PICARD'S ALGEBRAIC FUNCTIONS OF TWO VARIABLES.

Théorie des Fonctions algébriques de deux Variables indépendantes. Par ÉMILE PICARD, Membre de l'Institut, Professeur à l'Université de Paris, et GEORGES SIMART, Capitaine de Frégate, Répétiteur à l'École Polytechnique. Tome I, Paris, Gauthier-Villars et Fils, 1897. 8vo, vi+246 pp.

THE theory of functions of a single complex variable, the growth of which has been one of the striking features in the history of pure mathematics during this century, can, as it is well known, be developed from at least two tolerably distinct points of view. Cauchy and Riemann made extensive use both of methods of proof—such as integration—which may be conveniently called transcendental, and of geometrical reasoning; by such means they established not only results of a geometric or transcendental character but also others which, when once obtained, were naturally expressible in terms of pure algebra. Weierstrass and his school on the other hand have made scarcely any use of geometry, even for purposes of illustration, and have developed the subject from an almost purely algebraical standpoint, building it up systematically and logically from the most elementary notions. The former method has the interest which always attaches to investigations which connect two or more different branches of mathematics and use the methods of the one to solve problems in the other; and the geometric interpretations which continually present themselves are with most minds a valuable aid towards the clear comprehension of the theory. Partly owing to these reasons, partly owing to the well known difficulty of access to Weierstrass's ideas, most systematic treatises on the theory of functions which were published up to a few years ago expounded chiefly the ideas of Cauchy, Riemann and their followers. But, as the Weierstrassian methods have become more widely known, their severe simplicity, their unity, and their rigor have made many converts; and Weierstrass's dictum, that the theory of functions "must be built up on the basis of algebraic truths, and that it is not therefore the right way if conversely the 'transcendental' is employed to establish simple and fundamental algebraical propositions,"* is every year finding more general accept-

* Letter to Professor Schwarz, of October 3, 1875, Mathematische Werke, II, p. 235.

ance. Recent treatises illustrate this tendency. Dr. Forsyth's Theory of Functions and Messrs. Harkness and Morley's treatise on the same subject, each published some six years ago, were among the first books which gave an account of the subject from both points of view; and a like catholicity of treatment has shown itself in modern French and German treatises. More recently still Messrs. Harkness and Morley in their admirable Introduction to the Theory of Analytic Functions have employed Weierstrassian methods predominantly though not exclusively.

But the systematic methods which are possible and often desirable in expounding a subject which has been worked at with exceptional vigor and success for more than half a century, which has become a recognized subject of university teaching, and the boundaries between which and allied theories have been to some extent drawn, become inapplicable in dealing with a nascent subject, the difficulties of which have been only overcome here and there. As Weierstrass says in the letter already quoted: "It is evident that every path must be open to the investigator as long as he is still engaged in his inquiry."

The theory of functions of two or more independent variables, as at present known, belongs to this latter class of subjects, which are far too imperfect to admit of systematic treatment by a single method. The difficulties are such that any method which brings out results is welcome; even doubtful processes, which when carefully examined may be said rather to render their apparent conclusions plausible than to prove them, are of value, since results thus obtained are at any rate suggestive, and their accuracy can be tested at a later stage by other methods. In the same way it is permissible or even desirable to leave out of account exceptional cases in some particular part of the subject, reserving them for further treatment when some more results of a general character have been obtained, or to deal with special cases in the hope that they may throw light on a hitherto intractable general theory.

It is in this spirit that M. Picard and his collaborator, M. Simart, have written the present treatise; and in this spirit it should be read and appreciated. It is not, and from the nature of the case cannot be, a connected and systematic treatise dealing with functions of two variables, as Dr. Forsyth or Messrs. Harkness and Morley treat the corresponding case of one variable; its chapters may be described as a series of monographs dealing with those branches of the theory which have hitherto been most studied; but unlike the original memoirs on which they are based they are written on a uniform plan and the connection between the different topics treated of is, within certain limits, made as evident as the nature of the case admits.

