complex number arguments which is linear and homogeneous in each argument and is reversed in sign by the interchange of any two of its arguments as a linear alternate or simply an alternate. The author proposes to investigate the geometric relations of quaternion and higher complex numbers, and to determine the relations that exist between alternate identities and integrations through a given space and over the boundary of that space. An instance of such a connection has been already discussed by the author in the Proceedings of the Indiana Academy of Science, 1891.

A portion of the Tuesday afternoon session was devoted to a general discussion of the following topics: (1) *The accurate definition of the subject matter of mathematics;* (2) *The vocabulary of mathematics, the possibility of correcting and enriching it by coöperative action.*

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CONCERNING REGULAR TRIPLE SYSTEMS.

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§ 1.

Introduction. Definitions and notations.

A $\begin{array}{ll} k\text{-ad} & [l_1, \cdots, l_k] \\ k\text{-id} & \{l_1, \cdots, l_k\} \end{array}$

is an arrangement of k letters or elements $l_1 \cdots l_k$ in which the order $\begin{array}{l}\text{is not}\\ \text{is}\end{array}$ material. The letters are distinct.

A *triple system* Δ_t is an arrangement of t letters in 3-adic triples in such a way that every 2-adic pair appears exactly once in some triple of the triple system. There is no question of order of the triples of the triple system. t must have the form $t = 6m + 1$ or $t = 6m + 3$; t denotes always a number of such (say) *triple form.*

A triple system Δ_t is invariant under a certain (largest) substitution group G^t on its t elements; the Δ_t and the G^t belong each to the other; the G^t is a *triple group.*

In the definitive theory of triple systems and triple groups we must, among other things, determine whether a (any) particular group G^t of triple degree t is a triple group, and, if so, we must construct its various triple systems Δ_t.

A system Δ_t is *transitive* if its group G^t is transitive.

A transitive system Δ_t is *regular* if its group G^t contains a regular subgroup H^t_t of order t on the t elements of the Δ_t; the group H^t_t is a *group of regularity* of the *system* Δ_t; the system Δ_t has with respect to the group H^t_t the *regular aspect* $\Delta_t \mid H^t_t$. A regular group H^t_t is uniquely determined by the corresponding *abstract group* H_t.

Mr. Netto has exhibited, cyclic, or, as I say, *cyclic*-regular $\Delta_t(1°)$ for $t = 6m + 1 = p$ (p any prime) and (2°) for $t = 6m + 3 = 3p$ (p any prime of the form $p = 6k + 5$).

Mr. Heffter in the current number (vol. 49, part 1) of the *Mathematische Annalen*, has brought out clearly the "difference problems" underlying the problems of construction of the general cyclic-regular Δ_t for $t=6m+1$, $6m+3$. He then exhibits cyclic-regular $\Delta_t(3°)$ for $t=6m+1=12k+7 = 3p - 2$ (p any prime of the form $p = 4k + 3$ with the primitive root 2) and (4°) (by a slight modification of Mr. Netto's (2°) system) for $t = 6m + 3 = 3p$ (p any prime greater than 3).

Now a *cyclic* group H^t_t is the *simplest Abelian* or commutative group H^t_t.

In a paper, *Concerning Abelian-Regular Transitive Triple Systems*, forwarded to the *Annalen* three weeks ago, I have analyzed (*l. c.*, § 1) the general Abelian-regular Δ_t, and have exhibited (*l. c.*, (a) § 2, (b) §§ 3, 4) Abelian-regular Δ_t with respect to the abstract Abelian group H_t where

(a) t is any integer $t = 6m + 3$,

H_t is any Abelian group of order t having one invariant 3; or

(b) t is any integer $t = 6m + 1$ in which every prime factor of the form $p = 6k + 5$ enters an even number of times;

H_t is any Abelian group of order t in whose invariant-character every such prime enters always with the exponent 1.

The systems (a, b) are sweeping generalizations of the systems (2°, 4°; 1°) of Messrs. Netto and Heffter.

My analysis of the general Abelian-regular system Δ_t was so phrased as to admit of immediate application in the constructions (a, b). In this paper the general *regular triple system* Δ_t is subjected to a similar analysis (§ 2), on the basis of which a generalization of the system (a) is effected (§ 4).

§ 2.

Regular triple systems $\Delta_t \mid H_t^t$ *and the corresponding sextette separations* $\sigma_{m_1, m_2} \mid H_t$.

