no hesitation, add several well-known names, and many of lesser note, to those mentioned.

The error pointed out in the article in the *Educational Review* can hardly, save on the principle of *ipse dixit*, be called "the most serious error in Cajori's book." It is, however, quite pardonable for Professor Halsted to be partial to that article; he wrote it. As to the article in the *Nation*, that was a charming essay, but it was hardly a serious review of Cajori.

Since with the exception of a few pages in Ball we have no similar attempt at a synopsis of the whole of modern mathematical history up to the present time, it is quite safe to make the sweeping assertion that this portion of the work is "without a rival in the world, in any language." Nevertheless one may close the final chapters with disappointment.

Having no idea of the meaning of Professor Halsted's reference to Chasles' Christian name, or of his statement that "30 is perhaps a slip for 300," the writer ventures to pass them by. He also ventures to reassert his appreciation of the work under discussion as a popular exposition of the historical advance of mathematical science, as set forth in the closing paragraph of his review.

DAVID EUGENE SMITH.

Michigan State Normal School,
YPSILANTI, MICH.

ON ORTHOGONAL SUBSTITUTIONS.

BY PROF. HENRY TABER.

IN 1846, in *Crelle's Journal*, Cayley gave his well-known determination of the general proper orthogonal substitution of n variables rationally in terms of the minimum number of parameters. Subsequently, in *Crelle*, vol. 50, and in the *Philosophical Transactions* for 1858, Cayley expressed these results in the notation of matrices.

In accordance with the theory of matrices,* two linear substitutions are regarded as susceptible of being added or subtracted. If $(\phi)_{rs}$ denotes that coefficient of the linear substitution ϕ which appears in the rth row and sth column of its square array or matrix, the sum or difference of the linear substitutions ϕ and ψ is defined as follows:

$$(\phi \pm \psi)_{rs} = (\phi)_{rs} \pm (\psi)_{rs}.$$

Addition and subtraction of linear substitutions are then subject to the laws which hold for these processes when we deal with the symbols of ordinary algebra.

---

* See Cayley's "Memoir on the Theory of Matrices," *Phil. Trans.*, 1858.

Multiplication is taken as equivalent to the composition of substitutions; and consequently multiplication is associative and distributive, but not in general commutative.* The reciprocal ϕ^{-1} of any linear substitution ϕ is the linear substitution which multiplied by or into ϕ gives the identical substitution.

In the notation of matrices, Cayley's expression for the general proper orthogonal substitution of n variables is

$$(1 + \Upsilon)^{-1}(1 - \Upsilon),$$

in which 1 denotes the identical substitution (or matrix unity) and Υ is an arbitrary alternate (or skew symmetric) linear substitution, but such that the determinant of $1 + \Upsilon$ is not zero.

For if we denote by tr ϕ the substitution transverse (or conjugate) to ϕ, that is, the substitution whose matrix is obtained from the matrix of ϕ by interchanging its rows and columns, we then have

$$\begin{aligned} \operatorname{tr} \phi &= (1 - \operatorname{tr} \Upsilon)(1 + \operatorname{tr} \Upsilon)^{-1} \\ &= (1 + \Upsilon)(1 - \Upsilon)^{-1} \dagger \\ &= (1 - \Upsilon)^{-1}(1 + \Upsilon); \ddagger \end{aligned}$$

and therefore

$$\operatorname{tr} \phi = \phi^{-1},$$

that is

$$\phi \operatorname{tr} \phi = 1,$$

which is the condition that ϕ shall be orthogonal.

The orthogonal substitutions given by Cayley's expression are all proper. And every orthogonal substitution whose characteristic equation does not have -1 as a root is given by this expression; but an orthogonal substitution whose characteristic equation has -1 as a root cannot be put in this form.§

---

* If $\phi\psi = \psi\phi$, and if $f\phi$ and $F\psi$ are polynomials in ϕ and ψ respectively, then $f\phi \cdot F\psi = F\psi \cdot f\phi$. In particular, ϕ is commutative with any polynomial in ϕ.

† For $\operatorname{tr}(\phi\psi) = \operatorname{tr}\psi \cdot \operatorname{tr}\phi$, $\operatorname{tr}(\phi^{-1}) = (\operatorname{tr}\phi)^{-1}$, and $\operatorname{tr}(\phi + \psi) = \operatorname{tr}\phi + \operatorname{tr}\psi$; moreover, $\operatorname{tr} 1 = 1$, and $\operatorname{tr}\Upsilon = -\Upsilon$, by definition, since (employing the notation used above) $(\Upsilon)_{rs} = -(\Upsilon)_{sr}$.