The methods employed correspond for the most part to those used by Cauchy and by Riemann for the study of functions of one variable ; and the material of the book is to a large extent derived from memoirs by M. Picard himself and by M. Poincaré, while various results due to Noether, Clebsch, Betti, Cayley, Castelnuovo, Enriques and others have also been embodied. There is no reference to Weierstrass's theory of analytic functions of two or more variables, and no use is made of his methods.

It is well known that the theory of functions of one complex variable is intimately connected with geometry in two distinct ways. Some of the advantages of representing a complex variable $x(= \xi + i\xi')$ by a point in a plane were noticed more than a century ago ; an important extension of the method was made by Riemann in introducing the surfaces now associated with his name for the treatment of many valued functions : the complex variable is thus represented by a point not confined to a single plane but moving either on a system of superposed planes connected in a particular way or on a surface, such as the familiar anchor ring, contained in space of three dimensions. The "branching" of the function receives an interpretation in certain of the geometrical properties of the corresponding Riemann surface. It is characteristic of this method that a correspondence is established between a single complex variable x and a real two dimensional locus, and that, though the method is admirably adapted for the study of what we may conveniently call the qualitative relations between the dependent variable or function (y) and the independent variable (x), the quantitative variations of y receive no interpretation.

An entirely different method consists in interpreting the two variables x, y as coordinates of a point in a plane ; the functional relation between them gives a curve in the plane. But real points on the curve correspond only to real values of the variables ; complex values of the variables only receive an interpretation when we generalize our geometrical conceptions so as to include imaginary points, and the actual diagrammatic representation fails. In this method, then, we establish a correspondence between our pair of variables and a one dimensional locus ; and the quantitative variations of both are represented.

Thus if we are dealing with two variables x, y connected by the equation

$$y^2 = (1 - x^2)(1 - k^2x^2),$$

the first method employs the surface of an anchor ring, a double plane, or some other two dimensional locus, while the second interprets the equation as that of a plane quartic curve.

Some important differences arise when we pass from the case of one to that of two independent variables (x, y). The obvious extension of Riemann's geometric method requires for the representation of the four real variables involved in $x(= \xi + i\xi')$ and $y(= \eta + i\eta')$ a four dimensional locus,* and, just as in the case of one complex variable it may be convenient to work with a surface (like an anchor ring) contained in space of a higher number of dimensions, so this locus may be a curved "surface" in space of five dimensions. The second method of representation, however, only leads to an ordinary surface in space of three dimensions.

Thus, restricting ourselves to the study of *algebraic* functions of two variables, we have to deal on the one hand with the *Analysis situs* of hyperspace and on the other with the theory of ordinary algebraic surfaces.

The first two chapters of the book under review deal with some fundamental questions relating to the geometry of hyperspace and to integration in such space, these being necessary as an introduction to problems which are subsequently discussed.

If we restrict ourselves to ordinary space we have three kinds of integrals involving three variables, which present themselves commonly and are of geometrical importance, viz., line integrals of the type $\int(Pdx + Qdy + Rdz)$, surface integrals such as $\iint(Adydz + Bdzdx + Cdxdy)$ and volume integrals $\iiint Vdxdydz$. If we pass to space of n dimensions we have similarly m-tuple integrals in n variables, where m may have any integral value from 1 to n inclusive; the conditions of integrability of such integrals in the case of $m < n$, and the conditions that they should vanish when taken over a closed locus, relations between integrals of different orders, and allied questions are dealt with in a most suggestive though incomplete manner. The discussion of the *Analysis situs* of loci of any number of dimensions in space of a higher number of dimensions, which

* I use this word, though with some hesitation, as being on the whole the best English equivalent for *variété*.