We denote the t elements of the group H_t and the t elements of the regular system $\Delta_t \mid H_t^t$ without confusion by the same notation—$A_1, \cdots, A_t$. The substitution group H_t^t is made up of the substitutions

$$(1) \qquad A_j = \begin{pmatrix} A_1, \cdots\cdots, A_i, \cdots\cdots, A_t \\ A_1 A_j, \cdots, A_i A_j, \cdots, A_t A_j \end{pmatrix} \qquad (j = 1, 2, \cdots, t).$$

The regular system $\Delta_t \mid H_t^t$ is invariant under the t substitutions A_j (1) of the H_t^t. Hence the Δ_t contains with the triple $[A_{i_1} A_{i_2} A_{i_3}]$ the t triples (not necessarily all distinct)

$$(2) \qquad [A_{i_1} A_j, A_{i_2} A_j, A_{i_3} A_j] \qquad (j=1, 2, \cdots, t).$$

Writing the triple or 3-ad $[A_{i_1} A_{i_2} A_{i_3}]$ having regard to the order of the letters as a 3-id $\{A_{i_1} A_{i_2} A_{i_3}\}$ we see that the corresponding sextette

$$(3) \qquad \sigma\{A_{i_1} A_{i_2} A_{i_3}\} = \begin{Bmatrix} A_{i_2} A_{i_3}^{-1}, & A_{i_3} A_{i_1}^{-1}, & A_{i_1} A_{i_2}^{-1} \\ A_{i_3} A_{i_2}^{-1}, & A_{i_1} A_{i_3}^{-1}, & A_{i_2} A_{i_1}^{-1} \end{Bmatrix}$$

is an invariant and indeed a characteristic invariant for the 3-idic triples

$$\{A_{i_1} A_j, A_{i_2} A_j, A_{i_3} A_j\}$$

of this set (2).

Such a sextette $\sigma\{A_{i_1} A_{i_2} A_{i_3}\}$ has the necessary and sufficient form

$$(4) \qquad \sigma \mid H_t = \begin{Bmatrix} B_1, & B_2, & B_3 \\ B_1^{-1}, & B_2^{-1}, & B_3^{-1} \end{Bmatrix}$$

where*

$$(5) \qquad B_1 B_2 B_3 = I, \quad B_i \neq I, \quad B_i \neq B_j^{-1} \quad (i, j = 1, 2, 3),$$

* The group H_t is of odd order t. Every triple has three distinct letters. Two triples having two letters in common have also the third of each in common. The sextette $\sigma \mid H_t$ with $B_1 B_2 B_3 = I$ belongs to the 3-idic triples

$$(I, B_3^{-1}, B_2), \quad (B_3, I, B_1^{-1}), \quad (B_2^{-1}, B_1, I).$$

From these remarks one draws conclusions (5, 6) of the text.

the B's being elements and the I the identity element of the group H_t. There are in all two types of sextettes $\sigma \mid H_t$:

$$(6) \qquad \begin{array}{l} (1^\circ)\ B_1 B_2 B_3 \textit{ are distinct;} \\ (2^\circ)\ B_1 = B_2 = B_3. \end{array}$$

According as the sextette $\sigma \mid H_t$ is $\sigma_1 \mid H_t$ or $\sigma_2 \mid H_t$ of the type 1° or 2°, it contains six or two distinct elements and the corresponding set of triples (2) contains t or $\frac{1}{3}t$ triples and is indeed a (tactical) configuration* $Cf_1 \begin{pmatrix} t & 3 \\ 3 & t \end{pmatrix}$ or $Cf_2 \begin{pmatrix} t & 1 \\ 3 & \frac{1}{3}t \end{pmatrix}$ regular with respect to the group H_t^t of substitutions A_j (1). The type 2° occurs only if t has the form $t = 6m + 3$.

The system $\Delta_t \mid H_t^t$ is the composition of m_1 configurations $Cf_1 \mid H_t^t$ of type 1° and m_2 configurations $Cf_2 \mid H_t^t$ of type 2°, with distinct triples. Here $tm_1 + \frac{1}{3}tm_2 = \frac{1}{6}t(t-1)$, $\therefore 3m_1 + m_2 = \frac{1}{2}(t-1) = 3m$ or $3m+1$. Hence we have

$$(7) \qquad t = \begin{matrix} 6m+1 \\ 6m+3 \end{matrix}, \quad (m_1, m_2) = \begin{matrix} (m, 0) \\ (m - m', 1 + 3m') \end{matrix} \quad (0 \leqq m' \leqq m)$$

Corresponding to and characteristic of this *configuration-separation* $Cf_{m_1, m_2} \mid H_t^t$ of the system $\Delta_t \mid H_t^t$ is a *sextette-separation* $\sigma_{m_1, m_2} \mid H_t$ of the $t-1$ elements A $(A \neq I)$ into m_1 sextettes $\sigma_1 \mid H_t$ and m_2 sextettes $\sigma_2 \mid H_t$ (in which repetitions of elements occur only within the individual sextettes $\sigma_2 \mid H_t$).

Conversely, with respect to any abstract group H_t of order $t = 6m+1$, $6m+3$ any such sextette-separation $\sigma_{m_1, m_2} \mid H_t$ of the $t-1$ elements A $(A \neq I)$ serves uniquely to define a regular triple system $\Delta_t \mid H_t^t$.

We consider as essentially the same the six sextettes $\sigma_1 \mid H_t$ derived from the two

* I use the general matrix notation for configurations introduced in I (The General Tactical Configuration: Definition and Notation) of my paper *Tactical Memoranda* I–III (*American Journal of Mathematics*, vol. 18, pp. 264–303, 1896).

$$(8) \qquad \left\{ \begin{matrix} B_1 \;, & B_2 \;, & B_3 \\ B_2^{-1}, & B_2^{-1}, & B_3^{-1} \end{matrix} \right\}, \quad \left\{ \begin{matrix} B_1^{-1}, & B_3^{-1}, & B_2^{-1} \\ B_1 \;, & B_3 \;, & B_2 \end{matrix} \right\}$$

by cyclical permutation of their columns, since they arise from the six 3-idic triples corresponding to one 3-adic triple.

