$$( - )^{n/2} \frac{m!\, n!}{2^{m+n}} \times \text{the coefficient of } \alpha^m \beta^n \text{ in}$$

$$\exp \left[ \sum_2^\infty ( - )^r \frac{s_r}{r} \{ (2\alpha)^r - (\alpha + \beta)^r - (\alpha - \beta)^r \} \right].$$

This generating function is, to a few terms,

$$\exp\ [s_2(\alpha^2 - \beta^2) - 2\ \alpha s_3(\alpha^2 - \beta^2)$$
$$+ \tfrac{1}{2} s_4 (7\alpha^4 - 6\alpha^2\beta^2 - \beta^4) - 2\ \alpha s_5 (3\alpha^4 - 2\alpha^2\beta^2 - \beta^4) + \ ]$$

or

$$1 + s_2(\alpha^2 - \beta^2) - 2s_3(\alpha^3 - \alpha\beta^2)$$
$$+ \frac{s_2^2 + 7s_4}{2}\alpha^4 - (s_2^2 + 3s_4)\alpha^2\beta^2 + \frac{s_2^2 - s_4}{2}\beta^4$$
$$- 2(s_2 s_3 + 3s_5)\alpha^5 + 4(s_2 s_3 + s_5)\alpha^3\beta^2 - 2(s_2 s_3 - s_5)\alpha\beta^4 + \cdots$$

Thus, in tabular form, a few values for

$$\frac{2}{\pi} \int_0^{\pi/2} (\log 2 \cos \varphi)^m \varphi^n d\varphi$$

are :

| $n =$ | 0 | 2 | 4 |
|---|---|---|---|
| $m =$ 0* | 1 | $\frac{1}{2}s_2$ | $\frac{3}{4}(s_2^2 - s_4)$ |
| 1 | 0 | $-\frac{1}{2}s_3$ | $-\frac{3}{2}(s_2 s_3 - s_5)$ |
| 2 | $\frac{1}{2}s_2$ | $\frac{1}{4}(s_2^2 + 3s_4)$ | |
| 3 | $-\frac{3}{2}s_3$ | $-\frac{3}{2}(s_2 s_3 + s_5)$ | |
| 4 | $\frac{3}{4}(s_2^2 + 7s_4)$ | | |
| 5 | $-\frac{15}{2}(s_2 s_3 + 3s_5)$ | | |

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# ON THE ALGEBRAIC POTENTIAL CURVES.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, February 23, 1901.)

THE object of this paper is to derive the characteristic geometric properties of a class of curves which are of in-

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* The row for which $m = 0$ is of course merely a verification, leading to the known values

$$s_2 = \pi^2/6, \quad s_4 = \pi^4/90.$$

terest in connection with the theory of equations and of the potential function. Analytically, these curves are obtained by equating to zero the rational integral solutions $\varphi(x, y)$ of Laplace's equation

$$\triangle\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0,$$

or, what is equivalent, the real (or imaginary) parts of the rational integral functions of $x + iy$. Various geometric properties are given in Briot and Bouquet's Théorie des fonctions elliptiques (book IV, chapter II), but none are completely characteristic.

## § 1. *Apolarity with Respect to a Point Pair.*

A curve

$$\Phi \equiv a_x^n = 0,$$

is said to be apolar* to a conic

$$Q \equiv \sum P_{ik} u_i u_k \equiv u_P^2 = 0,$$

when every polar conic of the curve is circumscribed about an infinite number of triangles self-conjugate with respect to the conic $Q$; *i. e.*, when the bilinear covariant

$$S \equiv \sum P_{ik} \frac{\partial^2 \Phi}{\partial x_i \partial x_k} \equiv a_P^2 a_x^{n-2} \qquad (1)$$

vanishes identically.

Let the conic degenerate into a pair of points $A$, $B$,

$$A \equiv u_\alpha = 0, \quad B \equiv u_\beta = 0,$$

so that

$$Q = u_\alpha u_\beta ;$$

then

$$S = a_\alpha a_\beta a_x^{n-2},$$

which is the apolar covariant of the forms

$$u_\alpha \quad \text{and} \quad u_\beta a_x^{n-1},$$

or of

$$u_\beta \quad \text{and} \quad u_\alpha a_x^{n-1}.$$

It is easy to show, however, that a point and a curve can be apolar only when the curve consists of a set of straight lines passing through the point, so that

* Reye, *Crelle*, vol. 79 (1874), p. 159. For a convenient summary of the theory of apolar relations see Schlesinger, *Math. Annalen*, vol. 22 (1883), pp. 520–523.

