In a similar way we see that φ (x) can have no complex root whose pure imaginary part is negative.

$x_1, x_2, \cdots, x_k$ are, therefore, all real. Suppose one of them were greater than e_n. Call this one (or, if there are more than one, the greatest of them) x_1. Then the above equation again involves a contradiction since no term is negative or zero.

In the same way we see that no root can be less than e_1.

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INFLEXIONAL LINES, TRIPLETS, AND TRIANGLES ASSOCIATED WITH THE PLANE CUBIC CURVE.

BY PROFESSOR HENRY S. WHITE.

(Read before the American Mathematical Society at the Meeting of February 26, 1898.)

THE configuration of the nine inflexions of a nonsingular plane cubic and the twelve lines containing them three-and-three would seem too well known to merit discussion. It is the uniform mode, in such compends as I have seen, to show first that every line joining two inflexions meets the cubic again in a third inflexion; second, that through the nine inflexions there must lie in all twelve such lines; and thirdly, that three lines can be selected which contain all nine inflexions. These three lines are termed an inflexional triangle, and the entire twelve are thought of as constituting four inflexional triangles. But there is another arrangement of the nine lines remaining after the erasure of one inflexional triangle, which I have not happened to find mentioned, which yet seems the easiest and most natural mode of access to the inflexional triangle itself.

It shall be presupposed known that there are nine inflexional points, and that every line joining two of them contains also a third. Select two inflexional points A, B, and any third C not collinear with the first two. Call these three an *inflexional triplet*. Join them by three lines, and produce BC, CA, AB to meet the cubic in a second inflexional triplet, in the points A_1, B_1, C_1 respectively.*

Repeating the process upon these three, determine a third triplet A_2, B_2, C_2. From these, determine similarly a fourth triplet. Since its points cannot be additional inflexions, nine having been included already; and since they cannot be the points of the second triplet (as is evident from the figure) unless certain inflexions coincide, they must be the

*It is easily seen that A_1, B_1, C_1, and again A_2, B_2, C_2, are not collinear.

points of the first triplet. Repetition of the process would lead us through the same period of three triplets. We have seen accordingly that: *The nine inflexional points of a plane cubic may be regarded as vertices of three triangles, each of which is inscribed in one and circumscribed about another, the three thus forming a closed series.*

Recalling the exact order followed in the construction of the diagram, we notice that it contains no line joining A_1A or A_1A_2, but that the three lines meeting in A_1 do contain all inflexional points except A, A_2. Hence $\overline{A_1A}$ must contain A_2, and similarly $\overline{B_1B}$ contains B_2, $\overline{C_1C}$ contains C_2. (Of course a *real* diagram cannot show this.) Hence *the three inflexional lines not appearing in the diagram contain all nine points, and constitute by definition an inflexional triangle.* The three other inflexional triangles appear easily, from the notation employed. They contain respectively the lines:

$$\left\{\begin{matrix}\overline{AB}\\\overline{A_1B_1}\\\overline{A_2B_2}\end{matrix}\right\}, \quad \left\{\begin{matrix}\overline{AC}\\\overline{A_1C_1}\\\overline{A_2C_2}\end{matrix}\right\}, \text{ and } \left\{\begin{matrix}\overline{BC}\\\overline{B_1C_1}\\\overline{B_2C_2}\end{matrix}\right\}.$$

Starting with any three inflexions, A, B, C, which do not lie in a straight line, we have found directly a closed chain of three triplets, and indirectly one (excluded) inflexional triangle. Each triplet of the three would evidently lead to the same closed chain, and to the exclusion of the same inflexional triangle. But of such triplets there exist in all $\frac{9\cdot8\cdot6}{1\cdot2\cdot3} = 72$; and of closed chains, $\frac{72}{3} = 24$, while there are but 4 inflexional triangles. Therefore each of the four is excluded by 6 different closed chains of triplets. And our diagram shows this; for the same diagram results if we start from any of the five other triplets lying on triangles having A for a vertex: AB_1C_1, ABC_2, AC_1B_2, ACB_2, AB_1C_2. Since the diagram exhibits the nine lines belonging to three inflexional triangles, the same facts might have been stated thus: *The lines which constitute three inflexional triangles of a plane cubic may be regarded in six different ways as constituting a closed series of three triangles, each inscribed in the preceding (and circumscribed about the following).*

We may add: *In these arrangements each side of one inflexional triangle occurs four times in a triplet with each side of every other inflexional triangle.* This would be plain also from the four inflexional lines that meet, *e. g.*, $\overline{AB}$ and $\overline{AC}$, in inflexional points other than A.

The nonsingular plane cubic is one of an indefinitely great series of "elliptic normal curves" in spaces of three dimensions, four dimensions, etc., respectively. On each of these there exists a configuration analogous to that of the inflexion-system of the cubic. All such are mere matters of course when the points of the curve are represented by values of an elliptic integral of the first sort. But they are no less easy of discovery by the immediate extension of the foregoing method; and the closed chains of four tetrahedra, five pentahedra, etc. appear to be novel and interesting objects for the geometric imagination. Further, by the application of elliptic parameters to these objects an extensive theory may be evolved, peculiar and not devoid of profit.

NORTHWESTERN UNIVERSITY,
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ON THE INTERSECTIONS OF PLANE CURVES.

BY PROFESSOR CHARLOTTE ANGAS SCOTT.

LINEA Ordinis (n) occurrere potest aliæ ejusdem Ordinis in punctis n^2. Proinde duæ Lineæ Ordinis (n) per eadem puncta n^2 transire nonnunquam possunt; adeoque puncta data quorum numerus est $\frac{1}{2}(n^2+3n)$ non sufficiunt ad Lineam Ordinis (n) ita determinandam ut unica sit curva quæ per ea data puncta duci possit: Cum vero coefficientes in æquatione generali ad Lineam Ordinis (n) sint $\frac{1}{2}(n^2+3n)$, patet si plura dentur puncta, Lineam Ordinis (n) per ea forsan duci non posse et Problema reddi posse impossibile. Sic novem puncta non adeo plene determinant Lineam Ordinis tertii ac quinque Lineam Ordinis secundi, decem tamen ad Lineam tertii Ordinis determinandam nimia sunt.

MACLAURIN, *Geometria Organica*, 1720; Sect. V, Lemma III, Corol: II; p. 137.

Ensuite je ferai voir le défaut, qui se trouve dans ces conséquences, qui consiste dans une fort subtile précipitation du raisonnement, laquelle n'etant pas si facile à découvrir, nous doit rendre extrémement circonspects, principalement dans les autres sciences, afin que nous ne nous laissions pas séduire par de semblables contradictions apparentes. Car,