If in a sextette $\sigma_1 \mid H_t$ (4) we interchange

$$B_i, B_i^{-1} \; (i = 1, 2, 3),$$

the new say *reciprocal* sextette is a $\sigma_1 \mid H_t$ if and only if $B_1^{-1} B_2^{-1} B_3^{-1} = I$. This happens, for instance, always if H_t is an Abelian group. Two reciprocal sextettes of type 1° are essentially distinct.

§ 3

Explicit exhibition of a sextette-separation $\sigma_{0,\,1+3m} \mid H_{t=3^k=6m+3}$, where H_t is any group of order $t = 3^k = 6m + 3$ whose elements not the identity are all of period 3.

Corresponding to the $1 + 3m$ pairs of reciprocal elements A $(A \neq I)$ of the group H_t we have the separation of those elements (of period 3) into $1 + 3m$ sextettes $\sigma_2 \mid H_t$ of type 2°.

For the (cyclid) Abelian $H_{t=3^k}$ generated by k generators each of period 3 this separation underlies the Abelian-regular $\Delta_3^{\,k} \mid H_{3^k}^{3^k}$ whose group is the linear group modulo 3. (Netto, *Substitutionentheorie*, pp. 224–234.)

§ 4

Explicit exhibition of 2^m sextette-separations $\sigma_{m,1} \mid H_{t=6m+3}$, where H_t is any group of order $t{=}6m{+}3$ having a self-conjugate element A_0 of period 3 and a subgroup K_{2m+1} of order $2m{+}1$ not containing A_0.

C denotes always an element of the subgroup K_{2m+1}. The $2m$ elements $C(C \neq I)$ separate uniquely into m pairs of reciprocal elements C, C^{-1}. If the $C_i (i = 1, \cdots, m)$ form a system of representatives of these m pairs, then so do the $C_i^2 (i = 1, \cdots, m)$.

The group K_{2m+1} extends by A_0 to the group $H_{t=6m+3}$. Since A_0 is a self-conjugate element in H_t, it is commutative with every element C. The elements A of H_t have the form C, CA_0, $CA_0^{\,2}$.

The separation $\sigma_{m,1} \mid H_t$ of the $t - 1 = 6m + 2$ elements $A(C \neq I)$ of the H_t consists of the one sextette

$$(1) \qquad \sigma_2 \mid H_t = \left\{ \begin{matrix} A_0, A_0, A_0 \\ A_0^2, A_0^2, A_0^2 \end{matrix} \right\} \qquad (A_0^3 = I)$$

and of the m sextettes $\sigma_1^{(C_i)} \mid H_t$ depending upon a representative system $C_i = C_1, \cdots, C_m$ of the m pairs of reciprocal elements $C (C \neq I)$,

$$(2) \qquad \sigma_1^{(C)} \mid H_t = \left\{ \begin{matrix} CA_0, & CA_0^2, & C^{-2} \\ C^{-1}A_0^2, & C^{-1}A_0, & C^2 \end{matrix} \right\},$$

or

$$(3) \qquad \sigma_1^{(C)} \mid H_t = \left\{ \begin{matrix} CA_0^2, & CA, & C^{-2} \\ C^{-1}A_0, & C^{-1}A_0^2, & C^2 \end{matrix} \right\}.$$

The two sextettes $\sigma_1^{(C)}$, $\sigma_1^{(C^{-1})}$ of (2) or (3) are essentially the same, while $\sigma_1^{(C)}$ (2) and $\sigma_1^{(C)}$ (3) are essentially distinct (reciprocals).

According to the choice of (2) or (3) for each pair C, C^{-1} we have in all 2^m sextette-separations $\sigma_{m,1} \mid H_t$ and so 2^m regular $\Delta_t \mid H_t^t$ for every abstract group H_t of the character in question. In particular, since every Abelian group H_t with one invariant 3 is such a group H_t, we have the Abelian-regular $\Delta_t \mid H_t$ (§ 1, a).

For still more general types of groups H_t of order $t = 6m + 3$ we may by suitable modification of the preceding process exhibit a sextette-separation $\sigma_{m,1} \mid H_t$. Thus, for example, for those H_t with the following properties: (1) the H_t has an element A_0 of period 3, (2) the group I, A_0, A_0^2 extends to the H_t by the identity and certain $2m$ extenders C by pairs reciprocal, (3) a certain representative system $C_i (i, = 1, \cdots, m)$ of these m pairs is invariant under transformation by A_0, (4) the system $C_i A_0 C_i A_0^2$ $(i = 1, \cdots, m)$ is a representative system of these m pairs. This type of group H_t contains the type previously discussed and also, for instance, the group H_{21} generated by two generators A_0, C subject to the generational relations

$$A_0^3 = I, \; C^7 = I, A_0 C = C^2 A_0.$$

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