THEOREM I. *If a curve of the nth order is apolar to a point pair (considered as a degenerate conic), the first polar (and therefore any of the polars) of either of the points with respect to the curve consists of a set of lines passing through the other point; the converse is also true.*

Letting $n = 2$, it follows that a conic is apolar to a point pair when the two points are conjugate with respect to the conic. From the definition of apolarity we have then

THEOREM II. *If a curve is apolar to a point pair, the latter is self-conjugate with respect to all the polar conics of the curve; conversely, etc.*

Since from Theorem I both the first polar and the polar conic of either point have nodes, we have

THEOREM III. *If a curve is apolar to a point pair, both the Hessian and the Steinerian of the curve pass through the point pair; furthermore, these points correspond in the sense defined by Clebsch.**

## § 2. *Polar Properties of Potential Curves.*

Instead of an arbitrary point pair, consider now the pair of circular points at infinity $I$, $J$. The equation of this point pair in rectangular line coördinates may be written

$$u^2 + v^2 = 0\,;$$

so that, expressed in rectangular point coördinates, the covariant $S$ of the preceding section becomes

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}.$$

The vanishing of this expression, however, denotes that the curve $\varphi = 0$ is a potential curve. Therefore,

THEOREM IV. *Any potential curve is apolar to the fundamental conic of euclidean geometry consisting of the circular points at infinity; conversely, any curve which is apolar to this fundamental conic is a potential curve.*†

From Theorem I we have then

THEOREM V. *All the polar curves of a circular point with respect to a potential curve degenerate into sets of straight lines passing through the other circular point; conversely, etc.*

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*Clebsch, "Ueber einige von Steiner behandelte Kurven," *Crelle*, vol. 64, p. 288. The converse of the above theorem is not true.

† Cf. Clifford, "On the canonical form of spherical harmonics," Works, p. 234, for a statement concerning "nodal curves" on a sphere, which appears to have some connection with the above.

A conic is a potential curve when the circular points $I$, $J$ are conjugate with respect to it; this implies that the conic intersects the line at infinity in points which are harmonic with respect to $I$ and $J$, *i. e.*, that the asymptotes of the conic are rectangular. From Theorem II, we have then a statement of the characteristic property of potential curves which has the advantage of dealing only with real elements, as follows:

THEOREM VI. *A curve is a potential curve when, and only when, the polar conics of all points with respect to the curve are rectangular hyperbolas.*

From Theorem III we have a property, which is however not characteristic, *i. e.*, not restricted to the potential curves:

THEOREM VII. *The Hessian and the Steinerian curves of a potential curve pass through the circular points $I$, $J$; furthermore, these points correspond in the sense defined by Clebsch.*

Since the polar conics of a polar curve are the polar conics of the original curve, we have

THEOREM VIII. *All the polar curves of a potential curve are themselves potential curves.*

## § 3. *Focal Properties of Potential Curves.*

Consider any rational integral function of the $n$th order in $z$

$$(2) \qquad f(x+iy) \equiv \varphi(x, y) + i\psi(x, y),$$

together with the conjugate expression

$$(3) \qquad \bar{f}(x-iy) \equiv \varphi(x, y) - i\psi(x, y);$$

the equation of the potential curve $\varphi = 0$ may be written in the form

$$(4) \qquad f(x+iy) + \bar{f}(x-iy) = 0.$$

The two terms of the left hand member of this equation, equated separately to zero, represent sets of minimal lines, the first representing $n$ lines through $I$, and the second $n$ lines through $J$. Furthermore, equation (4) is unchanged when $f(z)$ is replaced by $f(z) + i\lambda$, where $\lambda$ is an arbitrary real constant. We have then

THEOREM IX. *The linear system of curves of the $n$th order determined by $2n$ miminal lines ($n$ through each of the circular points) is composed of potential curves; conversely, any potential curve may be obtained as a member of an infinite number of such linear systems.*

This theorem may be restated by using the fact that a curve of the $n$th class has $n^2$ foci, namely the intersections of the $n$ minimal tangents of one system with the $n$ minimal tangents of the other system, as follows :

THEOREM X. *Any curve of the nth order passing through the $n^2$ foci of a curve of the nth class is a potential curve; conversely, all potential curves may be obtained in this way—each potential curve passes through the foci of an infinite number of systems of confocal curves of the nth class.**

Thus for $n = 2$, we have that all the conics which pass through the two real and the two imaginary foci of a conic are rectangular hyperbolas.

