and let $\dfrac{\partial^2 f(x, y)}{\partial y \partial x}$ denote $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f(x, y)}{\partial x}\right)$.

Let (x_0, y_0) be any point for which the conditions of the theorem are fulfilled and let the lines $x = a$, $x = b$, $y = c$, $y = d$ bound a region of the plane enclosing the point (x_0, y_0) and so small that the conditions stated are satisfied throughout the interior of the rectangle and on its boundary. Under these conditions we have

$$\int_c^d dy \int_a^b dx \, \frac{\partial^2 f(x, y)}{\partial x \partial y} = f(b, d) - f(b, c) - f(a, d) + f(a, c),$$

$$\int_a^b dx \int_c^d dy \, \frac{\partial^2 f(x, y)}{\partial y \partial x} = f(b, d) - f(a, d) - f(b, c) + f(a, c).$$

But, under the conditions of the theorem,

$$\int_a^b dx \int_c^d dy \, \frac{\partial^2 f(x, y)}{\partial y \partial x} = \int_c^d dy \int_a^b dx \, \frac{\partial^2 f(x, y)}{\partial y \partial x}.$$

Hence $$\int_c^d dy \int_a^b dx \left(\frac{\partial^2 f(x, y)}{\partial x \partial y} - \frac{\partial^2 f(x, y)}{\partial y \partial x} \right) = 0.$$

Now, if a function, continuous in the neighborhood of a point (x_0, y_0), is such that its integral, extended over *any* rectangle enclosing this point, is zero, it is readily seen that the function cannot be positive or negative at the point (x_0, y_0). Hence

$$\frac{\partial^2 f(x, y)}{\partial x \partial y} - \frac{\partial^2 f(x, y)}{\partial y \partial x} = 0$$

at the point (x_0, y_0). But this was any point, and the theorem is proved.

SOME OBSERVATIONS ON THE MODERN THEORY OF POINT GROUPS.

BY MISS FRANCES HARDCASTLE.

THE origins of the theory of point groups are to be found in Brill and Noether's classic memoir (see infra) published nearly twenty-five years ago, but it is only within the last fifteen years that systematic attention has been given to the subject by the Italian mathematicians, Segre, Bertini, Castelnuovo and others; in their hands a series of isolated problems bids fair to develop into an organic theory, a theory, moreover, which furnishes links in thought between subjects as apparently diverse as projective geometry and the analytical theory of elimination, while maintaining the initial, direct connection between the theory of higher plane curves and the theory of functions. In the following pages I have briefly indicated some of the converging lines of the German and Italian work; in the first section, by a discussion of certain of the technical terms; in the second section, starting from the so-called Riemann-Roch equations, by the suggestion of certain lines of enquiry which may prove useful in the classification of algebraic curves. The appended bibliography is divided into two parts; the first contains those memoirs which may be considered as fundamental, from the historical as well as from the purely technical point of view, to the theory of point groups, including those more especially concerned with linear systems of *plane* curves; the second includes memoirs on the theory of curves in ordinary and in hyperspace, more or less directly connected with the theory of point groups.

§ 1.

The geometry of a space of a given number of dimensions treats of the characteristic properties of the configurations existing in that space and formed by spaces of a smaller number of dimensions. The geometry on a plane curve, a one-dimensional space, deals, therefore, with configurations formed of points and studies the laws which they follow. In this connection the curve is called the *base curve*, the configurations are *point groups*. A point group which presents itself very naturally is that formed by the intersections of the base curve with any other specified plane curve, and among such special attention has been directed to those obtained by means of the so-called *adjoint curves*—curves, namely, which have an $(i-1)$-fold point at every i-fold point of the base curve (whether satisfying further conditions or not). In the discussions of such point groups, those intersections of the adjoint curve with the base curve which necessarily coincide with the i-fold points of the latter* are excluded. A point group in this portion of the theory is composed of some or all of the remaining points of intersec-

* The i-fold points of the base curve are often referred to as the *base points*, the remaining points of intersection of an adjoint curve with the base curve as the *moveable points of intersection*.

