demonstration* that all the roots of the polynomial solution are real, the reader is referred to an article by Bôcher in the April number of the BULLETIN. The method which he has there employed I shall make use of to prove that the roots of the accessory polynomial φ are likewise real. Let P denote the polynomial solution and $x_1, \cdots, x_{n-1}$ the roots of its derivative which are, of course, real. If P be substituted in the differential equation and x be placed equal to a root a of φ, we get

$$P''(a) + \left(\frac{1-\lambda_1}{a-e_1} + \cdots + \frac{1-\lambda_r}{a-e_r}\right)P'(a) = 0,$$

or dividing by $P'(a)$,

$$\frac{1}{a-x_1} + \cdots + \frac{1}{a-x_{n-1}} + \frac{1-\lambda_1}{a-e_1} + \cdots + \frac{1-\lambda_r}{a-e_r} = 0.$$

If now a is an imaginary root $p + qi$ for which q is positive, the pure imaginary part of each fraction will have a negative sign. The equation therefore involves a contradiction. Hence

VIII. *The roots of the accessory polynomial φ of the differential equation* (8) *for a Stieltjes polynomial are all real and included between the two extreme singular points, e_1 and e_r.*

WESLEYAN UNIVERSITY,
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NOTE ON STOKES'S THEOREM IN CURVILINEAR CO-ORDINATES.

BY PROFESSOR ARTHUR GORDON WEBSTER.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

THE expression for the curl of a vector point-function, when required in terms of orthogonal curvilinear coördinates, is usually obtained by direct transformation from their values in rectangular coördinates. The proof of Stokes's theorem given in my Lectures on electricity and magnetism, due to Helmholtz, can be easily adapted to curvilinear coördinates so as to prove the theorem independently of rectangular coördinates.

Let P_1, P_2, P_3 be the projections of a vector P on the

* The proof given by Stieljes in the sixth volume of the Acta Mathematica is based upon mechanical considerations.

varying directions of the tangents to the coördinate lines at any point. Let the projections in the same directions of the arc ds of a curve connecting two points A and B be ds_1, ds_2, ds_3. The theorem concerns the line integral of the resolved tangential component of the vector along the given curve:

$$I = \int_A^B P \cos(P, ds) ds$$

$$= \int_A^B (P_1 ds_1 + P_2 ds_2 + P_3 ds_3).$$

But in terms of the curvilinear coördinates ρ_1, ρ_2, ρ_3 we have

$$ds_1 = \frac{d\rho_1}{h_1}, \quad ds_2 = \frac{d\rho_2}{h_2}, \quad ds_3 = \frac{d\rho_3}{h_3},$$

where

$$h_s^2 = \left(\frac{\partial \rho_s}{\partial x}\right)^2 + \left(\frac{\partial \rho_s}{\partial y}\right)^2 + \left(\frac{\partial \rho_s}{\partial z}\right)^2 \qquad (s = 1, 2, 3).$$

Let us now make an infinitesimal transformation of the curve, so that the transformed curve shall lie on a given surface containing A and B, and shall itself pass through those points. Then the change in the integral due to the infinitesimal changes $\delta\rho_1$, $\delta\rho_2$, $\delta\rho_3$ in the coördinates is

$$\delta I = \delta \int \left(\frac{P_1}{h_1} d\rho_1 + \frac{P_2}{h_2} d\rho_2 + \frac{P_3}{h_3} d\rho_3 \right)$$

$$= \int \Big[\delta \left(\frac{P_1}{h_1}\right) d\rho_1 + \delta \left(\frac{P_2}{h_2}\right) d\rho_2 + \delta \left(\frac{P_3}{h_3}\right) d\rho_3 + \frac{P_1}{h_1} d\delta\rho_1$$

$$+ \frac{P_2}{h_2} d\delta\rho_2 + \frac{P_3}{h_3} d\delta\rho_3 \Big].$$

The last three terms can be integrated by parts, giving

$$\int_A^B \frac{P_s}{h_s} d\delta\rho_s = \frac{P_s}{h_s}\delta\rho_s \Big|_A^B - \int \delta\rho_s d\left(\frac{P_s}{h_s}\right) \quad (s = 1, 2, 3),$$

and, the integrated terms vanishing at the limits,

$$\delta I = \int \Big[\delta \left(\frac{P_1}{h_1}\right) d\rho_1 + \delta \left(\frac{P_2}{h_2}\right) d\rho_2 + \delta \left(\frac{P_3}{h_3}\right) d\rho_3 - d\left(\frac{P_1}{h_1}\right) \delta\rho_1$$

$$- d\left(\frac{P_2}{h_2}\right) \delta\rho_2 - d\left(\frac{P_3}{h_3}\right) \delta\rho_3 \Big].$$

