REPORT ON THE THEORY OF PROJECTIVE INVARIANTS: THE CHIEF CONTRIBUTIONS OF A DECADE.

BY PROFESSOR H. S. WHITE.

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Introduction.

IF we find it useful to distinguish short periods in the development of a science, the theory of invariants may easily enough be considered to have passed a milestone in 1887. In that year was published the second part of Gordan's Vorlesungen über Invariantentheorie. The plan of this work was dominated by the intent to expound and exemplify worthily the famous Gordan theorem on the finiteness of the form system of one or more binary forms. Gordan had announced and proven this theorem of fundamental importance in 1868,* and had since that time simplified his methods at least twice; and his was still in 1887, with one exception, the only current proof of the theorem. The two proposed by Jordan† and Sylvester‡ seem to have been not enough simpler to secure currency. The statement is, in briefest form, this: *For every binary form there is a finite system of covariants, in terms of which all other covariants, infinite in number, can be expressed rationally and integrally.* Without recalling here the details of the argument, we may characterize it as depending altogether upon the nature of the operations which generate covariants.

The one exception, just referred to, was a radically new method devised by Mertens, published in vol. 100 of the *Journal für reine und angewandte Mathematik.* By inductive process, assuming the theorem true for any given set of forms, he proves that it must still hold true when the order of one of the forms is increased by a unit. This method is deserving of attentive consideration, by virtue of its simplicity and power as shown in this first application, and even more on account of the strong probability that it might have been so extended as to prove the corresponding theo-

* *Crelle*, vol. 69.
† *Liouville's Journal*, 3d series, vol. 2 (1876), p. 122.
‡ *Proc. Lond. Math. Soc.*, vol. 27 (1878), p. 11–13.

rem for systems of ternary forms, and so in due course for forms in four, five, or any number of homogeneous variables.

The extension of Gordan's theorem to forms in more than two variables had not been achieved in the twenty years during which it had been eagerly awaited as an imminent possibility. For particular cases it was known to be true. Already in the first volume of the *Mathematische Annalen* (1869) Gordan had given out the complete system of ground forms concomitant to the ternary cubic : the system of a quadric in any number of variables was known, and the systems of two and of three ternary quadrics had been worked out by Gordan* and Ciamberlini† respectively. For a special ternary quartic :

$$f_4 = x_1^3x_2 + x_2^3x_3 + x_3^3x_1,$$

Gordan established a finite system of ground forms, 54 in number.‡ For further forms or sets of forms the systems had not been computed, nor was there any known proof that Gordan's method would terminate in a finite number of steps. Here lay the chief obstacle to further progress in the theory.

Two other problems of a general nature were plainly in need of study ; the enumeration of covariants of given weight and of given order in three or more variables, and the construction of a systematic theory of syzygies. Why the former of these had not been completely solved it is hard to say. Sylvester's papers on the binary problem were practically concluded by Franklin's résumé in the third volume of the *American Journal of Mathematics* (1880), and these were obviously the model for subsequent investigation. And as to the latter, syzygies had been studied in connection with particular forms since the appearance of Cayley's second memoir on quantics (1856).§

It may fairly be claimed that the past decade has seen the solution of these three important problems. In many other points has the theory of invariants received valuable contributions, and in these three there is, of course, an immense amount of work needed in order to thoroughly possess the conquered territory. Postponing restrictions and qualifications and matters of secondary importance, let us consider briefly what has been done upon these three main questions.

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*See Clebsch-Lindemann, Vorlesungen über Geometrie, I. (1876), p. 288 ; or Osgood in *Amer. Journ. of Math.*, vol. 14 (1892), pp. 262–273.

†*Battaglini's Giornale*, vol. 24 (1886), pp. 141–157.

‡*Math. Annalen*, vol. 17, pp. 217–233.

§Collected Works, vol. 2, p. 250–275.

