S, merely converting the two integral curves into any other two curves on this surface. Invariant relations, it is shown, give projective properties of the integrating ruled surface. The differential equation of the sixth order, satisfied by the line coördinates of a generator of the ruled surface, is set up in a normal form. Then the necessary and sufficient conditions are discussed under which the generators of the ruled surface belong to a linear complex, either general or special, to a linear congruence with distinct or coincident directrices, and finally the conditions under which they form a surface of the second order. The reduction of the system to the semi-canonical form is found to be equivalent with the geometrical problem of finding the curved asymptotic lines on the surface, and a new proof of Serret's theorem about such lines is given.

The primitive continuous group of point transformations in two variables can, by a proper choice of the variables, be transformed into *projective groups* of the plane, a result Lie obtains after determining the canonical forms of the primitive groups. This fact can, however, be established from the general properties of such groups, and it is shown in Dr. Blichfeldt's paper, which will appear in the *Transactions*, that this point of view gives rise to a new determination of these primitive groups.

EDWARD KASNER,
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THE APRIL MEETING OF THE CHICAGO SECTION.

A REGULAR meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Saturday, April 6, 1901. The following members were present:

Professor W. W. Beman, Professor Oskar Bolza, Professor D. F. Campbell, Professor E. W. Davis, Professor Thomas F. Holgate, Professor E. H. Moore, Dr. F. R. Moulton, Miss Ida M. Schottenfels, Professor H. E. Slaught, Mr. Burke Smith.

The President of the Society, Professor E. H. Moore, occupied the chair. The following papers were presented and read:

(1) Dr. F. R. Moulton : "On the validity of the method of computing absolute perturbations."

(2) Professor George C. Chatburn : "A theorem in arrangements."

(3) Professor E. W. Davis : "A definition of mathematical probability."

(4) Dr. Jacob Westlund : "Note on multiply perfect numbers."

(5) Professor F. Mertens : "Zur linearen Transformation der θ-Reihen."

(6) Dr. J. H. McDonald : "The reduction of hyperelliptic integrals of genus two to elliptic integrals, by a transformation of order four."

(7) Professor L. E. Dickson : "Representation of linear groups as transitive substitution groups."

(8) Dr. G. A. Miller : "Determination of all the groups of order p^m which contain the abelian group of the type $(m-2, 1)$, p being any prime number."

(9) Mr. Robert E. Moritz : "A generalization of the differentiation process."

(10) Mr. Robert E. Moritz : "On ratients, with an extension of the ordinary calculus."

Professor Chatburn's paper and the two papers by Mr. Moritz were presented to the Society through Professor Davis, and in the absence of the authors were read by him ; Professor Mertens's paper was presented through Professor Moore ; in the absence of Dr. Westlund his paper was read by the Secretary. Dr. McDonald was introduced to the Section by Professor Bolza. Abstracts of the papers follow.

It is generally understood, and indeed it is usually stated in works on celestial mechanics,* that the methods used by astronomers are only approximate, or else that the convergence of the series employed is assumed. The former means to a mathematician that the functions from which the perturbations are computed do not represent the actual perturbations with rigor ; the latter leaves the validity of the whole process in doubt. In Dr. Moulton's paper it is asked whether the power series in the masses used in computing the perturbations are convergent, and whether they rigorously represent these perturbations. It is answered that, under certain conditions which are fulfilled in case of the major planets, the series in the masses, in actual use by as-

* Dziobek's Planeten-Bewegung, p. 167 ; Tisserand's Mécanique céleste, vol. 1, pp. 190 and 193–4 ; Brown's Lunar theory, preface, p. vi.

tronomers, are absolutely convergent for all values of the time not greater in numerical value than a fixed limit which can be determined, and that these series represent the perturbations exactly. This limit within which the series are certainly convergent is doubtless much smaller than the true radius of convergence. The results are equally true in Laplace's method of computing the absolute perturbations of the coördinates, and in Lagrange's method of the variation of parameters.