follows, leads up to the definitions and to properties of certain numbers, which are, from one point of view, the generalization of the number known as the connectivity in the case of an ordinary Riemann surface. The connectivity of such a surface depends on the number of curves of an assigned character which can be drawn on the surface. But in the case of a higher number of dimensions it becomes necessary to consider in an analogous way not only curves (one dimensional loci) but also loci of 2, 3, $\cdots, n-1$ dimensions, where n is the number of dimensions of the locus considered. We thus have the conception of connectivities of different orders $p_1, p_2, \cdots, p_m, \cdots, p_{n-1}$, where p_m depends on the number of m-dimensional loci of a certain character which can be drawn on the surface. These numbers are called the *numbers of Betti and Riemann*, as they occur in a memoir by the former* and in a posthumous fragment published in the collected works of the latter.† It will be convenient to refer to them simply as Bettian numbers. It may be remarked in passing that it is perhaps a little unfortunate that this particular notation should have been chosen by MM. Picard and Simart, as when we revert to the case of $n = 2$, the number p_1 does not reduce, as the notation would suggest, to the familiar p of the ordinary theory, but is the connectivity $2p + 1$. These Bettian numbers are also shown, as was to be expected, to play an important part in the theory of integrals on the surface, the fundamental property being that if we are dealing with an n-dimensional locus E_n contained in space of a higher number of dimensions, then p_m is the number of independent periods obtained by integrating round closed circuits an m-tuple integral

$$\int \cdots \int \sum X_{1, 2, \cdots, m} dx_1 dx_2 \cdots dx_m,$$

which satisfies the condition of integrability.

A remarkable property of Bettian numbers, which has, of course, no analogue in the theory of the ordinary Riemann surface, is that for a closed surface

$$p_m = p_{n-m},$$

a property established by M. Poincaré in an important memoir on the *Analysis situs* of hyperspace‡, but only proved by MM. Picard and Simart for the case of $m = 1$.

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* *Annali di Matematica*, vol. 4 (1870–1871).

† Fragment aus der Analysis Situs, Werke (2d edition), pp. 479-482.

‡ *Jour. de l'École polytechnique*, series 2, vol. 1 (1895).

The following chapter deals chiefly with double integrals of functions of two variables, which are in general taken to be complex. The fundamental theorem is M. Poincaré's generalization* of Cauchy's well known theorem on the contour integration of a holomorphic function. The corresponding theorem when the function has singularities inside the locus over which the integral is taken is only established for the case of rational functions; and it is shown that corresponding to the residues which occur in Cauchy's theorem we now have periods of abelian integrals corresponding to the curve obtained by equating to zero the denominator of the subject of integration. Here also it may be permitted to make a trifling criticism of the definition of a residue of a function of two variables which M. Picard adopts after M. Poincaré, as the quantity thus defined corresponds, in the case of one variable, not to the residue which Cauchy's work has made classical, but to the same multiplied by $2\pi i$. The chapter concludes with a short discussion of simple integrals of total differentials in two complex variables, viz., of integrals of the type

$$\int (Pdx + Qdy)$$

where P and Q satisfy a condition of integrability, and are, moreover, restricted to being rational functions, but a fuller treatment of the subject is postponed to a later chapter.

The next chapter introduces the second of the two methods which we have referred to as giving a geometric interpretation of a function of two variables, viz., by an algebraic surface in ordinary space. Almost any problem of analytical solid geometry might of course be regarded as belonging to the theory of functions of two variables, but, as in the case of two dimensions, the characteristic ideas of the theory of functions are connected primarily with certain special parts of the theory of surfaces. In particular the theory of the singular lines and points of a surface, and the method of birational transformation play a leading part. The problem of the reduction of the higher singularities of plane curves, *e g.*, multiple points where two or more branches touch or have contact of a higher order with one another, to double points with distinct tangents is, as is well known, of great importance for various questions connected with the theory of functions of one variable. It has been extensively studied during the last quarter of a century and

* "Sur les résidus des integrales doubles," *Acta Math.*, vol. 9 (1887).