§ 1. *Mertens' Demonstration and Hilbert's First Proof of Gordan's Theorem.*

It was evident that there was some advance movement under way when within thirty months (1886–88) there appeared two new brief and elegant demonstrations of Gordan's theorem for binary systems. The first depends upon symbolic expression of covariants, the second upon their expression in terms of the actual linear factors of the stem forms of the system.

According to Mertens, we can always adjoin a new linear form to any set of binary forms for which Gordan's theorem is known to be true, and it will still hold good for the enlarged set. And the set may be altered further by multiplying the linear form into any other form of the set, thus diminishing by one the number of forms, but raising by a unit the order of any one form. Evidently if unit changes of both these styles can be made without invalidating the theorem, we need only to know its truth for a single linear form—or, indeed, for a single form of order zero, a constant—before we can conclude its truth for a set of forms as many in number and as high in orders as we choose. And a linear form has no invariant or covariant except powers of itself: hence the theorem is universally true.

The first step, the adjunction of a linear form, is as follows: Gordan's development in series expresses any covariant in two sets of variables x_1, x_2 ; y_1, y_2, as a sum of a finite number of terms, each containing some power of the determinant $(y_2x_1 - y_1x_2)$ multiplied by a polar of some covariant which contains only the variables x_1, x_2. Now instead of y_1, y_2 insert the cogredient coefficient $-p_2, p_1$ of the adjoined linear form $(p_1x_1 + p_2x_2)$. Thus every covariant containing p_1, p_2 will be expressed in terms of $(p_1x_1 + p_2x_2)$ and a finite number of what we may still call polars derived from covariants not containing p_1, p_2. Accordingly these latter, by hypothesis finite in number, together with their polars in $(-p_2, p_1)$ and the form $(p_1x_1 + p_2x_2)$ itself, constitute the form system of the enlarged set of binary stem forms.*

The second step is almost equally simple, but introduces as auxiliary a Diophantine system of equations. If a form g is the product of a form f and a linear form p, then all covariants of a set including g are included among those of a set including f and p and all members of the first set except g. But the converse is not true: at most, only those

* Compare proof of the same result in Clebsch's *Binäre Formen*, §55.

covariants of $(\cdots, f, p)$ can be covariants of $(\cdots, g)$ which are of equal degree in the coefficients of f and of p. This condition yields a single equation, linear in the exponents of those fundamental covariants which appear in any term of a reducible covariant of $(\cdots, g)$. To state this more explicitly, call $F(\cdots, g)$ any covariant of the set indicated. In it replace coefficients of g by those of the product $f \cdot p$; $F(\cdots, g) \equiv F(\cdots, fp)$. The result is by hypothesis expressible in terms of a finite number of fundamental covariants of $(\cdots, f, p)$

$$F(\cdots, f \cdot p) = \sum c A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n} \cdot B_1^{\beta_1} B_2^{\beta_2} \cdots B_\nu^{\beta_\nu}.$$

Here every A_i is understood to contain the coefficients of f to a degree higher than those of p by some number, either zero or positive, which we may call a_i, the excess of A_i; let b_i denote similarly the defect of B_i. The equation of condition is then evidently

$$a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_n \alpha_n = b_1 \beta_1 + b_2 \beta_2 + \cdots + b_\nu \beta_\nu.$$

This Diophantine equation has a finite number of linearly independent solutions in integers not negative, and to each corresponds one possible term in the above expansion; and to each reducible solution corresponds a term that can be factored into two or more of the non-reducible sort. But these non-reducible factors, though constituting a basis for covariants of the set $(\cdots, g)$, appear to involve still the irrational factor p of g. The means of removing this difficulty is not far to seek,* and so the theorem is proven for a set containing instead of f a form one degree higher, g.

