high scientific value, but it is also well arranged and very clearly written, and furnishes an excellent introduction into the theory of Abelian functions.

Harkness and Morley's treatise has rendered the theory of functions accessible to every one who wishes to acquire a thorough knowledge of the subject. The great merits of this valuable work will secure it a high rank in modern mathematical literature. H. MASCHKE.

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