has been completely solved in a variety of ways by means of birational transformations. The corresponding problem in three dimensions is complicated by the existence of singularities of two distinct types, multiple lines and multiple points, while the latter may either be isolated or lie on the former. In reducing these singularities use is made of methods of projecting from space of higher dimensions which were, I believe, first used by Clifford* and have been extensively developed by modern Italian geometers. Just as, for example, a nodal plane cubic curve can be treated as the projection of a twisted cubic ; so in general, a surface can be treated as the projection of a two dimensional locus in space of four or more dimensions with fewer singularities. The most general result of this kind established by MM. Picard and Simart is that any algebraical surface can be converted by projections into a "surface" in five dimensional space without singularities, and then, by further projections, into a surface in ordinary space, the only singularities of which are double lines with triple points on them. Thus isolated multiple points and lines with multiplicity higher than two can be removed. This projection establishes a one-one correspondence between the two loci, and when put into algebraical language is equivalent to birational transformation. Hence in problems in which the order of a surface is unimportant and we are dealing only with properties which are unaffected by birational transformation, it becomes unnecessary to consider singularities other than those just named.

The remainder of the chapter deals with the connectivity of algebraical surfaces. Since, as we have seen, the two complex variables (x, y) correspond to four real variables, there are three connectivities, p_1, p_2, p_3, to be considered, which, however, reduce to two in virtue of the equation $p_1 = p_3$. Purely geometrical methods of treatment, such as can be employed in the case of ordinary Riemann surfaces, soon become unmanageable owing to the difficulties of realizing hyperspace, and have to be supplemented by algebraical and transcendental processes ; among the latter is the employment of a certain linear ordinary differential equations, from which the number p_1 for the surface can be obtained. A very remarkable difference between the numbers p_1, p_2, which at the same time emphasizes the difference between the cases of a plane curve and of a surface, is established. In the case of a plane curve C_n, represented by

* "On the classification of loci," *Phil. Trans.* (1878) and Math. Papers, pp. 305–331

$f(x, y) = 0$, it is well known that each additional singularity diminishes the genus (deficiency, genre, Geschlecht) of the curve, and consequently the connectivity of the corresponding Riemann surface; in particular the Riemann surface of a non-singular C_n is necessarily multiply connected if $n > 2$. The general effect of singularities on the connectivity of order $2 (p_2)$ of an algebraical surface is of the same character, though not capable of such simple expression; the addition of a new singularity in general diminishes the number of possible two dimensional closed loci that can be drawn on the real four dimensional locus, which represents (in the sense already explained) the algebraic equation $f(x, y, z) = 0$. But the addition of a new singularity has, in general, an effect of the opposite kind on the connectivity of order one (p_1), as it tends to increase the possible number of closed curves that can be drawn; and in particular so far from a non-singular surface possessing the maximum number of such curves it is shown that it cannot possess any, and it is therefore simply connected.

The following chapter (V), on integrals of the first kind of total differentials, deals with one of the most interesting parts of the whole subject, the main outlines of it being due to M. Picard himself. In connection with a plane algebraic curve, $f(x,y) = 0$, we have abelian integrals $\int R(x,y)\,dx$ (where R is a rational function), which have been classified into the familiar three kinds. When we pass from a plane curve to a surface, $f(x,y,z) = 0$, we have to inquire in what direction we are to look for the equivalent of an abelian integral. One obvious generalization is to replace the single integral $\int R(x,y)dx$ by a double integral $\int\int R(x,y,z)dxdy$. Another and quite different generalization consists in taking an integral of a total differential, viz., $\int (Pdx + Qdy)$ where P and Q are rational functions of x,y,z, which satisfy a condition of integrability. These integrals like ordinary abelian integrals fall into three classes according to the nature of their infinities, and a special interest attaches to those of the first kind, which are finite at every point of the surface. Integrals of this kind were first introduced by M. Picard fifteen years ago*; and a very large part of what is now known about them is due to him. Though they do not seem to have been extensively studied, their importance both in geometry and differential equations may be seen by reference

* In a short paper in the *Comptes rendus*, December 1, 1884; soon followed by the memoir. "Sur les intégrales de différentielles totales algébriques de première espèce," *Liouville's Journal*; series 4, vol. 1 (1885).

to Humbert's memoir on hyperelliptic surfaces* and to Painlevé's Stockholm lectures on differential equations†.