The potential curves $\varphi = 0$, $\psi = 0$, obtained in the decomposition of a function of $x + iy$, may be termed *conjugate*† *potential curves*, since the functions $\varphi$ and $\psi$ are conjugate. From (2) and (3) we have

$$(5) \qquad \begin{aligned} 2\varphi &= f(x+iy) + \bar{f}(x-iy), \\ 2i\psi &= f(x-iy) - \bar{f}(x-iy) \,; \end{aligned}$$

therefore the curves belong to a linear system of the kind considered above. Furthermore they intersect orthogonally.‡

THEOREM XI. *Conjugate potential curves of the nth order intersect orthogonally in the foci of a system of confocal curves of the nth class; conversely, two curves of the nth order which intersect orthogonally in the foci of a curve of the nth class are conjugate potential curves.*

The properties stated in Theorems IV and X being definitive for the same class of curves, it follows that these properties are equivalent. From this equivalence we may pass to a more general result relating to the apolarity of a curve and a point pair ; it is necessary merely to project the circular points into an arbitrary pair of points, the potential curves transforming into curves which are apolar to this pair. Therefore, if through each of two points $A$, $B$, $n$ straight lines are drawn, any curve of the $n$th order passing through the $n^2$ points so determined is apolar to the pair $A, B$ ; moreover this construction yields all the apolar curves. This result may be restated :

THEOREM XII. *A curve of the nth order is apolar to a pair of points, $A$, $B$ when, and only when, it is possible to find upon*

* The number of parameters in such a confocal system is $\frac{1}{2}n(n-1)$ ; so that the number of curves from which any potential curve may be derived as a focal curve is $\infty^{\frac{1}{2}(n^2-n+2)}$.

† The term conjugate, of course, here refers to the properties of the functions $\phi$, $\psi$, and is not synonymous with the term apolar.

‡ Briot et Bouquet, p. 223.

*the curve $n^2$ points lying by n's upon n lines through A and at the same time upon n lines through B.**

## § 4. *The Asymptotes.*

Briot and Bouquet prove† that the $n$ asymptotes of a potential curve of the $n$th order are real, concurrent, and disposed symmetrically about their common point, the angle between consecutive asymptotes being $2\pi/n$. This property, however, imposes only $2n - 3$ independent conditions, while the number imposed by the equation‡ $\triangle\varphi \equiv 0$ is $\frac{1}{2}n(n-1)$; so that in general (*i. e.*, if $n > 3$) the above relation between the asymptotes is not peculiar to the potential curves, and Briot and Bouquet's theorem cannot be converted. As to the case $n = 2$ §, it has been shown in § 2 that the potential conics are the rectangular hyperbolas, so that the relation between the asymptotes is characteristic. The same is true in the case $n = 3$, as may be shown by taking a coördinate system with its origin at the point of concurrence of the asymptotes and its axis of abscissas coinciding with one of the asymptotes, and verifying the condition $\triangle\varphi = 0$. Therefore, *the potential cubics may be defined as those cubics which have three real concurrent asymptotes intersecting at angles of 120°.*

In all cases the *point of concurrence O of the asymptotes is a center* of the curve, *i. e.*, if any line is drawn through $O$, the sum of the distances measured from $O$ of the points of intersection lying on one side of $O$ is the same as the corresponding sum for the points on the other side. This follows from the fact that when the origin of coördinates is taken at $O$, all the terms of order $n - 1$ disappear. ‖

From the potential curves we may pass by projection to the curves which are apolar to any point pair $A$, $B$. The asymptotes of the potential curve are transformed into a set of concurrent lines tangent to the new curve at the points $P_1, P_2, \cdots, P_n$ where the line joining $A$, $B$ cuts the curve. From the equality of the angles between consecutive asymptotes, the anharmonic ratios

* When one such set of $n^2$ points exists there is necessarily an infinite number of sets.

† L. c., p. 227.

‡ The number of parameters in the potential curve of the $n$th order is $2n$, so that of all the curves of the $n$th order which pass through $2n$ assigned points one and only one is potential.

§ The case $n = 1$ is trivial since all straight lines are potential curves.

‖ Briot et Bouquet, p. 226; Salmon-Fiedler, Höhere Kurven, 2d ed. p. 145.

$$(A\ B\ P_1\ P_2),\quad (A\ B\ P_2\ P_3),\quad \cdots,\quad (A\ B\ P_n\ P_1)$$

are equal, and therefore the set of points $P_1, \cdots, P_n$ is apolar to the pair $A$, $B$.

