The last number in the third row is the greatest common divisor of the two last numbers in the first row, and the last number in the second row is their least common multiple. The last number but one in the third row is the greatest common divisor of 3240 and 33750, and 405000 is their least common multiple. The last number but two in the third row is the greatest common divisor of 2700 in the first row and 405000 in the second row, and 405000 a little to the left is the least common multiple of the same numbers. The second number in the third row is the greatest common divisor of 720 and 405000, and 910000 is their least common multiple. The first number in the third row is the greatest common divisor of 480 and 910000, and $1820000 = e_6$ is their least common multiple, etc., etc.

M. Stieltjes gives other expressions for the same numbers, for which I refer to his paper. We have now worked our way through the first chapter* with the figure of the great Greek mathematician overshadowing us all the while. The second chapter begins under the auspices of Gauss.

JOSEPH DE PEROTT.

CLARK UNIVERSITY, *May* 23, 1895.

NOTE ON HÖLDER'S THEOREM CONCERNING THE CONSTANCY OF FACTOR-GROUPS.†

BY MR. GEORGE L. BROWN.

HÖLDER'S proof of the constancy of the factor-groups for the different series of composition of a group is based upon the following lemma:‡ *If a group G possesses two different maximal self-conjugate subgroups A and B, having C as their greatest common subgroup, then the quotient-groups $G \mid A$ and $B \mid C$, likewise $G \mid B$ and $A \mid C$, are holoedrically isomorphic.*

The proof of this lemma may be very much simplified by making use of the following theorem, due to Giudice:§ *If A and B are two commutative groups of orders p and q respectively, having C, of order r, as their greatest common subgroup, then the order of the group $F = \{A, B\}$ formed by combining the operations of A and B in every possible way is $\frac{pq}{r}$.*

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* I had no desire to exhaust the subject. The reader will find many propositions which I omitted in Grassmann's Arithmetik and Stieltjes' paper.

† *Math. Annalen*, vol. 34, pp. 34–37.

‡ Hölder, l. c., § 9.

§ Netto, Theory of Substitutions, § 38; F. Giudice, *Palermo Rend.* vol. 1, pp. 222, 223.

For let

$$A = [1,\ a_2,\ a_3\ .\ .\ .\ a_p];$$
$$B = [1,\ b_2,\ b_3\ .\ .\ .\ b_q];$$
$$C = [1,\ c_2,\ c_3\ .\ .\ .\ c_r].$$

Form the rectangular table for B with C as a basis:*

$$(1)\qquad \begin{array}{llll} 1\ , & c_2\ , & c_3\ .\ .\ .\ . & c_r \\ \beta_2\ , & c_2\beta_2\ , & c_3\beta_2\ .\ .\ . & c_r\beta_2 \\ \beta_3\ , & c_2\beta_3\ , & c_3\beta_3\ .\ .\ . & c_r\beta_3 \\ .\ .\ . & .\ .\ . & .\ .\ . & .\ .\ . \\ \beta_{\frac{q}{r}}\ , & c_2\beta_{\frac{q}{r}}\ , & c_3\beta_{\frac{q}{r}}\ .\ .\ . & c_r\beta_{\frac{q}{r}}. \end{array}$$

If now the operations of A be multiplied through by these *same multipliers,* $1,\ \beta_2,\ \beta_3\ .\ .\ .\ \beta_{\frac{q}{r}}$, then the $\frac{pq}{r}$ operations

$$(2)\qquad \begin{array}{llll} 1\ , & a_2\ , & a_3\ .\ .\ . & a_p \\ \beta_2\ , & a_2\beta_2\ , & a_3\beta_2\ .\ .\ . & a_p\beta_2 \\ \beta_3\ , & a_2\beta_3\ , & a_3\beta_3\ .\ .\ . & a_p\beta_3 \\ .\ .\ . & .\ .\ . & .\ .\ . & .\ .\ . \\ \beta_{\frac{q}{r}}\ , & a_2\beta_{\frac{q}{r}}\ , & a_3\beta_{\frac{q}{r}}\ .\ .\ . & a_p\beta_{\frac{q}{r}} \end{array}$$

can easily be proved to be all distinct and to form a group $F = \{A, B\}$ containing all possible combinations of the operations of A and B. Hence the order of $F = \{A, B\}$ is $\frac{pq}{r}$.