tion of an adjoint curve and the base curve. Of fundamental importance in these investigations is the *Theorem of Residuation,* which introduces the notions of *residual* and *coresidual point groups* and establishes a characteristic property of such point groups. The definitions are as follows: That point group G_Q of Q points, which together with a point group G_R of R points, makes up the whole number of intersections (always excluding the base points) of the base curve with an adjoint curve of arbitrarily assumed order, is said to be *residual to* G_R and vice versa; two point groups G_R, $G_{R'}$ are said to be *coresidual with respect to a third point group* G_Q when each is residual to G_Q; and it is noteworthy that this definition applies whether the orders of the adjoint curves which pass through the pairs of point groups G_Q, G_R and G_Q, $G_{R'}$ respectively, be the same or different; thus the coresidual point groups G_R, $G_{R'}$ do not necessarily contain the same number of points. The theorem in its simplest form is thus worded: *if the point groups* G_R, $G_{R'}, \cdots$ *are coresidual with respect to a point group* G_Q, *they are also coresidual with respect to every other point group* $G_{Q'}$, *residual to any one of their number.**

The proof of the theorem of residuation is such that it is capable of immediate extension by introducing, in the place of a single point group, the wider conception of a *system of point groups,* determined on the base curve by a system of adjoint curves. A system of point groups, namely, G_R, G_R, $\cdots$, is determined on the base curve by means of a system of adjoint curves, in which every curve is constrained to pass through a given point group G_Q, (G_R, $G_{R'}$, $\cdots$ form, in fact, a coresidual system in which $R = R' = \cdots$, numerically); or, we may say, this system of adjoint curves is determined by G_Q and determines G_R, $G_{R'}$, $\cdots$. The theorem of residuation then shows that, *if* $G_{Q'}$ *be any point group coresidual to* G_Q *with respect to* G_R, *the system of adjoint curves determined by* G_Q *is equivalent to that determined by* $G_{Q'}$, *the same system of point groups* G_R, $G_{R'}$, $\cdots$ *being determined on the base curve by either system of adjoint curves.* The equation of a system of adjoint curves involves one or more variable parameters, which may, without invalidating the truth of the theorem, enter in any specified degree; the most simple case, that of *linear systems,* however, is the only one which

*The nomenclature is due to Sylvester (cf. Salmon's Higher Plane Curves, 2d edit. (1873), p. 131), who proved the theorem for cubics; the notation, G_Q, was introduced by Brill and Noether in the memoir referred to above, viz.: "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie," *Math. Annalen.*, vol. 7 (1873).

has been at all fully worked out. And the importance of the theorem of residuation lies in the fact that the number r of linearly independent curves of a linear system of adjoint curves determined by *any* one of a coresidual system of point groups, G_Q, $G_{Q'}$, $\cdots$ is the same; this number r is therefore characteristic of a unique coresidual system of point groups, G_R, $G_{R'}$, $\cdots$ and may be affixed as an index to the symbol denoting any one of these point groups; *e. g.*, G^r_R. If it is not essential to fix the attention on any one point group of such a coresidual system, but rather on the linear system itself (of which the single point group is an element) *the symbol g^r_R is employed.*

This is the original, limited meaning of this symbol as introduced by Brill and Noether (*loc. cit.*). Later, in the Italian memoirs, the symbol is used to denote *any* system of point groups on a base curve, whether determined by adjoint curves or not and whether these adjoint curves, if employed, satisfy further conditions or not; it simply means that the system of point groups is r-ply infinite and that each element is composed of a group of R points on the base curve; *the order of the system g^r_R is R, the dimensions r.* And then, if g^r_R be a *linear* system (whether determined by adjoint curves or not) a distinction is drawn between "*complete*" *systems* (serie completa, Vollschaar) and "*partial*" *systems* (serie parziale, Teilschaar),* viz., a linear system is complete if it is not contained in another of the *same* order but of *higher* dimensions; partial, if it is so contained.

