NOTE ON THE TRANSFORMATION OF A GROUP INTO ITS CANONICAL FORM.

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LET

$$X_j \equiv \sum_1^n {}_k \xi_{jk}(x_1, \cdots, x_n) \frac{\partial}{\partial x_k} \quad (j = 1, 2, \cdots, r),$$

where the ξ's are analytic functions of n independent variables $x_1, \cdots, x_n$), denote r independent infinitesimal transformations of a given r parameter group. The finite equations of the one-parameter groups generated by each of the infinitesimal transformations X_j $(j = 1, 2, \cdots, r)$ may be obtained by integration of the r simultaneous systems

$$\frac{dx_1'}{\xi_{j1}(x_1', \cdots, x_n')} = \cdots = \frac{dx_n'}{\xi_{jn}(x_1', \cdots, x_n')} = da$$
$$(j = 1, 2, \cdots, r),$$

subject to the condition that $x_i' = x_i$ $(i = 1, 2, \cdots, n)$ for $a = 0$, a being an arbitrary parameter. Let the integrals of these simultananeous systems be represented by the equations

$$x_i' = f_{ik}(x_1, \cdots, x_n, a) \quad (i = 1, 2, \cdots, n\,;\ k = 1, 2, \cdots, r).$$

Performing upon the manifold $x_1, \cdots, x_n$ a general transformation T_1 of the one parameter group generated by X_1 we obtain the manifold $x_1', \cdots, x_n'$; performing upon this latter manifold a general transformation T_2 generated by X_2, we obtain the manifold $x_1'', \cdots, x_n''$, etc. Thus we have

$$x_1' = f_{11}(x_1, \cdots, x_n, a_1), \qquad \cdots\ x_n' = f_{n1}(x_1, \cdots, x_n, a_1),$$
$$x_1'' = f_{12}(x_1', \cdots, x_n', a_2), \qquad \cdots\ x_n'' = f_{n2}(x_1', \cdots, x_n', a_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_1^{(r)} = f_{1r}(x_1^{(r-1}\ \cdots, x_n^{(r-1)}, a_r), \cdots\ x_n^{(r)} = f_{nr}(x_1^{(r-1)}, \cdots, x_n^{(r-1)}, a_r),$$

where $a_1, \cdots, a_r$ are arbitrary parameters. Eliminating $x_1', \cdots, x_n^{(r-1)}$ between these equations, we have

$$x_i^{(r)} = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \quad (i = 1, 2, \cdots, n),$$

or, if we denote the transformed variables $x_1^{(r)}, \cdots, x_n^{(r)}$ by $x_1', \cdots, x_n'$ respectively,

$$(1) \qquad x_i' = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \quad (i = 1, 2, \cdots, n),$$

in which the parameters $a_1, \cdots, a_r$ are all essential. These equations define ∞^r transformations of the given r-parameter group.*

This method will now be applied to finding the finite equations of the group whose infinitesimal transformations are

$$X_1 \equiv \frac{\partial}{\partial x_2}, \quad X_2 \equiv x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}.$$

The infinitesimal transformation $\frac{\partial}{\partial x_2}$ generates the one-parameter group whose finite equations are

$$(2) \qquad x_1' = x_1, \quad x_2' = x_2 + a_1,$$

and similarly $x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}$ generates the one-parameter group whose finite equations are

$$(3) \qquad x_1'' = x_1' + a_2, \quad x_2'' = x_2' e^{a_2}.$$

Eliminating x_1', x_2' between equations (2) and (3) we have

$$x_1'' = x_1 + a_2, \quad x_2'' = x_2 e^{a_2} + a_1 e^{a_2},$$

or, replacing x_1'', x_2'' by x_1', x_2',

$$(4) \qquad x_1' = x_1 + a_2, \quad x_2' = x_2 e^{a_2} + a_1 e^{a_2}.$$

These equations define a transformation T_a of the given group G. Similarly, the equations defining a transformation T_b of G are

$$(5) \qquad x_1'' = x_1' + b_2, \quad x_2'' = x_2' e^{b_2} + b_1 e^{b_2}.$$

The transformation $T_b T_a$, obtained by the composition of the transformations T_a and T_b in the order named, is defined by the equations

$$(6) \qquad x_1'' = x_1 + a_2 + b_2, \quad x_2'' = x_2 e^{a_2+b_2} + a_1 e^{a_2+b_2} + b_1 e^{b_2}.$$

* Lie, Continuierliche Gruppen, pp. 192–197.