‡ $(1 - \Upsilon)^{-1}$ can be expressed as a polynomial in powers of Υ; it is therefore commutative with $1 + \Upsilon$. See note above.

§ If -1 is not a root of the characteristic equation of the orthogonal substitution ϕ, $1 + \phi$ has a reciprocal, and we may put

$$\Upsilon = (1 - \phi)(1 + \phi)^{-1}.$$

About a third of a century after Cayley's paper of 1846, the investigation was completed by Frobenius, who showed, *Crelle*, vol. 84, that every proper orthogonal substitution not given by Cayley's expression is given by the limit of Cayley's expression when Υ is infinite. That is, let ϕ be any proper orthogonal substitution whose characteristic equation has -1 as a root (in which case ϕ is not given by Cayley's expression), then we can always find a skew symmetric linear substitution Υ_ρ whose coefficients are rational functions of a parameter ρ, of which one at least is infinite for $\rho = 0$, such that $(1+\Upsilon_\rho)^{-1}(1-\Upsilon_\rho)$ can be made as nearly as we please equal to ϕ by taking ρ sufficiently small. Moreover, we have

$$\phi = L_{\rho=0}(1+\Upsilon_\rho)^{-1}(1-\Upsilon_\rho).$$

In what follows we shall be concerned principally with a consequence of this theorem.

By a well-known theorem every orthogonal substitution whatever is equal to the product of Cayley's expression into a symmetric orthogonal substitution. That is, if ϕ is orthogonal, we can always put

$$\phi = \phi_0(1+\Upsilon)^{-1}(1-\Upsilon),$$

in which Υ is skew symmetric, and $\phi_0 = \text{tr}\ \phi_0$, $\phi_0^2 = 1$ (whence follows $\phi_0\ \text{tr}\ \phi_0 = 1$). If the characteristic equation of ϕ has not -1 as a root, we may put $\phi_0 = 1$.

We may so choose ϕ_0 and Υ that they shall be commutative.* In which case

$$\phi^2 = \phi_0^2[(1+\Upsilon)^{-1}(1-\Upsilon)]^2 = [(1+\Upsilon)^{-1}(1-\Upsilon)]^2.$$

From which we obtain

$$\begin{aligned}\text{tr}\,\Upsilon &= (1+\text{tr}\ \phi)^{-1}(1-\text{tr}\ \phi)\\ &= (1+\phi^{-1})^{-1}(1-\phi^{-1})\\ &= (\phi+1)^{-1}\,\phi\,.\,\phi^{-1}(\phi-1)\\ &= (\phi-1)(\phi+1)^{-1}\\ &= -\Upsilon;\end{aligned}$$

whence it follows that Υ is skew symmetric. We also have

$$(1+\Upsilon)\,\phi = 1-\Upsilon.$$

But since

$$1+\Upsilon = [(1+\phi)+(1-\phi)](1+\phi)^{-1} = 2(1+\phi)^{-1},$$

the determinant of $1+\Upsilon$ is not zero. Therefore

$$\phi = (1+\Upsilon)^{-1}(1-\Upsilon).$$

* If -1 is a root of multiplicity m of the characteristic equation of ϕ,

Whence it follows that every orthogonal substitution which is the second power or square of an orthogonal substitution is given by the square of Cayley's expression. Thus the square of every improper orthogonal substitution is also the square of a proper orthogonal substitution.

But every proper orthogonal substitution cannot be put equal to the square of Cayley's expression. *Exempli gratia,* the proper orthogonal substitution given on p. 123 of vol. 16 of the *American Journal of Mathematics* is not the second power of any orthogonal substitution whatever.

We are thus led to designate an orthogonal substitution as of the first or second kind according as it is or is not the square of an orthogonal substitution.

In a paper read at last year's Mathematical Congress at Chicago I have shown that every real proper orthogonal substitution of any number of variables, and every imaginary proper orthogonal substitution of two or three variables, also every proper symmetric orthogonal substitution, is given by the square of Cayley's expression; and in a recent number of the *American Journal of Mathematics,* that every orthogonal substitution given by Cayley's expression is also given by the square of Cayley's expression.* Each of the proper orthogonal substitutions enumerated above is therefore the second power of a (proper) orthogonal substitution.