A problem of capital importance which presents itself at the outset is that of ascertaining what surfaces admit of such integrals. This is by no means as simple as the corresponding problem in two dimensions. For whereas every plane curve of genus p admits of exactly p integrals of the first kind, so that only rational curves admit of no such integrals, the corresponding problem for surfaces is much more complicated and has by no means been completely solved. It is shown by a singularly beautiful piece of analysis (given in M. Picard's earliest paper on the subject), that the determination of surfaces, of a given order n, which admit of integrals of the first kind, depends on the integration of a linear partial differential equation. In homogeneous coordinates this equation can be expressed in the form

$$\theta_1 \frac{\partial f}{\partial x} + \theta_2 \frac{\partial f}{\partial y} + \theta_3 \frac{\partial f}{\partial z} + \theta_4 \frac{\partial f}{\partial t} = 0,$$

where the coefficients θ are quantics of order $n-3$, which satisfy the equation

$$\frac{\partial \theta_1}{\partial x} + \frac{\partial \theta_2}{\partial y} + \frac{\partial \theta_3}{\partial z} + \frac{\partial \theta_4}{\partial t} = 0.$$

For the case of $n = 4$ the integration of the equation presents no difficulties, except that the discussion of certain exceptional cases requires a little care. Two quartic surfaces are found which satisfy the required conditions, and it is stated without detailed proof that there are no others but these and their projections and certain cones. There is a trivial exception to this conclusion, which M. Picard appears to have overlooked, as a third surface can be found which, though it can easily be deduced by a limiting process from one of his surfaces, is not strictly included in it in the form in which it is given. In the case when $n > 4$, the integration of the differential equation presents formidable difficulties which do not appear to have been overcome. Several general results are, however, given which throw light on the existence or non-existence of surfaces of the kind considered. Thus in the case of a cone every abelian integral relative to a plane section (not passing

* *Liouville's Journal*, series 4, vol. 9 (1883).

† Leçons sur la théorie analytique des équations différentielles; Paris, 1897.

through the vertex) is obviously an integral of a total differential relative to the cone, so that if the section of the cone is a curve of genus p, we have p integrals of the first kind. As also a birational transformation converts an integral of the first kind of a total differential into an integral of the same character, surfaces which can be obtained from non-rational cones by a birational transformation will possess such integrals. On the other hand, a rational (unicursal) surface—the coordinates of a point on which are rational functions of two parameters—evidently possesses no such integrals, since the integral of a rational function is necessarily infinite for some values of the parameters. It follows that no surfaces of order 2 or 3 can possess any such integrals. Further, since one of our integrals must reduce to an abelian integral of the first kind for any algebraic curve on the surface, it follows that our integral must vanish identically along any rational (unicursal) curve on the surface; if, therefore the surface contains a family of rational curves, and there is an integral of the first kind, $\int(Pdx + Qdy)$, then $Pdx + Qdy = 0$ must be the differential equation of the family. MM. Picard and Simart also show that an integral of the first kind can only exist if the connectivity of the first order (p_1) is at least 3; and therefore cannot exist on a non-singular surface, for which it has already been shown that $p_1 = 1$, a result which is established independently by means of the differential equation.

Another subject of interest discussed in this chapter is the existence of functionally independent integrals of the first kind. An abelian integral associated with a plane curve is a function of one variable only, and therefore all such integrals are functions of one another; the ordinary theory deals only with *linear* independence. But integrals on a surface are in general functions of two independent variables, and we have therefore to consider cases of functional as well as of linear independence. Except in certain cases of hyperelliptic surfaces (which however are not discussed in the book) the results hitherto obtained are chiefly negative. The integrals of the first kind on a cone or on a surface obtained from it by a birational transformation as well as on any surface containing a family of rational curves, are obviously functions of one another; and it is further shown that there cannot be two functionally independent integrals on quartic surfaces or on certain classes of quintic surfaces.