Having analyzed at such length Merten's demonstration, we can state Hilbert's first proof with fewer words.† Instead of considering explicitly a single actual linear factor p of the stem form g, he considers all the factors

$$(x_1 - e_1 x_2)(x_1 - e_2 x_2) \cdots (x_1 - e_n x_2).$$

Every covariant is expressible rationally and integrally in these factors and in differences of the quantities $e_1, \cdots, e_n$ arising from the same stem form or from different stem forms. But the exponents of powers of such differences are parameters conditioned by a set of Diophantine equations, since the covariant is of equal degrees in the quanti-

* Precisely this rationalization is the most novel and ingenious feature of Mertens's proof.

† This demonstration is fully stated in Elliott's Algebra of Quantics, pp. 193–203. First published in *Math. Annalen*, vol. 33 (1888), p. 223.

ties e_1, e_2, $\cdots$, e_n, etc. To the independent integral solutions of these equations correspond a finite number of irrational integral functions of coefficients; and power products of these with all sets of exponents that can be formed within a certain finite range of integers are readily combined with their conjugates to yield a numerous but still finite system of covariants, a basis for the reduction of all others. This proof is simpler than that of Mertens in that, while each requires two logical steps, Hilbert's does not leave a subsequent step-by-step induction to be considered. Notwithstanding this, I venture to express a personal opinion that Merten's proof is the more powerful *as it stands*; for while there is very little difference in the ease of their application to covariants of a single stem form, for a greater number the method of Mertens divides the proof into its simplest possible elementary steps, all alike, while Hilbert prefers to consolidate it into a single argumentative process not repeated. Further, Mertens, employing factors which may be symbolic only, offers a possible opportunity for extension to forms in more than two variables, an opportunity not so readily discerned in Hilbert's use of *actual* factors.

§ 2. *Hilbert's General Proof of Gordan's Theorem for Forms in n Variables.*

It was an agreeable surprise to learn that the elaborate proofs of Gordan's theorem formerly current could be replaced by one occupying not more than four quarto pages. Gordan's series, required as a foundation in Merten's proof, was applicable to ternary forms; and it seemed entirely possible that by this attack might come the next considerable extension of the theorem. It is certain that no one was prepared for the announcement which came in December, 1888,* that the theorem could be established, by uniform method, for forms in any desired number of variables. No wonder that some learned heads shook in doubt over the sweeping generalizations of the enthusiastic young Dr. Hilbert from Königsberg. At length the most incredulous were obliged to concede that he had exemplified the maxim: Generalize your problem and solve it. Then it was apparent why the feat had not been accomplished before: investigators had been using tools much too fine for the work. Hilbert cast aside all needless limitations, and asked directly: If an infinite system of forms be given, containing a finite

**Göttinger Nachrichten*, 1888, pp. 450–457, and *Math Annalen*, vol. 36 (1890), pp. 521–529.

number of variables, under what conditions does a finite set of forms exist, a *basis*, in terms of which all the others are expressible as linear combinations with rational integral functions of the same quantities for coefficients? The answer was: Always. The argument need not be given here, for it is brief and simple, and is accessible either in Dr. Story's improved form in vol. 41 of the *Mathematische Annalen*,* in Meyer's, "Bericht über den gegenwärtigen Stand der Invarianten-Theorie,"† or in Weber's Algebra,‡ as well as in Hilbert's own memorable paper.§ We observe only that it proceeds inductively from the case of n quantities to $n + 1$; and that in the application to invariants it is a matter of indifference whether those quantities are variables, usually so called, or whether a part or all of them are coefficients of stem forms. The transformations to which the quantities are subject play no part in the argument until after the existence of a finite basis is established.

The application of the principal proposition is not of itself, however, sufficient for the requirements of Gordan's theorem. Every covariant F is reduced to the form

$$F \equiv A_1F_1 + A_2F_2 + \cdots + A_rF_r,$$

where $F_1, F_2, \cdots, F_r$ are covariants, but where the coefficients $A_1, A_2, \cdots, A_r$ are not known to be such. To transform the identical equation so as to substitute covariants for the A's without modifying the F's save by numerical factors, Hilbert devised or adapted a scheme of much intrinsic beauty, for which Dr. Story substitutes a most ingeniously contrived explicit differential operator.|| The properties of this operator and its structural character, as necessary or arbitrary, are matters of importance that have not yet been discussed.

