NOTE ON THE GENERALIZATION OF POINCARÉ AND GOURSAT'S PROOF OF A THEOREM OF WEIERSTRASS'S.

BY PROFESSOR W. F. OSGOOD.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, Mass., August 19, 1898.)

WEIERSTRASS* enunciated the theorem that to an arbitrary (single-leaved) continuum A there correspond single-valued functions, analytic at every point of A, but not capable of being continued beyond the boundary of A. The first published proof of this theorem appears in a memoir by Runge,† in which the author considers the problem in its most general form and constructs a proof by methods so valuable in themselves that the paper is an important contribution to the general theory of functions. As early, however, as 1881 Poincaré‡ and almost simultaneously with him, but independently,§ Goursat|| had devised a proof for the case that the continuum is bounded by a curve that has curvature everywhere,—at least, this requirement is made explicitly by Poincaré, and Goursat needs a part of it. The two authors thus assume the theorem with virtually the same restrictions, and their proofs, which are identical in substance, differ only slightly in form. At that time the subject of Cantor's sets of points was new to mathematicians,—in fact, this very problem and problems of a similar character treated by Mittag-Leffler¶ contributed largely toward making Cantor's theory known by showing some of its applications in the general theory of functions,—and so this proof of Poincaré and Goursat failed to receive the generalization of which its nature ren-

* *Berliner Monatsberichte*, Aug., 1880; Zur Functionenlehre, 1886, p. 92; Mathematische Werke, vol. 2, p. 223.

† "Zur Theorie der eindeutigen analytischen Functionen," *Acta Math.*, vol. 6 (1885).

‡ "Sur les fonctions à espaces lacunaires," *Acta soc. sci. Fennicæ*, vol. 12, Helsingfors; reprinted, under the same title, with additional matter, in the *Amer. Journ.*, vol. 14 (1892). Cf. also Hermite, Cours d'Analyse, 3d ed., p. 157, or 4th ed., p. 171.

§ Cf. second reference under ||.

|| *Comptes Rendus*, 13 March. 1882; the proof there given is reproduced in greater detail in a paper entitled, "Sur les fonctions à espaces lacunaires," *Bull. d. Sci. Math.*, vol. 22 (1887).

¶ "Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendente;" *Acta Math.*, vol. 4 (1884).

ders it readily capable. The object of this note is to give this generalization.

It may be remarked that Mittag-Leffler had the necessary means at his disposal for the proof of Weierstrass's theorem in his memoir of 1884 above referred to. Let Q be an arbitrary, isolated set of points, infinite in number, and let Q' be the derivative of Q. Let Q' form the boundary of a continuum A, consisting of but one piece, within which the points Q lie. Mittag-Leffler showed that there exists a single-valued function, analytic within A, which vanishes at each point of Q, and nowhere else, and which therefore has in each point of Q' an essential singularity. In order to obtain Weierstrass's theorem from this result it is sufficient to show that, an arbitrary continuum A, consisting of a single piece, being given, it is always possible to choose an isolated set of points Q, lying within A, in such a manner that the derivative, Q', coincides with the boundary of A. This theorem had been proved by Bendixson,* and Mittag-Leffler, in considering a theorem relating to the approximate representation of a given function, refers to this paper.† But that which is essential in Weierstrass's theorem is that it is an *existence theorem*—a continuum A is arbitrarily given and there exist functions having A as their domain of definition. This point of view, for the most general case, is lacking in Mittag-Leffler's paper. Mittag-Leffler nowhere refers to Weierstrass's theorem in its general form.

Let C be any set of points whatever forming the complete boundary of a continuum A lying in the x-plane, and let $x = \infty$ be an interior point of A. A set of points b_0, b_1, $\cdots$, each belonging to C, shall now be chosen in such a manner that each point of C not itself a b-point shall have b-points clustering about it, and that furthermore, if these are taken as the b-points of Poincaré's formula (1), p. 203,‡ the proof there given that the function $\varphi(x)$ cannot be continued beyond A will hold with certain modifications to be pointed out.