Integrals of the second kind of total differentials, which form the chief part of the subject matter of chapter

VI, are in some respects both simpler and less interesting than those of the first kind. The characteristic of such integrals is that they are algebraically infinite at one or more points of the surface with which they are associated; consequently rational functions of the variables form an obvious though trivial class of such integrals, which it is in general unnecessary to consider further. The number of linearly independent integrals of the second kind, rational functions being ignored, is shown to be less by 1 than the number p_1 which expresses the connectivity of the first order. On the other hand the number of such integrals is shown to depend upon certain linear differential equations, and the study of these equations thus leads to a method of determining p_1, a number which it is not in general practicable to obtain by purely geometric methods.

A short discussion of integrals of the third kind concludes the chapter, but no results of special interest are obtained.

Double integrals which are always finite on an algebraic surface are also said to be of the first kind. The properties of these integrals and a number of important geometric questions with which they are intimately associated form the subject matter of the last two chapters. These integrals were first introduced by Noether,* and have subsequently been studied chiefly by M. Picard and, in connection with hyperelliptic surfaces, by Humbert. They form the most direct analogue of abelian integrals of the first kind and are closely connected with certain parts of the theory of birational transformation of surfaces. In the theory of plane curves there is only one known number p which is invariant for such transformation; it can be defined in at least four ways, viz., (1) in terms of the connectivity of the associated Riemann surface by means of the equation $2p + 1 = N$, (2) by means of the formula $p = \frac{1}{2}(n-1)(n-2) - 2\delta$, connecting it with the order of the curve and the number of double points (or equivalent higher singularities), (3) as equal to the number of linearly independent integrals of the first kind, (4) as equal to the number of adjoint polynomials of order $n - 3$. We have already seen that when we extend to surfaces the ideas involved in the first of these definitions we meet with two distinct Bettian numbers p_1, p_2. Corresponding to the third and fourth definitions we have for a surface of order

* "Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde von beliebig vielen Dimensionen," *Math. Annalen*, vol. 2 (1870).

n an invariant number* (p_g), called the geometric genus (*genre géometrique*) which is equal to the number of linearly independent double integrals of the first kind, as well as to the number of linearly independent adjoint surfaces of order $n - 4$, the definition of adjoint being appropriately modified. Corresponding to the second definition we have a number (p_n), also invariant, called the numerical genus (Cayley's deficiency), which in the case when the surface has only singularities of the simplest kinds is defined by the formula

$$p_n = \tfrac{1}{6}(n-1)(n-2)(n-3) - \Sigma\tfrac{1}{6}i(i-1)(i-2)$$
$$- (n-4)d + 2t + \pi - 1$$

where the second term on the right is due to isolated i-tuple points, and the subsequent terms to a double curve of order d and genus π with t triple points on it. Analogy with the plane case might lead us to suppose that we should always have $p_g = p_n$; but this proves not to be the case. By the definition p_g is positive or zero; but it was pointed out by Cayley in 1871† that p_n might be negative, and that in particular for a ruled surface p_n was equal to the genus of a plane section taken negatively. For example, in the case of a quartic cone without double lines, the formula just quoted gives

$$p_n = 1 - 4 = -3.$$

Subsequent investigation has shown that, though in a certain large sense the two numbers are generally equal, there are important classes of surfaces for which $p_g > p_n$, while the inequality is never reversed. A slight sketch is given of some of the leading geometrical properties with which these numbers are associated, and of their connection with double integrals of the first kind. The theory of double integrals is also applied to prove the invariance for birational transformation of still another "genus" of surfaces (*Curvengeschlecht, second genre*), which can be defined as the genus (in the ordinary sense) of the variable curves of intersection of the original surface and the adjoint surfaces of order $n - 4$; with this are again associated other numbers showing the fundamental property of invariance.