Thus after twenty-one years the question raised by Gordan's early success in binary forms is definitively settled, nor has there appeared as yet any proposal for a radically different proof of the theorem.

One important consequence of Hilbert's first theorem should be cited at this point. The generalization of a Diophantine system of equations will evidently arise by using, instead of constant coefficients and integral solutions,

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* P. 471.

† *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 1 (1892), p. 145.

‡ Vol. 2 (1896), p. 165.

§ "Ueber die Theorie der algebraischen Formen," *Math. Annalen*, vol. 36, p. 475.

|| l. c., p. 488.

homogeneous polynomials for coefficients, and requiring as solutions sets of homogeneous polynomials, rational in some prescribed domain. For such generalized sets of equations Hilbert shows that only a finite number of solutions are independent, in the sense that all others are compounded linearly from them, with coefficients rational in the same domain. The application of this corollary is primarily to the theory of syzygies.

§ 3. *Deruyts' Researches in Enumeration of Covariants of Given Characteristics.*

To state and to solve in the most general form the enumerative problem, the lowest case of which Cayley and Sylvester had at last brought to a conclusion, this was the aim of the series of studies which Deruyts consolidated into his now classic book.* Two features of the work first engage our interest. Discussing forms in n variables, he does not follow the older practice, due to Clebsch, of admitting $n-1$ different sets of variables, each set contragredient to one other (unless dual to itself) and cogredient to certain determinants formed from two, three, etc., rows of the variables of simplest type. Capelli's proposal is adopted instead, to employ $n-1$ sets of variables all cogredient. The older practice has intrenched itself in analytic geometry, and cannot be dislodged; but the alternative is undoubtedly better from the point of view of algebra. The second point is, that the whole argument is conducted by the aid of the Clebsch-Aronhold symbolic notation, and it is difficult to see how the end could be attained without this auxiliary.

Dr. Story has generalized Sylvester's semi-invariants by calling them differentiants, and distinguishing as many kinds of differentiants as there are pairs of variables in the set of n. Thus an xy-differentiant is invariant of the substitution

$$\begin{aligned} x &= x' + \lambda y' \\ y &= y', \end{aligned}$$

the other variables remaining unchanged. Deruyts, on the other hand, uses the term semi-invariant, but enlarges the substitution with respect to which it is invariable. He defines semi-invariant as a function (properly qualified) which is not altered by a substitution in whose matrix all coefficients on one side of the principal diagonal are zero.

*"Essai d'une théorie générale des formes algébriques," par Jacques Deruyts; Bruxelles, F. Hayez, 1891, 8vo, pp. vi + 156.

Let this be the lower (left-hand) side; such a linear substitution is designated aptly by $S_{\frac{1}{2}}$. Semi-invariants are important for this reason, that each is the source of a *primary covariant*, which is uniquely determined by it; conversely every primary covariant has only one source. For example, the mixed concomitant or connex θ which occurs in the theory of the ternary cubic

$$\theta = (abu)^2 a_x b_x$$

is replaced in this theory by the primary covariant

$$\begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}^2 a_x b_x,$$

whose source is the semi-invariant

$$a_1 b_1 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2.$$

If we replace symbols of the cubic by those of its Hessian, we have a source which contains symbolic factors of all types that are possible in semi-invariants of ternary forms: the covariant becomes

$$(a_1 b_2 c_3)^2 (d_1 e_2 f_3)^2 \begin{vmatrix} a_x & d_x \\ a_y & d_y \end{vmatrix} \begin{vmatrix} b_x & e_x \\ b_y & e_y \end{vmatrix} c_x f_x$$