If this is equivalent to a transformation T_c of G, we also have

$$(7)\qquad x_1'' = x_1 + c_2, \quad x_2'' = x_2 e^{c_2} + c_1 e^{c_2}.$$

Therefore

$$(8)\qquad \begin{aligned} c_1 &= a_1 + b_1 e^{-a_2} \equiv \varphi_1(a, b), \\ c_2 &= a_2 + b_2 \quad\;\; \equiv \varphi_2(a, b). \end{aligned}$$

The c's are finite for every finite system of values of the a's and b's, but the transformation $T_b T_a$, or T_c, may not be generated by an infinitesimal transformation of the group, as will appear later.

In order to transform the finite equations

$$(1)\qquad x_i' = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \quad (i = 1, 2, \cdots, n)$$

of the group into their canonical form, Lie proceeds as follows.* By differentiation and elimination we obtain the differential equations

$$\frac{\partial x_i'}{\partial a_k} = \sum_1^r {}_j \psi_{jk}(a_1, \cdots, a_r) \xi_{ji}(x_1', \cdots, x'_n)$$
$$(i = 1, 2, \cdots, n;\; k = 1, 2, \cdots, r),$$

which are satisfied identically by equations (1) above. Since the determinant of the $\psi_{jk} \not\equiv 0$, these equations may be written in the form

$$\xi_{ji}(x_1', \cdots, x_n') = \sum_1^r {}_k \alpha_{jk}(a_1, \cdots, a_r) \frac{\partial x_i'}{\partial a_k}$$
$$(i = 1, 2, \cdots, n;\; j = 1, 2, \cdots, r),$$

where the determinant of the $\alpha_{jk}(a) \not\equiv 0$, and no linear relation of the form $e_1 \xi_{1i}(x') + \cdots + e_r \xi_{ri}(x') \equiv 0$, with constant coefficients e persists simultaneously for $i = 1, 2, \cdots, n$. The canonical form of the finite equations of the group can now be obtained by integration of the simultaneous system

$$\frac{dx_1'}{\sum_1^r {}_j \lambda_j \xi_{j1}(x')} = \cdots = \frac{dx_n'}{\sum_1^r {}_j \lambda_j \xi_{jn}(x')} = dt,$$

* Transformationsgruppen, vol. 3, pp. 609–611.

that is to say, by integration of the simultaneous system

$$\frac{dx_i'}{dt} = \sum_{k,j}^{1\cdots r} \lambda_j a_{jk}(a_1, \cdots, a_r) \frac{\partial x_i'}{\partial a_k} \quad (i = 1, 2, \cdots, n),$$

subject to the condition that $x_i' = x_i (i = 1, 2, \cdots, n)$ for $t = 0$. Consequently $a_1, \cdots, a_r$, considered as functions of t, satisfy the simultaneous system

$$(9) \qquad \frac{da_k}{dt} = \sum_1^r {}_j \lambda_j a_{jk}(a_1, \cdots, a_r) \qquad (k = 1, 2, \cdots, r).$$

Since $x_i' = x_i (i = 1, 2, \cdots, n)$ for $t = 0$, $a_1, \cdots, a_r$ must assume the values $a_1^0, \cdots, a_r^0$ for $t = 0$, where $a_1^0, \cdots, a_r^0$ are the parameters which furnish the identical transformation. Integrating the simultaneous system (9) subject to the condition that $a_k = a_k^0 (k = 1, 2, \cdots, r)$ for $t = 0$, we obtain the integrals

$$a_k = M_k(\lambda_1 t, \cdots, \lambda_r t) \qquad (k = 1, 2, \cdots, r),$$

or, if we denote the parameters $\lambda_1 t, \cdots, \lambda_r t$ by $\mu_1, \cdots, \mu_r$, respectively, we have

$$a_k = M_k(\mu_1, \cdots, \mu_r) \qquad (k = 1, 2, \cdots, r).$$

Inserting these values of $a_1, \cdots, a_r$ in equations (1), we have

$$x_i' = f_i(x_1, \cdots, x_n, M_1(\mu), \cdots, M_r(\mu)) \qquad (i = 1, 2, \cdots, n),$$

which are the *canonical* equations of the given group.