Performing the operations denoted by δ and d, six of the eighteen terms cancel, and there remain the terms

$$\delta I = \int \Big[(\delta\rho_2 d\rho_3 - \delta\rho_3 d\rho_2) \left\{ \frac{\partial}{\partial\rho_2}\left(\frac{P_3}{h_3}\right) - \frac{\partial}{\partial\rho_3}\left(\frac{P_2}{h_2}\right) \right\}$$

$$+ (\delta\rho_3 d\rho_1 - \delta\rho_1 d\rho_3) \left\{ \frac{\partial}{\partial\rho_3}\left(\frac{P_1}{h_1}\right) - \frac{\partial}{\partial\rho_1}\left(\frac{P_3}{h_3}\right) \right\}$$

$$+ (\delta\rho_1 d\rho_2 - \delta\rho_2 d\rho_1) \left\{ \frac{\partial}{\partial\rho_1}\left(\frac{P_2}{h_2}\right) - \frac{\partial}{\partial\rho_2}\left(\frac{P_1}{h_1}\right) \right\} \Big]$$

Now the changes $\delta\rho_2$, $d\rho_2$, $\delta\rho_3$, $d\rho_3$ in the coördinates correspond to distances

$$\frac{\delta\rho_2}{h_2}, \quad \frac{d\rho_2}{h_2}, \quad \frac{\delta\rho_3}{h_3}, \quad \frac{d\rho_3}{h_3},$$

measured along the coördinate lines, and the determinant of these distances

$$\frac{1}{h_2 h_3}\{\delta\rho_2 d\rho_3 - d\rho_3 \delta\rho_2\}$$

is equal to the area of the projection on the surface ρ_1 of the infinitesimal parallelogram swept over by the arc ds during the transformation. Calling this area dS, and its normal n, we have

$$\frac{1}{h_2 h_3}(\delta\rho_2 d\rho_3 - \delta\rho_1 d\rho_3) = \cos(nn_1) dS, \quad \text{etc.}$$

Now, repeating the transformation so that the given curve 1 is transformed into a second given curve 2 joining AB, the total change in the line integral is represented by the surface integral over the surface lying between the curves

$$\int \delta I = I_2 - I_1$$

$$= \int\int \Big[h_2 h_3 \left\{ \frac{\partial}{\partial\rho_2}\left(\frac{P_3}{h_3}\right) - \frac{\partial}{\partial\rho_3}\left(\frac{P_2}{h_2}\right) \right\} \cos(nn_1)$$

$$+ h_3 h_1 \left\{ \frac{\partial}{\partial\rho_3}\left(\frac{P_1}{h_1}\right) - \frac{\partial}{\partial\rho_1}\left(\frac{P_1}{h_3}\right) \right\} \cos(nn_2)$$

$$+ h_1 h_2 \left\{ \frac{\partial}{\partial\rho_1}\left(\frac{P_2}{h_2}\right) - \frac{\partial}{\partial\rho_2}\left(\frac{P_1}{h_1}\right) \right\} \cos(nn_3) \Big] dS.$$

But the difference of the line-integrals $I_2 - I_1$ is the line-integral around the closed contour 21, so that we have the line-integral of the tangential component of the vector P around the closed contour proved equal to the surface-integral, over a surface bounded by the contour, of the normal component of a vector Ω whose components are

$$\omega_1 = h_2 h_3 \left\{ \frac{\partial}{\partial \rho_2}\left(\frac{P_3}{h_3}\right) - \frac{\partial}{\partial \rho_3}\left(\frac{P_2}{h_2}\right) \right\},$$

$$\omega_2 = h_3 h_1 \left\{ \frac{\partial}{\partial h_3}\left(\frac{P_1}{h_1}\right) - \frac{\partial}{\partial \rho_1}\left(\frac{P_3}{h_3}\right) \right\},$$

$$\omega_3 = h_0 h_2 \left\{ \frac{\partial}{\partial \rho_1}\left(\frac{P_2}{h_2}\right) - \frac{\partial}{\partial \rho_1}\left(\frac{P_1}{h_1}\right) \right\}.$$

The vector Ω is called the *curl* of P.

ON THE STEINER POINTS OF PASCAL'S HEXAGON.

BY DR. VIRGIL SYNDER.

THE proof given by v. Staudt* of the conjugate nature of M, N with regard to the conic for which M, N are associated Steiner points is perhaps rigorous, but unnecessarily long, and the most important statement† is only proved for the particular case in which the two triads of points defining the hexagon are linearly perspective.

He gives a second proof in article 8 of the same paper which is much shorter, but involves imaginary elements.

The following proof is much more simple and direct than either, and shows clearly which of Steiner's points are associated as "Gegenpunkte."

Let A_1, A_2, A_3 and B_1, B_2, B_3 be two triads of points lying on the same conic; these points can be made projective in six ways, namely

$$\begin{pmatrix} A_1 A_2 A_3 \\ B_1 B_2 B_3 \end{pmatrix} \quad \begin{pmatrix} A_2 A_3 A_1 \\ B_1 B_2 B_3 \end{pmatrix} \quad \begin{pmatrix} A_3 A_1 A_2 \\ B_1 B_2 B_3 \end{pmatrix}$$

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* Ueber die Steiner'schen Gegenpunkte * * *, *Crelle's Journal*, vol. 62.

† "Weil ferner P, Q harmonïsch getrennt sind durch M und seine Polare * * *."