and its source is the semi-invariant

$$(a_1 b_2 c_3)^2 (d_1 e_2 f_3)^2 (a_1 d_2)(b_1 e_2) c_1 f_1.$$

Now the number of linearly independent covariants of given degrees and orders is the same as the number of primary covariants of like degrees and orders; and that again is equal to the number of different semi-invariants of corresponding type. From the covariants so determined, all others can be derived by the aid of polar processes and identical covariants such as $(x_1 y_2 z_3)$ or $(x_1 y_2 z_3 w_4)$, etc. The problem so reduced Deruyts has solved by means of partition numbers, so that the enumeration of *irreducible* covariants for given weights presupposes that for all lower weights and degrees. As an illustration of the computations used, consider the well known quadric covariants of second degree in the coefficients of each of two quadric forms in three variables

$$(ab\alpha)\,(ab\beta)\alpha_x\beta_x, \quad (\alpha\beta a)\,(\alpha\beta b)\,a_x b_x,$$
$$(ab\alpha)^2\beta_x^2, \qquad (\alpha\beta a)^2 b_x^2,$$

which are connected by the single linear relation

$$(aba)^2\beta_x^2 - (a\beta a)^2 b_x^2$$
$$\equiv (aba)\,(ab\beta)a_x\beta_x - (a\beta a)\,(a\beta b)a_x b_x.$$

There should be found, as the number of covariants of these degrees and order, three. Deruyts' final formula gives

$$\begin{aligned}[422] &= \{422\}_\Delta \\ &= \{422\} + \{530\} + \{611\} \\ &- \{431\} - \{512\} - \{620\} \\ &= 24 + 6 + 6 - 16 - 12 - 5 = 3.\end{aligned}$$

The expansion of a three rowed determinant $\triangle$ serves as a mnemonic for the formula. So completely is the problem resolved, that one can determine the required number, not only when the stem forms and concomitants contain $n-1$ sets each of cogredient variables, but even when more different sets are present, and when different groups of sets are subject to independent linear substitutions.

The next desideratum in this direction is a table showing the nature of the irreducible covariants in the system of the ternary quartic, a similar one for the quaternary cubic, and later for the simplest simultaneous systems and for stem forms that contain more than one set of variables. For the use of geometers, also, it would appear worth while still to consider whether the general problem might not be solved if Clebsch's $n-1$ sets of correlated variables were admitted in the stem forms, and whether the problem so stated can be made to depend in any way upon the results of the work of Deruyts.*

§4. *Hilbert's Theorem upon Syzygies of Higher Orders.*

Between the irreducible ground forms of a system arising from one or more stem forms there exist relations called syzygies. All terms of such a relation being collected in one member of the equation, the aggregate is termed a *syzygant.* A syzygant of the first kind is identically equal to zero, not when it is expressed in terms of the ground forms, but only when these are further reduced to terms of

* M. Deruyts kindly informs me (Dec. 10, 1898) that he has considered this question in a special memoir, and has found that for ternary forms there is a one-to-one correspondence between covariants in the two sorts of reduced systems, but that this is not so for stem forms in more than three variables. See his essay: "Sur la réduction des fonctions invariantes dans le système des variables géomètriques," *Bull. de l'Acad. roy. de Belgique*, 3d series, vol. 24 (1892), pp. 558–571.

the coefficients of the stem forms. Syzygants of the first kind form a system similar to that of the covariants, out of which may be selected, as Hilbert proves, a reduced system of ground syzygants, in terms of which all others may be expressed as rational and integral functions. For the annihilation of this first kind of syzygant, to repeat what was just now said, the coefficients and variables of the stem forms constitute the requisite domain of rationality. But if the ground forms of the system of covariants be taken as constituting a second domain of rationality, in this new domain there will be annihilated certain linear functions of the syzygants of the first kind, and these are called syzygants of the second kind. Preserving, thenceforward, the same second domain of rationality, there rise successive kinds of syzygants one beyond another, each linear in the coefficients occurring in the kind next lower in rank. Recalling the theorem or corollary cited at the close of §2, we understand that each of these kinds of syzygants must constitute a finite system, in the sense that its ground forms are finite in number. For the first system there have been published two exhaustive methods of discovery, the first by Study in his concise and comprehensive "Methoden zur Theorie der ternären Formen,"* the second by Stroh,† in addition to that arising from the "typical representation" of binary forms. Upon kinds higher than the first there has been done practically no detail work. So much the more noteworthy is therefore the fundamental theorem disclosed by Hilbert (l. c., p. 492, Theorem III.) that *the number of kinds of syzygants is always finite*. If m denote for any given system the number of ground forms, then the successive kinds of syzygants are not more than $m + 1$ in number.