With reference to the relation between the values of R and r in a linear g^r_R, it can be proved† that, the deficiency of the base curve being a known quantity p, a g^r_R for which $R - r \leqq p - 1$ can always be determined on the base curve by means of a system of adjoint curves of order $n - 3$ (where n is the order of the base curve); and this holds whether R be less than or equal to the whole number of intersections (excluding the base points) of a curve of the system with the base curve; such systems of point groups are said to be *special systems.* It can also be proved that the linear system determined on the base curve by a system of adjoint curves of any order which satisfies no further condition than those due to its passage through the base points (and possibly through ordinary points on the base curve) is always complete; ‡ thus a special system *if* determined by such a system of adjoint curves (of order $n - 3$) is com-

* Cf. Segre, *Annali di matematica*, vol. 22, p. 65, (1894).

† Brill and Noether, *loc. cit.*, p. 278.

‡ Cf. Bertini, *Annali di matematica*, vol. 22, p. 9, (1894).

plete, but need not necessarily be so. Moreover it follows * from the theorem of residuation and from the nature of a special system that there are exactly p linearly independent adjoint curves of order $n-3$; each of these cuts the base curve in $2p-2$ points. In other words, then, there exists on the base curve a g_{2p-2}^{p-1} which is *special* and *complete;* this system has been termed the *canonical system* (serie canonica).†

The above definition of a special system is not that originally given by Brill and Noether; they defined‡ a special point group G_R^r as one for which

$$r > R - p + 1, \text{ i. e., } R - r < p - 1,$$

thus omitting the case in which $R - r = p - 1$. Lindemann has adopted this original definition in his edition of Clebsch's Geometrie§, published in 1876, but in Brill and Noether's report on the theory of algebraic functions ||, published in 1894, the wider definition is adopted and the distinction between the two cases is effected by the introduction of the term *extraordinary point group* (ausgezeichnete Gruppe) to designate any point group which presents *fewer* conditions to a given system of curves (whether adjoint curves, or not) than the number of points it contains; a point group for which $R - r < p - 1$, is *extraordinary with respect to the linear system of adjoint curves of order* $n - 3$, it is a particular case of an extraordinary point group, being, in fact, an *extraordinary, special* point group; a point group for which $R - r = p - 1$, on the other hand, is *special* but not extraordinary. The Italian mathematicians use the wider definition, sometimes in the form: g_R^r *is a special point group if* $R - r \leqq p - 1$ ¶; sometimes in the equivalent form: g_R^r *is a special point group if it can be determined on the base curve by a linear system of adjoint curves of order* $n - 3$**. Moreover, instead of marking out a given point group as extraordinary *with respect to a given system of curves*, they say that a given system of curves is *regular* (regolare)†† *with respect to a given point group* when the conditions which every curve of the system must satisfy in order to pass through this point group are all independent, and if the system is *not* regular, *i. e.*, if the conditions

* Brill and Noether, *loc. cit.*, p. 280. Cf. also Bertini, *loc. cit.*, p. 17.

† Segre, *loc. cit.*, p. 118.

‡ *loc. cit.* p. 280.

§ p. 699.

|| *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 3, pp. 542, 543.

¶ Segre, *loc. cit.*, p. 15.

** Bertini, *loc. cit.*, p. 111.

†† Castelnuovo, *Mem. d. Reale Accad. di Torino*, vol. 42, (1891).

are not all independent (in the German terminology, *if the given point group is extraordinary with respect to the given system of curves*) the number of conditions which follow from the rest is called the *excess* (sovrabbondanza)* *of the system of curves.*