If ϕ is any proper orthogonal substitution of the second kind, by the theorem of Frobenius, a skew symmetric linear substitution Υ_ρ can be found whose coefficients are rational functions of a parameter ρ such that

$$\phi_\rho = (1 + \Upsilon_\rho)^{-1}(1 - \Upsilon_\rho)$$

can be made as nearly as we please equal to ϕ by taking ρ sufficiently small. And since ϕ_ρ is given by Cayley's expression, we can find a proper orthogonal substitution ψ_ρ whose coefficients are algebraic functions of ρ such that

$$\phi_\rho = \psi_\rho\psi_\rho. \dagger$$

and if the roots of this equation other than -1 are $g_1, g_2, \ldots g_i$, then we may put

$$\phi_0 = 1 - 2\Phi,$$

in which

$$\Phi = \frac{(\phi+1)^m-(g_1+1)^m}{-(g_1+1)^m} \cdot \frac{(\phi+1)^m-(g_2+1)^m}{-(g_2+1)^m} \cdots \frac{(\phi+1)^m-(g^i+1)^m}{-(g_i+1)^m}.$$

* For if ϕ is an orthogonal substitution given by Cayley's expression, it is the square of an orthogonal substitution. But then by the preceding theorem ϕ is given by the square of Cayley's expression. The coefficients of the orthogonal substitution given by Cayley's expression of which ϕ is the second power are algebraic functions of the coefficients of ϕ.

† See preceding note.

Therefore, by taking ρ sufficiently small we can make the second power of ψ_ρ as nearly equal to ϕ as we please. Whence it follows that an orthogonal substitution of the first kind can always be found whose second power shall be as nearly as we please equal to any proper orthogonal substitution of the second kind (itself not the square of any orthogonal substitution).

As an example, let ϕ be the proper orthogonal substitution whose matrix is

$$\begin{array}{llll} \frac{1}{2c^2}-1, & \frac{\sqrt{-1}}{2c^2}, & 0, & \frac{1}{c} \\ \frac{\sqrt{-1}}{2c^2}, & -\frac{1}{2c^2}-1, & 0, & \frac{\sqrt{-1}}{c} \\ 0, & 0, & -1, & 0 \\ -\frac{1}{c}, & \frac{-\sqrt{-1}}{c}, & 0, & -1 \end{array}$$

in which c is an arbitrary constant. This orthogonal substitution is not the square of any orthogonal substitution whatever. But if ψ_ρ designates the orthogonal substitution whose matrix is

$$\begin{array}{llll} \frac{\rho}{r}+\frac{1}{8\rho r^3c^2}, & \frac{1}{r}+\frac{i}{8\rho r^3c^2}, & \frac{-i}{4\rho rc}+\frac{1-\rho i}{4r^3c}, & \frac{1}{4\rho rc}-\frac{i+\rho}{4r^3c} \\ \frac{-1}{r}+\frac{i}{8\rho r^3c^2}, & \frac{\rho}{r}-\frac{1}{8\rho r^3c^2}, & \frac{1}{4\rho rc}+\frac{i+\rho}{4r^3c}, & \frac{i}{4\rho rc}+\frac{1-\rho i}{4r^3c} \\ \frac{i}{4\rho rc}+\frac{1+\rho i}{4r^3c}, & \frac{-1}{4\rho rc}+\frac{i-\rho}{4r^3c}, & \frac{\rho}{r}, & \frac{1}{r} \\ \frac{-1}{4\rho rc}-\frac{i-\rho}{4r^3c}, & \frac{-i}{4\rho rc}+\frac{1+\rho i}{4r^3c}, & \frac{-1}{r}, & \frac{\rho}{r} \end{array}$$

in which $r=\sqrt{1+\rho^2}$ and $i=\sqrt{-1}$, then the matrix of $\phi_\rho=\psi_\rho^2$ is

$$\begin{array}{llll} \frac{\rho^4-1}{r^4}+\frac{1}{2c^2r^4}, & \frac{2\rho(\rho^2+1)}{r^4}+\frac{i}{2c^2r^4}, & \frac{-\rho i(\rho+i)}{cr^4}, & \frac{1-\rho i}{cr^4} \\ \frac{-2\rho(\rho^2+1)}{r^4}+\frac{i}{2cr^4}, & \frac{\rho^4-1}{r^4}-\frac{1}{2c^2r^4}, & \frac{\rho(\rho+i)}{cr^4}, & \frac{i(1-\rho i)}{cr^4} \\ \frac{\rho(1+\rho i)}{cr^4}, & \frac{\rho i(1+\rho i)}{cr^4}, & \frac{\rho^4-1}{r^4}, & \frac{2\rho(1+\rho^2)}{r^4} \\ \frac{-1+\rho i}{cr^4}, & \frac{-i(1+\rho i)}{cr^4}, & \frac{-2\rho(1+\rho^2)}{r^4}, & \frac{\rho^4-1}{r^4} \end{array}$$

and if ρ is taken sufficiently small $\phi_\rho = \psi_\rho^2$ may be made as nearly as we please equal to ϕ.