The proof, as in the case of the theorem discussed in §2, is entirely divorced from the processes peculiar to the theory of invariants, concerning itself only with rational integral functions as such, and yielding therefore as much to the theory of algebraic loci as to the knowledge of invariants. Although this proof is, as both Hilbert,‡ and Franz Meyer§ testify, "nicht mühelos," yet it is possible to convey briefly some idea of the scheme employed. Suppose arbitrary polynomials in m homogeneous variables to be denoted by

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* Leipzig, Teubner, 1889. See p. 97.

† "Ueber die symbolische Darstellung der Grundsyzyganten einer binären Form sechster Ordnung, u. s. w.," *Math. Annalen*, vol. 36 (1890), pp. 262–303.

‡ l. c., p. 492.

§ *Jahresbericht der deutschen Mathematiker-Vereingung*, vol. 1, p. 148.

$$\begin{Bmatrix} F_{11}, & F_{12}, & \cdots & F_{1m_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ F_{n1}, & F_{n2}, & \cdots & F_{nm_1} \end{Bmatrix},$$

and undetermined polynomials, rational in the same domain, to be denoted by $X_1, X_2, \cdots, X_{m_1}$ seek solutions to the set of equations

$$\begin{array}{l} F_{11}X_1 + F_{12}X_2 + \cdots + F_{1m_1}X_{m_1} = 0 \\ F_{21}X_1 + F_{22}X_2 + \cdots + F_{2m_1}X_{m_1} = 0 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\qquad \vdots \\ F_{n1}X_1 + F_{n2}X_2 \quad \cdots + F_{nm_1}X_{m_1} = 0. \end{array}$$

(We pause to observe that if the F_{ik} are syzygants of the first kind, every solution of one such equation gives a syzygant of the second kind.) Suppose this set of equations fully solved, and a minimum sufficient set of fundamental solutions determined, two steps theoretically possible. Arrange in a rectangle these solutions, with values of X_1 in the first row, of X_2 in the second, etc. Denote this array by the symbols

$$\begin{Bmatrix} F'_{11} & \cdots & F'_{1m_2} \\ \vdots & & \vdots \\ F'_{m_1 1} & \cdots & F'_{m_1 m_2} \end{Bmatrix}, \text{ values of } \begin{Bmatrix} X_1 \\ \vdots \\ X_{m_1} \end{Bmatrix}.$$

Now seek relations linear in these horizontal rows of F's, just as before among the original F's. (Such relations will have for significant members the coefficients in syzygants of second kind, in the present application.) Continue this process, forming a second derived set, a third derived set, and so on until, if ever, the last set admit no solutions.

To see that this interruption will come, divide the solution of every set into two parts, such that the first part are immediately discoverable and help in reducing the order of the others in respect to a selected variable, say the last or m-th; while the second part depend for their determination upon a new set of similar equations, containing only the first $m-1$ variables. The device is so chosen that the second of these auxiliary sets is derived from the solutions of the first auxiliary set in the same way as each set in the principal series is derived from its preceding set in the same series. Argue now by induction: if the second series depending upon $m-1$ variables, is interrupted by default of solution after $m-2$ steps, then the principal series must be interrupted after $m-1$ steps. For complete assurance examine the form of the first part of solutions of the principal set, which will be after $m-1$ steps the *complete* array of solutions. It is