In Brill and Noether's paper, the main object of which was the algebraic formulation of theorems known to be true from the theory of functions, point groups, as such, only present themselves incidentally and attention is almost exclusively directed to the special systems, and, moreover, to complete special systems. It was there shown for the first time that complete special systems of point groups always present themselves in pairs in such a manner that the existence of one system carries with it the existence of the complementary system. Their statement of this theorem is as follows: *if through a point group* G_Q^q *on the base curve where* $q = Q - p + 1 + r$ *(r being a positive integer less than* $p - 1$) *an adjoint curve of order* $n - 3$ *be passed, it will intersect the base curve in* $2p - 2 - Q = R$ *more points which again belong to a special system* g_R^r, *the value of r being* $R - p + 1 + q$ *derived from the equation above.* Brill and Noether call this theorem the *Riemann-Roch Theorem* but Klein points out that the *Brill-Noether Theorem of Reciprocity* would be the more appropriate title. The equations involved may be written

$$Q + R = 2p - 2, \qquad (i)$$
$$Q - R = 2(q - r), \qquad (ii)$$

and are usually referred to in this form.†

§ 2.

Two problems present themselves with reference to complete special systems of point groups regarded in connection with equations (*i*) and (*ii*).

(*a*) On a given base curve to enumerate all possible combinations of values of Q, q—whence R, r are known—and to show how each such g_Q^q is determined.

* Castelnuovo, *Ibid.*, p. 9. F. S. Macaulay in a paper on "Point Groups in Relation to Curves" (*Lond. Math. Soc. Proc.*, vol. 26, pp. 495–544, 1895), defined *excess* with reference to the point group not to the system of curves (p. 499) but, with care, no ambiguity need arise from this; Professor Klein in his lectures on Riemann's surfaces delivered in Göttingen, in 1891–92, employed the term *Ueberschuss*; it is used with reference to the *point group*, but turns out to be *greater by unity than the excess*, owing to the way in which it is originally defined (see lithoraphed lectures on Riemann'sche Flächen, I., p. 108; II., pp. 76, 77).

† *loc. cit.*, I., p. 108.

(β) To find on which base curves any given special system g^q_Q—and hence also its known complementary system g^r_R—can be determined and how.

By saying that a base curve is *given* is meant, not only that n and p are known, but also what particular combination of double points and other (ordinary) higher singularities produce the given value of p. And to "show how" means to decide the geometrical laws which govern every point group of the g^q_Q, *e. g.*, that 3 points of every G^q_Q lie on a straight line, that 2 points of every G^q_Q and a singularity of the base curve lie on a straight line, etc.

In the first place, it is necessary to establish a limitation to the possible combinations of values of Q and q, since equations (i) and (ii), for a given value of p, admit of an infinite number of such combinations. The fact that the questions involved in a one-one transformation of the base curve were of primary importance in Brill and Noether's paper led them in the second part of their memoir to deal exclusively with those base curves which are said to be "general" for a given deficiency. And the necessary limitation on the values of Q and q for a special system g^q_Q on such a base curve is found to be $p-(q+1)(r+1)\geq 0$. But if we regard the problem as formulated in (α) we do not necessarily confine our attention to "general" base curves and a limitation to the values of Q, q may exist, which, *while including all combinations permitted by Brill and Noether's inequality, permit others which theirs excludes.* Such a limitation was first established in the theory of abelian functions by Clifford,* if $Q\geq 2q$. It can be found without reference to abelian functions as follows: †

The Theorem of Reciprocity shows that by constraining the $(p-1)$-ply infinite system of adjoint curves of order $n-3$ to pass through the Q points of any G^q_Q belonging to a given g^q_Q, we determine on the base curve a g^r_R, and that by constraining the $(p-1)$-ply infinite system of adjoint curves of order $n-3$ to pass through the R points of any G^r_R of this g^r_R we determine on the base curve the same g^q_Q initially given. If we add the degrees of freedom of the element G^q_Q to the degrees of freedom of the system g^r_R determined by it, the sum cannot *exceed* the number of degrees of freedom of the system of adjoint curves of order $n-3$ by means of which both the g^q_Q and the g^r_R are determined, *i. e.*, $q+r\leq p-1$. And by equations (i) and (ii) this is

* "On the Classification of Loci," *Phil. Trans.*, 1878. (Math. Papers, p. 329.)