So long as ρ is not zero, the orthogonal substitution ϕ_ρ is of the first kind, that is, ϕ_ρ is the second power of the orthogonal substitution ψ_ρ. But in the limit for $\rho = 0$, $\phi_\rho = \phi$ is of the second kind, that is, ϕ_ρ, for $\rho = 0$, is not the square of any orthogonal substitution.

The orthogonal substitutions of the first kind have certain interesting properties. Any orthogonal substitution of this kind has an orthogonal mth root (m being any positive integer)—meaning by the mth root of a linear substitution ϕ the linear substitution ψ satisfying the equation $\phi = \psi^m$.

This theorem may be proved as follows. Let e^θ denote the infinite series $\sum_0^\infty r \dfrac{\theta^r}{r!}$, convergent for any linear substitution. For any linear substitution ϕ whose determinant does not vanish, a polynomial $f(\phi)$ in integer powers of ϕ can be found such that $\phi = e^{f(\phi)}$. If now ϕ is any orthogonal substitution of the first kind, we have

$$\phi = [(1 + \Upsilon)^{-1}(1 - \Upsilon)]^2,$$

in which Υ is skew symmetric and such that the determinant of $1 - \Upsilon$ does not vanish. Therefore we may find a polynomial $\vartheta = f(\Upsilon)$ in powers of Υ such that

$$1 - \Upsilon = e^\vartheta.$$

Taking the transverse of either side we have

$$1 + \Upsilon = 1 - \text{tr}\,\Upsilon = e^{\text{tr}\,\vartheta}.$$

Whence it follows that

$$(1 + \Upsilon)^{-1}(1 - \Upsilon) = (e^{\text{tr}\,\vartheta})^{-1}e^\vartheta = e^{-\text{tr}\,\vartheta}e^\vartheta = e^{-\text{tr}\,\vartheta + \vartheta},$$

since ϑ and $\text{tr}\,\vartheta = f(\text{tr}\,\Upsilon) = f(-\Upsilon)$* are both polynomials in Υ, and therefore are commutative.

Let $-f(-\Upsilon) + f(\Upsilon) = -\text{tr}\,\vartheta + \vartheta = \frac{1}{2}\theta$; we then have

$$\phi = (e^{\frac{1}{2}\theta})^2 = e^\theta,$$

in which θ is skew symmetric. If now

$$\psi = e^{\frac{1}{m}\theta},$$

---

* See note, p. 253.

in which m is any positive integer, then, since $\frac{1}{m}\theta$ is also skew symmetric,

$$\begin{aligned}\psi \text{ tr } \psi &= e^{\frac{1}{m}\theta} \text{ tr } e^{\frac{1}{m}\theta} \\ &= e^{\frac{1}{m}\theta} e^{\frac{1}{m}\text{tr } \theta} \\ &= e^{\frac{1}{m}\theta} e^{-\frac{1}{m}\theta} \\ &= e^{\frac{1}{m}\theta - \frac{1}{m}\theta} \\ &= 1.\end{aligned}$$

Moreover,

$$\phi = e^{\theta} = \left(e^{\frac{1}{m}\theta}\right)^m = \psi^m.$$

Therefore ϕ has an mth root which is orthogonal.

By taking m sufficiently great, the coefficients of $\frac{1}{m}\theta$ may be made as small as we please;* and therefore $\psi = e^{\frac{1}{m}\theta}$ may be made to approach as near as we please to the identical substitution. But we still have $\phi = \psi^m$. Whence it follows that every orthogonal substitution of the first kind can be generated by the repetition of the same infinitesimal orthogonal substitution.

An infinitesimal orthogonal substitution cannot have -1 as a root of its characteristic equation. The infinitesimal orthogonal substitutions are therefore of the first kind; and can all be put equal to e^{θ}, in which θ is infinitesimal and skew symmetric. But any power of an orthogonal substitution of the first kind is of that kind. Therefore the substitutions generated by the repetition of the same infinitesimal orthogonal substitution are of the first kind. Consequently orthogonal substitutions of the second kind cannot be generated in this way.