THEOREM XIII. *If a curve is apolar to a point pair, the line through the pair intersects the curve in a set of points apolar to the pair, and the tangents to the curve at these points are concurrent.*

The converse is true only for conics and cubics.

## § 4. *Connection with the Theory of Equations.*

Consider the general equation of the $n$th degree in one unknown

$$(6)\qquad f(z) \equiv A_0 z^n + A_1 z^{n-1} + \cdots + A_n = 0 \qquad (A_k = b_k + ic_k),$$

with the $n$ roots

$$z_\alpha = x_\alpha + iy_\alpha \qquad (\alpha = 1, 2, \cdots, n).$$

The conjugate equation

$$\bar{f}(z) \equiv \bar{A}_0 z^n + \bar{A}_1 z^{n-1} + \cdots + \bar{A}_n = 0 \qquad (\bar{A}_k = b_k - ic_k),$$

then has the roots

$$(7)\qquad \bar{z}_\alpha = x_\alpha - iy_\alpha \qquad (\alpha = 1, 2, \cdots, n).$$

The complete solution of equation (6) is equivalent directly to the real solution of the system

$$(8)\qquad \varphi(x, y) = 0, \qquad \psi(x, y) = 0,$$

where $\varphi$ and $\psi$ are the real and imaginary parts of $f(x + iy)$. A problem which then presents itself, namely, the complete solution of this system, is virtually answered in Theorem X: the solutions, by (5), are obtained by solving the linear equations

$$x + iy = z_\alpha, \quad x - iy = \bar{z}_\beta \qquad (\alpha, \beta = 1, 2, \cdots, n).$$

THEOREM XIV. *The complete solution of the auxiliary system* (8) *connected with the equation* (6) *is*

$$x_{\alpha\beta} = \frac{z_\alpha + \bar{z}_\beta}{2}, \quad y_{\alpha\beta} = \frac{z_\alpha - \bar{z}_\beta}{2i} \qquad (\alpha, \beta = 1, 2, \cdots, n).$$

The $n$ real solutions $x_\alpha$, $y_\alpha$ are obtained by letting $\beta = \alpha$; the remaining solutions may be expressed in terms of these as follows:

$$x_{\alpha\beta} = \frac{x_\alpha + x_\beta}{2} + i\frac{y_\alpha - y_\beta}{2}, \quad y_{\alpha\beta} = \frac{y_\alpha + y_\beta}{2} - i\frac{x_\alpha - x_\beta}{2}.$$

As is well known, the $n$ quantities

$$R(z_\alpha) \qquad (\alpha = 1, 2, \cdots, n),$$

where $R$ denotes any rational function, satisfy an equation of the $n$th degree

$$F_n(t) = 0,$$

whose coefficients are rational in $A_0, A_1, \cdots, A_n$. This, however, no longer holds when we consider, instead of rational functions of the roots, rational functions of the real and imaginary parts of the roots; but if we consider the $n^2$ quantities

$$R(x_{\alpha\beta}, y_{\alpha\beta}),$$

they will satisfy an equation of the $n^2$ degree with coefficients which are rational in terms of the coefficients of $\varphi$ and $\psi$, *i. e.*, in terms of $b_0, \cdots, b_n, c_0, \cdots, c_n$. Therefore,

THEOREM XV. *The $n$ quantities*

$$R(x_1, y_1), \quad R(x_2, y_2), \cdots, R(x_n, y_n),$$

*are the real roots of an equation of degree $n^2$ with coefficients which are rational in terms of the real and imaginary parts of the coefficients in* (1) *; the remaining roots of the equation being*

$$R\left(\frac{x_\alpha + x_\beta}{2} + i\frac{y_\alpha - y_\beta}{2}, \quad \frac{y_\alpha + y_\beta}{2} + i\frac{x_\alpha - x_\beta}{2}\right), \quad (\alpha \neq \beta).$$

This result may easily be extended to functions of any number of roots $R(x_1, y_1, x_2, y_2, \cdots)$, and Theorem XIV may be extended to any system of simultaneous equations.

COLUMBIA UNIVERSITY,
*February* 25, 1901.

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## ALTERNATING CURRENT PHENOMENA.

*Alternating Current Phenomena.* By C. P. STEINMETZ. New York, Office of the Electrical World. Third Edition, 1900. Pp. xx + 525.

To electrical engineers Mr. Steinmetz's book is immediately conspicuous by reason of two distinguishing characteristics :