† Cf. also Bertini: *Annali di matematica*, vol. 22, p. 20.

equivalent to $Q - 2q \geq 0$. Now, since $Q = p - 1 + q - r$, $Q - 2q = p - (q + 1)(r + 1) + qr$ and hence Clifford's theorem permits all combinations of values of Q and q included in Brill and Noether's formula and also permits other combinations for which $p - (q + 1)(r + 1)$ is negative, provided this quantity is numerically less than qr. For instance, if $p = 6$, $g_3^1\, g_7^3$ makes $p - (q + 1)(r + 1)$ negative, but this pair of special systems can be found on a sextic *with a triple point and a double point*, for it is determined by the degenerate system of adjoint cubics consisting of a straight line through the triple point, cutting the base curve in a G_3' and a conic through the triple point and the double point cutting the base curve in a G_7^3. Again, if $p = 7$, $g_4^1\, g_8^3$ makes $p - (q + 1)(r + 1)$ negative, but, nevertheless, exists on a sextic with 3 double points, being determined by a straight line through one double point and a conic through the remaining two.

This restriction, $Q \geq 2q$, applies to every base curve, but on a *given* base curve, it may be possible to find further restrictions of a similar character, viz., $Q \geq xq$, where x has some value other than 2. There is, however, a unique character about the value $x = 2$, for this reason:

let $$Q = xq + k, \quad (k \gtreqless 0);$$

then since $$r = p - 1 + q - Q,$$

$$R - xr = 2p - 2 - Q - x(p - 1 + q - Q)$$

$$= (2 - x)(p - 1 - Q) + k.$$

Thus, if $x = 2$, $R \gtreqless xr$ according as $Q \gtreqless xq$, but if $x > 2$, this is only true when $Q = p - 1$; if $x > 2$ and $Q < p - 1$, R *must* be $< xr$, *if* $Q \leq xq$; the other possible cases, however, give no *necessary* limits to the values of R dependent on the values of Q.

As a particular example of such a restriction on a particular base curve, it can be shown* that no pair of complementary special systems g_Q^q, g_R^r *for which* $Q < 3q$ *and* $R < 3r$ can exist on a curve of the nth degree with an $(n - 3)$-fold point, T triple points and $2n - 5 - p - 3T$ double points.

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* Hardcastle: *Lond. Math. Soc. Proc.*, vol. 29.

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NOTE ON CONTACT TRANSFORMATIONS.

BY PROFESSOR EDGAR ODELL LOVETT.

IN a paper "Ueber die geodätische Krümmung der auf einer Fläche gezogenen Curven und ihre Aenderung bei beliebiger Transformation der Fläche" in the Zeitschrift für Mathematik und Physik, Vol. XXXVII., 1892, Dr. Mehmke, Schlömilch's successor as one of the editors of that journal, derives some interesting theorems relative to the change in the geodesic curvature of a curve on a surface when the latter is subjected to point transformations. Generalizations of these theorems are sought and are introduced by the following paragraph: "Die obigen Sätze erfahren eine Erweiterung und finden zugleich ihren natürlichen Abschluss beim Uebergange zu Transformationen, welche aus der Anwendung des Lie'schen Begriffes der Berührungstransformationen auf die Flächentheorie hervorgehen. Wir können jedem Linienelement auf einer gegebenen Fläche ein bestimmtes Linienelement auf einer anderen gegebenen Fläche so zuordnen, dass je zwei, im Sinne des Herrn Lie vereinigt liegenden Elementen wieder zwei vereinigt liegende Elemente entsprechen. Dann wird jede auf der ersten Fläche gezogene Curve sich in eine Curve (in besonderen Fällen auch in einen Punkt) der zweiten Fläche verwandeln und zwei sich berührende Curven werden im Allgemeinen in zwei sich ebenfalls berührende Curven übergehen.* Die punktweise Transformation einer Fläche ist hierin als besonderer Fall enthalten."

This is correct if the footnote and last sentence be omitted. These call for modification in order to correct misappre-

* "Derartige Transformationen von Flächen sind meines Wissens noch nicht untersucht worden. Sie lassen sich, wie leicht zu sehen ist, den räumlichen Berührungstransformationen keineswegs unterordnen."