The method of proof is an adaptation of Cauchy's calcul des limites,* which has been used with great success in the well known works of Fuchs, Frobenius, Weierstrass, Poincaré, and many others, in the theory of differential equations. The particular modification used here is the same extension as that employed by Poincaré in his memoir in the *Acta Mathematica*, volume XIII. Indeed, Poincaré's results are closely related to those attained here; but, having another object in view, his investigation does not regard the validity of the precise method used by astronomers.

The second part of the paper is devoted to a discussion of the geometrical and physical meaning of the terms of the various orders with respect to each mass separately. It is shown that there is a perfect correspondence between the terms of the power series and the successive corrections to the elliptic orbits which, from a general mathematical point of view, seem to lead most naturally toward the true orbit. The conclusions are that the methods used by astronomers are, under certain limitations, perfectly valid, and that "successive approximations" means successive approximations toward the true numerical values by means of series which represent the functions exactly.

Professor Chatburn's theorem in arrangements gives a method by which c things (say cards numbered consecutively) can be divided into groups, and then rearranged group by group, redivided and rearranged for n or more times, so that the final position of any one of the things shall be that given by the terms of an arithmetical progression in which unity is the first term, c the last term, and p the number of terms.

Professor Davis offered the following definition for mathematical probability: the direct probability that the occasion A shall lead to the event B is the value toward which tends

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* *Comptes rendus*, vol. 14.

the ratio of the number of A's that are followed by B, to the total number of A's, as these recur without limit. He also stated a corresponding definition for inverse probability.

Defining a multiply perfect number as one which is an exact divisor of the sum of all its divisors, the multiplicity being the quotient, Dr. Westlund shows in his note that the only numbers of multiplicity 3 of the form $m = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3$ where p_1, p_2, p_3 are three distinct primes, are the two numbers $2^3 \cdot 3 \cdot 5$ and $2^5 \cdot 3 \cdot 7$.

Professor Mertens makes a direct and elegant determination of the twenty-fourth root of unity, fundamental in the theory of the linear transformations of the θ-series. His paper will be published in the *Transactions.*

The reduction of hyperelliptic integrals of genus two to elliptic integrals is effected by a special involution of order four containing a form which is the square of a quadratic. From the general Weierstrass-Picard theorem it is known that if there exists one reducible integral, there exists also a second having the same irrationality and reducible by a transformation of the same order. The problem treated in Dr. McDonald's paper is: Given one integral, to find the other. It is first shown that the special involution of order four may be uniquely determined by a cubic and a quadratic form. Among the covariants of this system of cubic and quadratic there exists another cubic and quadratic having with the first mutual relations analogous to, but more perfect than, the relations between a single cubic form and its cubic covariant. The involution which is determined by this second cubic and quadratic is one which will deduce a second hyperelliptic integral having the same irrationality as the first but a different numerator, which must be the one indicated by the Weierstrass-Picard theorem. The relations between the two numerators of reducible integrals and the corresponding involutions are a generalization of those found by Professor Bolza for the reduction of order three, and which also exist for the reduction of order two.

In Professor Dickson's paper, which is to be published in the *American Journal of Mathematics*, it is pointed out that the group of all linear homogeneous substitutions S on m indices with coefficients in the $GF[p^n]$ may be represented as a transitive substitution group on $p^{nm} - 1$ letters; the simple group defined by taking the substitutions S fractionally may be represented as a doubly transitive substitution group on $(p^{nm}-1)/(p^n-1)$ letters. In the former case we may employ as the $p^{nm}-1$ letters the totality of functions $\lambda \equiv \lambda_1\xi_1 + \lambda_2\xi_2 + \cdots + \lambda_m\xi_m$, in which the λ_2 run independently through the series of marks of the field, the case $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$ being excluded.