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\begin{array}{llllllll}
 & \varphi_{11} - x_n & \varphi_{12} & \varphi_{13} & \cdots & \varphi_{1\mu} & \varphi_{1\mu+1} & \cdots \\
 & \varphi_{21} & \varphi_{22} - x_n & \varphi_{23} & \cdots & \varphi_{2\mu} & \varphi_{2,\mu+1} & \cdots \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
 & \varphi_{\mu 1} & \varphi_{\mu 2} & \varphi_{\mu 3} & \cdots & \varphi_{\mu\mu} - x_n & \varphi_{\mu,\mu+1} & \cdots \\
\text{values of} & x_1, & x_2, & x_3, & \cdots & x_\mu, & x_{\mu+1} & \cdots
\end{array}
$$

Here the φ_{ik} denote functions of the first $n-1$ variables. If these be taken by columns for coefficients of the next derived set of equations, the determinant of the first μ not vanishing, obviously there can be no solutions.

The theorem is, therefore, true for any number of variables so soon as it is true for a single variable, *i. e.*, for equations in numerical constants. If we limit the field of rationality to the natural domain, this is in effect to reduce the inquiry to the question : whether linearly independent sets of solutions of a set of Diophantine equations are linearly independent ! Hilbert chooses rather to establish a foundation proof for binary equations, introducing unnecessary complication, excusable on account of the elegance of his independent demonstration.

On this particular part of Hilbert's ample contributions to the theory of invariants I have dwelt at some length, though giving only the bare outline, and omitting entirely the critical points, first, because it has received less notice and excited less discursive activity than the more elementary theorems announced in the same essay ; and because, in the second place, it serves admirably to illustrate the statement that it is time for the theory of invariants to attach itself firmly to the most modern developments of algebra. The sequel to this essay, a treatise on the production of complete fundamental systems of covariants,* has pushed the frontier in this direction a long distance ahead, establishing the main thesis : that a finite number of trials of perfectly definite kind will always lead to the knowledge of the complete system of ground forms when the stem forms are given. Kronecker's theory of entire algebraic functions proves itself indispensable and effective, and the argument leading up to the definition of canonical null forms is likely to become the standard concrete illustration of Kronecker's highly abstract theory.

§ 5. *Miscellaneous Topics.*

One other question of principal moment has been discussed, by Maurer, in vol. 107 of the *Journal für reine und*

* D. Hilbert : "Ueber die vollen Invariantensysteme." *Math. Annalen*, vol. 42 (1893), pp. 313-373.

angewandte Mathematik :* the division of stem forms and systems into classes according to the number of conditions satisfied by their coefficients, and equivalence of forms within classes. The subject seems to promise more interest when better developed.

Not less important for the growth of the science than original articles is the preparation of treatises and text books. Of these, at least two of high grade beside that of Deruyts, have come to my notice within this decade, those of Study and Elliott already cited above. Study's book introduces substantial improvements in notation, and gives precision to the notion of rationality, and is full of originality in every chapter. Elliott's is intended less for advanced students, but is admirable pedagogically. Others that I have seen announced are evidently elementary books for beginners.

Of value higher than text books, as every scholar understands, are exhaustive *résumés* and reference compends. For such a work, replete with description, discriminating and impartial in its estimates, students of invariants are indebted to Professor W. Franz Meyer, now of the University of Königsberg.† Undertaken at the instance of the Deutsche Mathematiker-Vereinigung, it has placed in a favorable light the utility of such coöperative organizations.

As showing the ample attention that is paid to this department of mathematics, it is of interest to note the number of titles, reviewed in the *Jahrbücher über die Fortschritte der Mathematik*, which can fairly be classified under theory of invariants. There were :

in 1889,	46 titles,
" 1890,	42 "
" 1891,	41 "
" 1892,	38 "
" 1893 and '94,	50 "
" 1895,	30 "
" 1896,	30 "

Of special papers in this field, since the important one of Story's referred to above, the most interesting one produced in this country is without doubt the recent essay by F. Morley in vol. 49 of the *Mathematische Annalen*, wherein he gives the long wished for geometric construction of the linear

*Ueber Invarianten-Theorie, pp. 89–116.