The orthogonal substitutions of the second kind cannot have a root of even index. But every orthogonal substitution of the second kind has a root with any odd index. For

---

* Employing the notation of p. 251 we have $\left(\frac{1}{m}\theta\right)_{rs} = \frac{1}{m}\left(\theta\right)_{rs}$.

by the theorem given on p. 253 any orthogonal substitution ϕ of the second kind can be put equal to

$$\phi_0(1+\Upsilon)^{-1}(1-\Upsilon),$$

in which ϕ_0 is symmetric and orthogonal, and Υ is skew symmetric; and moreover ϕ_0 and Υ are commutative. If now ϑ is a polynomial in Υ satisfying the equation

$$1+\Upsilon=e^{\vartheta},$$

then $\theta=\vartheta-\operatorname{tr}\vartheta$ is skew symmetric and

$$(1+\Upsilon)^{-1}(1-\Upsilon)=e^{\theta}.$$

But then since θ is a polynomial in Υ, it is commutative with ϕ_0. Consequently, if

$$\psi=\phi_0 e^{\frac{1}{2m+1}\theta}$$

(m being any positive integer),

$$\begin{aligned}\psi \,.\, \operatorname{tr}\psi &= \phi_0 e^{\frac{1}{2m+1}\theta} e^{-\frac{1}{2m+1}\theta}\phi_0\\ &= e^{\frac{1}{2m+1}\theta} e^{-\frac{1}{2m+1}\theta}\\ &=1,\end{aligned}$$

and

$$\begin{aligned}\phi &= \phi_0(1+\Upsilon)^{-1}(1-\Upsilon)\\ &= \phi_0 e^{\theta}\\ &= \left(\phi_0 e^{\frac{1}{2m+1}\theta}\right)^{2m+1}\\ &= \psi^{2m+1}.\end{aligned}$$

These results may be summarized as follows. Every proper orthogonal substitution is of the first or second kind according as it is or is not the second power of an orthogonal substitution. All proper orthogonal substitutions of two or three variables, all real proper orthogonal substitutions of any number of variables, all orthogonal substitutions given by Cayley's expression (including all infinitesimal orthogonal substitutions), and all proper symmetric orthogonal substitutions are of the first kind. Every orthogonal substitution of the first kind is given by the square of Cayley's expression, can be generated by the repetition of the same infinitesimal orthogonal substitution, and has an orthogonal mth root for any index m. No orthogonal substitution of the second kind can be generated by the repetition of the same infinitesimal orthogonal substitution. Every orthogonal substitution of the second kind has an orthogonal root with any odd index; and no orthogonal substitution of this kind has an orthogonal root with even index. But, corresponding to any proper orthogonal substitution ϕ of the second kind, can always be found an orthogonal substitution ψ_ρ of the first kind whose coefficients are algebraic functions of a parameter ρ such that, by taking ρ sufficiently small, the $2m$th power of ψ_ρ can be made as nearly as we please equal to ϕ. Moreover, we have

$$\phi = L_{\rho=0}\psi_\rho^{2m},$$

but not

$$\phi = \left[L_{\rho=0}\psi_\rho\right]^{2m}.$$

[An exactly similar theory holds for the linear substitutions which automorphically transform a bilinear form with cogrediant variables.]

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, June 2, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Mr. William Eimbeck, U. S. Coast and Geodetic Survey, Washington, D. C.; Professor Herman J. Gaertner, Indiana Normal College, Covington, Indiana; Mr. Henry Volkman Gummere, Swarthmore College, Swarthmore, Pa.; Mr. George Herbert Ling, Columbia College, New York. The by-laws were amended in accordance with the recommendations of the council, the amendments to go into effect July 1, 1894.

Dr. Henry Taber read a paper entitled "On orthogonal substitutions." This paper appears in the present number of the BULLETIN, see p. 251.

THE council of the Society, influenced by the high importance of most of the papers presented to the Mathematical Congress at Chicago in 1893, by the desirability of their publication collectively, prepared, as they were, to a large extent, for the purpose of giving a general survey of the present state of knowledge throughout almost the entire range of mathematics, and by a sense of the honor conferred upon America by the contributions of so many distinguished mathematicians resident abroad, has resolved to undertake the publication of