For a subgroup defined by an invariant $\psi(\xi_1, \cdots, \xi_m)$, the above representation leads to an intransitive substitution group. To illustrate the method of obtaining a transitive representation, consider the orthogonal group. If an orthogonal substitution replaces ξ_1 by $\omega \equiv \alpha_1\xi_1 + \cdots + \alpha_m\zeta_m$ then $\alpha_1^2 + \cdots + \alpha_m^2 = 1$; inversely, for any such set of marks α_2, the orthogonal group contains a substitution replacing ξ_1 by ω. Under an orthogonal substitution the above functions λ are permuted in such a way that the totality of functions λ in which $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_m^2$ is a constant mark c are permuted transitively. Except when $\psi S = 3$, the case $c = 0$ leads to a representation upon fewer letters than any case $c \neq 0$. For $c = 0$, we combine the p^{n-1} functions λ, in which the ratios $\lambda_1 : \lambda_2 : \cdots : \lambda_m$ are fixed, into one symbol $\xi\lambda\xi$. We obtain for $m > 2$ a transitive representation of the orthogonal quotient group. According as m is odd or even, this minimum number of letters is $(p^{n(m-1)}-1)/(p^n-1)$ or

$$(p^{nm/2} - \varepsilon)(p^{n(m/2-1)} + \varepsilon)/(p^n - 1)$$

$$[\varepsilon \equiv (-1)^{m(p^n-1)/4}]$$

For $m = 3$, the number is $p^n + 1$, in accord with the known isomorphism of the orthogonal quotient group with the linear fractional group in one variable.

These methods are then applied to the representation of every linear group defined by a quadratic invariant. The first hypoabelian group on $2m$ indices in the $GF[2^n]$ is represented as a transitive substitution group on either of the following numbers of letters:

$$(2^{xm} - 1)(2^{n(m-1)} + 1)/(2^n - 1), \ (2^{nm} - 1)2^{n(m-1)},$$

the second number being larger than the first if $n > 1$ and smaller if $n = 1$. For the case $n = 1$, the small number is $2^{2m-1} - 2^{m-1}$, in accord with Jordan's proof of the isomorphism of the first hypoabelian group with the groups studied by Steiner in dealing with problems of contact of curves of the m^{th} order The second hypoabelian group is represented as a transitive substitution group on $(2^{nm} + 1)(2^{n(m-1)} - 1)/(2^n - 1)$ letters.

In a paper to be presented to the *Mathematische Annalen*, the method is applied to the hyperorthogonal groups with the invariant $\sum_{1=1}^{m}\xi_i\bar{\xi}_i$.

Dr. Miller's paper deals in the first place with the group of isomorphisms of the abelian group H of type $(m-2, 1)$. This study is facilitated by the use of two theorems (proved in the introduction), viz.: The only operators that transform each subgroup of an abelian group into itself are those which transform each operator of the group into the same power, and the number of invariant operators in the group of isomorphisms of an abelian group is equal to the number of natural numbers which do not exceed the highest order of operators of the group and are prime to this order.

The group of isomorphisms of H contains p^3-1 operators of order p (when p is odd) forming, with identity, the non-abelian group of order p^3. When $p=2$ there are 15 operators of order 2 in the group of isomorphisms of H. These generate a group of order 32 containing 16 operators of order four. When $p>3$ there are seven non-abelian groups of order p^m which contain the abelian group of type $(m-2, 1)$; when $p=3$ there are eight such groups and the number of these groups is seventeen when $p=2$. The paper will be published in the *Transactions*.

Mr. Moritz's first paper, which will be published in the *American Journal of Mathematics*, is an extension of the ideas presented in a paper by the same author, of which an abstract is found on page 185, volume 6, of the BULLETIN. Starting with an abstract conception of an indeterminate, he arrives at the conclusion that the process of differentiation does not occupy the unique position which it is commonly supposed to occupy, but that there are other limiting processes. The equation $\lim_{h=1} \log_h \frac{F(x^h)}{F(x)} = \frac{gy}{gx}$ defines such a process, to which the term *quotientiation* is applied, $\frac{gy}{gx}$ being the quotiential coefficient of $y=F(x)$ with respect to x. The laws of this process are examined and the various rules for its application developed. It is suggested that a quotiential in which quotientiation takes the place of differentiation in the ordinary calculus, is not only conceivable, but might be applied, though with less ease than the differential calculus, to the investigation and interpretation of natural phenomena.