†" Bericht über den gegenwärtigen Stand der projektiven Invarianten-Theorie im letzten Vierteljahrhundert ;" in the *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 1 (1892), pp. 79–292.

covariants of a binary quintic. This skilful synthesis removes from geometers the reproach which it is said Clebsch used in his lectures to cast upon them, in that none of them had yet been able to derive uniquely and symmetrically a sixth point from five given points on a straight line. The zeros of the quintic are denoted in Professor Morley's construction by five arbitrary points upon a conic.

§6. *Desiderata, and Remarks upon Courses of Instruction.**

Two things appear to me as proximate possibilities, and essentials to uniform advancement. Those familiar with Lie's group theory and interested in differential invariants will no doubt criticise this choice, which is perhaps in a narrower field.

(1) The working out of complete systems of syzygies of the first kind, second kind, and all higher kinds which occur among the covariants of binary forms of lowest orders. For the quintic and sextic the first kind are already tabulated by Stroh. This work will give tangible examples for the understanding and estimation of Hilbert's great theory.

(2) The revision of complete form systems already known, with the object of discovering subordinate systems among them. The most obvious point of attack, if we except the suggestive processes used in Clebsch and Gordan's classic treatise on the ternary cubic in vol. 6 of the *Mathematische Annalen*,† is offered by polars, symmetric in two sets of variables, derived from binary covariants of even order, and by ternary concomitants whose order and class are equal. By using these as transformers, systems of covariants can certainly be determined which are closed ; and particular covariants ought to be looked for, which shall be automorphic under such transformations. The next step would be, by transformers which are analogous to these in all save that they raise the order of the operand, to produce infinite series of covariants and to discover their recurrent laws. These again might be expected to develop some subordinate closed systems, and others probably coextensive with the complete form system.

Finally, there is to be remembered the least explored and most fascinating portion of the field, equally promising to

* With regard to the following paragraph it should be explained that the Sectional Committee of Section A of the American Association for the Advancement of Science expressly requested of authors of reports the formulation of problems suitable for coöperative attack and of pedagogic theses inviting discussion. H. S. W.

† Pp. 436–512 (1873).

the analyst and the geometer, the realm of irrational covariants.

Upon the question of courses of instruction I wish to formulate two propositions:

(1) A first or elementary course in invariant theory ought never to be restricted to binary forms.

(2) Preliminary to or concurrent with an advanced course, there should be given courses in the theory of substitution groups or abstract groups, and in the algebra of modular systems and of entire functions.

EVANSTON, ILL.,
August, 1898.

REYE'S GEOMETRIE DER LAGE.

Lectures on the Geometry of Position. By THEODOR REYE, Professor of Mathematics in the University of Strassburg. Translated and edited by THOMAS F. HOLGATE, M.A., PH.D., Professor of Applied Mathematics in Northwestern University. Part I. New York, The Macmillan Company, 1898. 8vo, xix + 248 pp.

THE true geometry of position has hardly been accessible in English up to the present time. Townsend's Modern Geometry and Lachlan's Modern Pure Geometry are vitiated by the use of the circle, they are essentially metric; Cremona's Projective Geometry, in Leudesdorf's translation, is curiously uninteresting and unattractive, and does not seem to take the student sufficiently into the heart of the subject. Russell's Pure Geometry follows the French treatment of cross ratio, which is based on apparently metric relations, though it is shown that these relations are such that the metric quality is eliminated. Thus while it is a thoroughly useful book, it only gradually frees the student from the limitations of Euclidean geometry, instead of enabling him to walk at liberty from the first. It is possibly one of the easiest books to read on the subject; grafting the new ideas on to those already established, it expresses the unknown in terms of the known, whereas the more correct and satisfactory treatment, building up geometry *ab initio*, is apt to strike a student at first as an elaborate and artificial expression of the known in terms of the unknown. But while the grafting of projective geometry on