We also see from this that *the index of A under F is the same as that of C under B,* the index of B under F the same as that of C under A.

Hölder's lemma can now be proved as follows: *Let A and B be maximal self-conjugate subgroups of G, and C be their greatest common subgroup;* then

1) Since A and B are self-conjugate under G, they are commutative. *The group $F = \{A, B\}$* formed by combining the operations of A and B is self-conjugate under G, and consequently *is identical with G,* since by hypothesis A and B are maximal self-conjugate under G. Also, by the preceding theorem, the index of A under G is equal to that of C under B, the index of B under G equal to that of C under A.

2) Since C is common to A and B, its transform by any operation of G is also common to A and B. Consequently, *C is self-conjugate under G,* and therefore *under A and B.*†

* Netto, Theory of Substitutions, § 41.

† Netto, Theory of Substitutions, § 88; Hölder, ***Math. Annalen,*** vol. 34, p. 34.

3) The quotient groups *$G \mid A$ and $B \mid C$ are holoedrically isomorphic.* For from Table (1) is immediately derived the group $B \mid C$, isomorphic with B. Let

$$B \mid C = [1, s_2, s_3, \ldots s_{\frac{q}{r}}].$$

Then if s_ρ corresponds to β_ρ, $c_2\beta_\rho$, $c_3\beta_\rho$, etc., of B
and s_σ " " β_σ, $c_2\beta_\sigma$, $c_3\beta_\sigma$, etc.,

to the product of any two operations of B, one taken from each of these rows, will correspond the product $s_\rho s_\sigma$. Taking β_ρ and β_σ as representatives of their respective rows, we have, if

$$\beta_\rho\beta_\sigma = c_\mu\beta_\tau,$$
$$s_\rho s_\sigma = s_\tau \text{ in the group } B \mid C,$$

where s_τ corresponds to β_τ, $c_2\beta_\tau$, $c_3\beta_\tau$, etc., of B.

In like manner the group $G \mid A$ is derived from Table (2), in each row of which are found the operations of the corresponding row of Table (1).

Let $$G \mid A = [1, s'_2, s'_3 \ldots s'_{\frac{q}{r}}];$$

then if s'_ρ corresponds to β_ρ, $a_2\beta_\rho$, $a_3\beta_\rho$, etc.,
and s'_σ " " β_σ, $a_2\beta_\sigma$, $a_3\beta_\sigma$, etc.,

we have, if

$$\beta_\rho\beta_\sigma = a_\nu, \beta_{\tau'},$$
$$s'_\rho s'_\sigma = s'_{\tau'} \text{ in the group } G \mid A.$$

But $\beta_\rho\beta_\sigma = c_\mu\beta_\tau$; and since C is a subgroup of A, it follows that $a_\nu = c_\mu$, and $\tau' = \tau$; that is, since

$$\beta_\rho\beta_\sigma = a_\nu\beta_{\tau'} = c_\mu\beta_\tau,$$
$$s'_\rho s'_\sigma = s'_\tau \text{ in the group } G \mid A,$$

where s'_τ corresponds to β_τ, $a_2\beta_\tau$, $a_3\beta_\tau$, etc., of G.

Hence the quotient-groups $G \mid A$ and $B \mid C$ have the same multiplication table and are therefore *holoedrically* isomorphic. In like manner $G \mid B$ is *holoedrically* isomorphic with $A \mid C$.

Moreover, since A and B are maximal self-conjugate under G, $G \mid A$ and $B \mid C$ are simple.* Consequently $A \mid C$ and $B \mid C$ are simple, and therefore C is maximal self-conjugate under A and B.

These relations known, the proof, as given by Hölder,† that the factor-groups are constant for all the series of composition of a group, easily follows by induction.

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* Hölder, *Math. Annalen*, vol. 34, p. 37, § 9.
† Hölder, *Math. Annalen*, vol. 34, p. 37, § 10.