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* First introduced by Clebsch, "Sur les surfaces algébriques," *Comptes rendus*, vol. 67 (December, 1868).

† "On the deficiency of certain surfaces," *Math. Annalen*, vol. 3; Coll. Math. Papers, vol. 8, no. 524.

The final chapter is chiefly geometrical, dealing partly with some of the questions already referred to, and especially with certain topics in which the theory of curves in space is associated with that of surfaces. It is necessarily fragmentary in character, and may, perhaps, be regarded in some sense as an introduction to the more purely geometric treatment of surfaces which M. Picard promises us in his second volume.†

It is difficult to give in a few words a critical verdict on a book which embraces as many subjects as the volume under review. It is undoubtedly rather fragmentary; and the reader who goes straight through it, and does not merely pick out the chapters which deal with topics of which he has made the acquaintance elsewhere, may be to some extent inclined to complain that the connection between different parts of the book is not made more evident. One would find it rather a relief—to take one illustration—if a clearer idea were given of the relation between the Bettian numbers which are dealt with in the earlier chapters, and the various "genera" which figure prominently towards the end, all of them in some sense analogues of the familiar and unique p of two dimensional geometry. And again a reader whose tastes lie in the direction of thoroughness in investigations may have some cause for complaint on account of the frequency with which the authors refuse to discuss any but the most straightforward cases of the various problems; it is difficult to estimate the value of a theorem which only professes to be "generally" true.

But this scrappiness, which I believe very largely to be due to the present state of our knowledge of functions of two variables, has great counter-balancing advantages. There is hardly a chapter in the book which does not suggest to any intelligent reader a variety of problems which cry for solution. Some of these are probably worthy the attention and require the skill of M. Picard's equals, while to others any competent analyst or geometer might devote some attention with reasonable expectation of obtaining results of interest. Above all things the book is supremely interesting; for my own part, at least, I can recall no book that

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* Introduced by Noether in his paper: "Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde," *Math. Annalen*, vol. 8 (1875).

† The reader to whom the general geometrical questions touched on in this and the preceding chapters are not familiar may be referred to a most fascinating article by Castelnuovo and Enriques, "Sur quelques récents résultats dans la théorie des surfaces algébriques" (*Math. Annalen*, vol. 48 (1897)).

I have read with such pleasure since the days when I first met with Dr. Salmon's incomparable treatise on conic sections.

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NOTE ON PAGE'S ORDINARY DIFFERENTIAL EQUATIONS.

AN interesting review of this elementary text book was given by Professor Lovett in the BULLETIN, April, 1898. As the suggestions offered in the review cited are mainly of a general nature and appeal especially to those teachers familiar with the larger works of Lie, and hence able to make the desirable amplifications, it would seem worth while to address to the average reader or teacher of this text a few critical remarks of detailed character. Since my first acquaintance with Lie's groups and theories of integration, I have had the desire to introduce a class of mature students to the theory of ordinary and partial differential equations through the medium of continuous groups. Having used* the text by Page, I am more than ever convinced that the proper method (and one that will come more and more into vogue) of attacking differential equations is that which employs the powerful machinery—so simple when once mastered—set up and perfected by the illustrious Lie.

Being in full sympathy with the aims of the text, I was glad to find that, on the whole, the task had been well executed. I trust that in a second edition all objections that prove to be well grounded will be eradicated and that the errata, too numerous for an elementary text, will be corrected.

There is a curious mistake on p. 6, where the tangents to every integral curve of an ordinary differential equation are said to pass through the origin ! This is indeed the case for the only example given in the paragraph concerned. The answer to Ex. (19), p. 9, should be

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* During a year's graduate course in continuous groups, we devoted two months to the reading of Page's text, finding it a very practical supplement to a course of lectures on the general theory.