Let the x-plane be divided up into squares, 2^{-n} on a side, by lines drawn parallel to the axes of reals and pure imaginaries, these axes themselves belonging to these lines. In a square containing points of A and, in its interior or on its boundary, a point of C, let a point x' of A be

* "Un théorème auxilaire de la théorie des ensembles;" *Bihang till Kongl. Svenska Vet. Ak. Handlingar*, vol. 9, No. 7 (1884).

† l. c., p. 45.

‡ *Amer. Jour.*, l. c.

chosen in such a manner that its distance from some points of C is not greater than its distance from the boundary of the square. About x' as center draw a circle so small that it contains no points of C, and let its radius grow till one or more points of C lie on its circumference, but none inside it. Choose one of these points of C as a point b. It will lie within or on the boundary of the square, since the circle does not extend beyond the boundary of the square. The number of squares corresponding to a given value of n, in which such b-points lie, is finite.

Now begin with $n = 1$ and denote the corresponding b-points, taken in any order, by $b_0, b_1, \cdots, b_{r_1-1}$. Next, let $n = 2$. In each of those new squares that contain points of A and C, but no one of the points $b_0, \cdots b_{r_1-1}$, mark a new point b as above defined, and denote these new b-points, taken in any order, by $b_{r_1}, b_{r_1+1}, \cdots, b_{r_2-1}$. And so on indefinitely. Then every point of C not itself a b-point has b-points clustering about it.

Form Poincaré's function :

$$\sum_{n=0}^{\infty} \frac{A_n}{x - b_n} = \varphi(x),$$

where the series of constants $\sum_{n=0}^{\infty} A_n$ converges absolutely. Let x_0 be any point of A, R the radius of that circle about x_0 as center that contains at least one point of C on its circumference, but no point of C in its interior. Then Poincaré's analysis applies, without modification, to the proof that $\varphi(x)$ is analytic at least within this circle. The special case that $x_0 = \infty$ presents no difficulty.

Passing now to the proof that $\varphi(x)$ is not analytic throughout a larger circle about x_0, Poincaré's assumption that, to begin with, x_0 shall lie on a normal to C in a point b_k shall be modified as follows : To the point b_k as defined in the present note there corresponded a point x' of A such that the circle about x' passing through b_k contained no point of C in its interior, but possibly an infinite number of points of C on its circumference. As point x_0 we will therefore choose a point lying on the line joining x' with b_k and situated between these points. The circle described with this point x_0 as center and passing through b_k will contain no point of C in its interior and only one point, namely, b_k, on its circumference. To the power series

$$-\varphi(x) = \sum_{q=0}^{\infty} B_q (x - x_0)^q$$

representing $-\varphi(x)$ within this circle all of Poincaré's analysis applies without modification. Hence this circle is the true circle of convergence for this series.

Finally, for the case that x_0 is any point of A, Poincaré's reasoning, with the modification just given, still holds, and the theorem is thus established that $\varphi(x)$ is analytic in A, but cannot be continued beyond A.

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SUPPLEMENTARY NOTE ON A SINGLE-VALUED FUNCTION WITH A NATURAL BOUNDARY, WHOSE INVERSE IS ALSO SINGLE-VALUED.

BY PROFESSOR W. F. OSGOOD.

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IN the June number of the BULLETIN I gave an example of a single-valued function with a natural boundary, the inverse of which is also single-valued. The function employed was the following:

$$f(z) = z + \frac{z^{a+2}}{(a+1)(a+2)} + \frac{z^{a^2+2}}{(a^2+1)(a^2+2)} + \cdots,$$

where a is a positive integer greater than unity. This function is continuous within and on the boundary of the unit circle, is analytic within this circle, and cannot be continued analytically beyond it.

I am indebted to Professor Hurwitz for an exceedingly simple proof of the principal theorem of my note, namely, that the inverse function is single-valued. The point to be established is that, z, z' being any two distinct points within or on the unit circle,

$$f(z) \neq f(z').$$

This follows at once by the application of a method employed by Professor Fredholm* to show that the inverse of the function

* *Cf.* Verhandlungen des ersten internationalen Mathematiker-Kongresses in Zürich vom 9. bis 11. August 1897; herausgegeben von Dr. Ferdinand Rudio, Professor am eidgenössischen Polytechnikum; Teubner, 1898; p. 109.