A further attempt at setting up other forms fails apparently, owing to the unsymmetric nature of the process a^b. If however $a^{\log b}$ is looked upon as derived by successive multiplication of a, just as $a \times b$ is obtained by successive additions of a, the difficulty vanishes completely. In fact, using De Morgan's form of the algebraic processes in which the process of the nth order is defined by $\log^{-n} (\log^n a + \log^n b)$, its inverse by $\log^{-n} (\log^n a - \log^n b)$, where n may be negative as well as positive, and where the 0th and 1st process are respectively identical with the ordinary addition and multiplication process, the writer finds that for every pair of consecutive processes there exists a limiting process, which when applied to any function $y = F(x)$ gives rise to a function $\frac{d_n y}{d_n x} = \log^{-n}\left(\frac{d \log^n y}{d \log^n x}\right)$, where n is the lower order of the processes involved. $\frac{d_n y}{d_n x}$ is called the *ratient* in the nth process of y with respect to x. Since n may be negative as well as positive, the chain of ratients can be extended in either direction without limit, the zero ratients being the ordinary differential coefficients. It is shown how any ratient may be expressed in terms of the ratient of any higher or lower order process. Successive ratients assume the form $\frac{d_n^r y}{d_n x^r} = \log^{-n} \frac{d^r \log^n y}{(d \log^n x)^r}$.

Mr. Moritz's second paper is based upon the definition of De Morgan's processes, and that of ratients as developed by the author in his paper on "Generalization of the differentiation process." A formula is developed for the ratient in the nth process of two or more functions combined by the mth process, from which the differentiation formulæ for a product of functions, powers of functions, or generally of functions combined by the process of any order, positive or negative, follow among others as simple corollaries. It is then shown that the fundamental theorems regarding the differentiation of a sum or a product hold for ratients in general when sums and products are replaced respectively by operands in the nth and $(n + 1)$th processes.

The number system for the nth process is examined and it is found that every De Morgan process possesses a distinctive number system whose properties are identical with those of the ordinary number system. To the imaginary i in the ordinary algebra correspond new imaginaries, to the transcendentals π and e correspond new transcendentals. In fact the transcendentals corresponding to π and e in the nth process are algebraic numbers borrowed from the $(n-1)$th and $(n+1)$th processes respectively. The totality of numbers may thus be conceived as arranged in an infinite number of strata, the ordinary algebraic numbers forming a single stratum.

With the extension of the differentiation process, and the stratification of the number body, all the results of the ordinary calculus, for example Taylor's theorem, can be expressed in terms of any two consecutive processes.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILL.

THE VALUE OF $\int_0^{\pi/2} (\log 2 \cos \varphi)^m \varphi^n d\varphi$.

BY PROFESSOR F. MORLEY.

(Read before the American Mathematical Society, April 29, 1899.)

THE integral in question, in which m is any positive integer, and n is any even positive integer or zero, is given in effect, for the cases $m = 1$, $n = 0$, in Williamson's Integral Calculus, § 118 of the second edition, being taken from a paper by H. G. (presumably Harvey Goodwin, Bishop of Carlisle) in volume 3 of the *Quarterly Journal of Mathematics*. Further it is given in effect, for the case $m = 2$, $n = 0$, in Wolstenholme's Problems, p. 332 of the second edition. It seems worth while to show how it can be expressed in general, in terms of the constants $s_r = \sum_1^\infty 1/n^r$, which may be regarded as known.

We know that when p and q are real

$$1 + \frac{p \cdot q}{1 \cdot 1} + \frac{p(p-1)q(q-1)}{1 \cdot 2 \cdot 1 \cdot 2} + \cdots = \frac{\Gamma(1+p+q)}{\Gamma(1+p)\Gamma(1+q)},$$

when $p + q > -1$ (Forsyth, Differential equations, p. 197). But the left member is the constant term in the product of the series

$$(1+x)^p = 1 + px + \cdots$$

and

$$(1 + 1/x)^q = 1 + q/x + \cdots.$$