Consider the application of this method to the finite equations (4) on page 281. Since the equations

$$(4) \qquad \begin{aligned} x_1' &= x_1 + a_2 &&\equiv f_1(x, a), \\ x_2' &= x_2 e^{a_2} + a_1 e^{a_2} &&\equiv f_2(x, a) \end{aligned}$$

define ∞^2 transformations T_a which constitute a group, functional equations persist of the form

$$f_i(f(x, a), b) = f_i(x, \varphi(a, b)) \qquad (i = 1, 2),$$

or

$$(10) \qquad f_i(x', b) = f_i(x, c) \qquad (i = 1, 2).$$

The functions φ_1, φ_2 in equations (8) are independent of one another with respect to b_1, b_2, for

$$\frac{\partial \varphi_h}{\partial b_j} = \begin{vmatrix} e^{-a_2}, & 0 \\ 0, & 1 \end{vmatrix}$$

is not identically zero. Therefore we may regard x_1, x_2, a_1, a_2, c_1, c_2 as independent variables, and x_1', x_2', x_1'', x_2'', b_1, b_2 as dependent variables. Thus the differentiation of the functional equations (10), that is, of

$$x_1' + b_2 = x_1 + c_2,$$

$$x_2' e^{b_2} + b_1 e^{b_2} = x_2 e^{c_2} + c_1 e^{c_2},$$

with respect to the a's gives

$$\frac{\partial x_1'}{\partial a_1} + \frac{\partial b_2}{\partial a_1} = 0, \qquad \frac{\partial x_1'}{\partial a_2} + \frac{\partial b_2}{\partial a_2} = 0,$$

$$(11) \qquad e^{b_2}\frac{\partial x_2'}{\partial a_1} + (x_2' + b_1)e^{b_2}\frac{\partial b_2}{\partial a_1} + e^{b_2}\frac{\partial b_1}{\partial a_1} = 0,$$

$$e^{b_2}\frac{\partial x_2'}{\partial a_2} + (x_2' + b_1)e^{b_2}\frac{\partial b_2}{\partial a_2} + e^{b_2}\frac{\partial b_1}{\partial a_2} = 0.$$

In order to obtain expressions for $\frac{\partial b_j}{\partial a_k}$, we differentiate equations (8) with respect to a_1, a_2, and thus obtain

$$0 = \frac{\partial b_2}{\partial a_1}, \qquad 0 = 1 + \frac{\partial b_2}{\partial a_2}, \qquad 0 = 1 + e^{-a_2}\frac{\partial b_1}{\partial a_1},$$

$$0 = - b_1 e^{-a_2} + e^{-a_2}\frac{\partial b_1}{\partial a_2},$$

whence

$$\frac{\partial b_1}{\partial a_1} = - e^{a_2}, \quad \frac{\partial b_1}{\partial a_2} = b_1, \quad \frac{\partial b_2}{\partial a_1} = 0, \quad \frac{\partial b_2}{\partial a_2} = - 1.$$

Inserting these values, equations (11) become

$$(12) \qquad \frac{\partial x_1'}{\partial a_1} = 0, \quad \frac{\partial x_1'}{\partial a_2} = 1, \quad \frac{\partial x_2'}{\partial a_1} = e^{a_2}, \quad \frac{\partial x_2'}{\partial a_2} = x_2'.$$

These equations are of the form

$$\frac{\partial x_i'}{\partial a_k} = \Psi_{1k}(a, b)\Phi_{1i}(x', b) + \Psi_{2k}(a, b)\Phi_{2i}(x', b)$$
$$(i = 1, 2; \; k = 1, 2),$$

where

$$\Psi_{ik}(a,\ b) \equiv \frac{\partial b_i}{\partial a_k} \quad (i=1,\ 2\ ;\ k=1,\ 2).$$

Therefore

$$\Psi_{11} = -e^{a_2}, \quad \Psi_{12} = b_1, \quad \Psi_{21} = 0, \quad \Psi_{22} = -1,$$

and, consequently,

$$\Phi_{11} = 0, \quad \Phi_{12} = -1, \quad \Phi_{21} = -1, \quad \Phi_{22} = -x_2' - b_1.$$

If we insert in equations (12) the values of x_1', x_2' derived from

$$(4) \qquad x_2' = x_2 e^{a_2} + a_1 e^{a_2}, \qquad x_1' = x_1 + a_2,$$

they become equations between the independent quantities x_1, x_2, a_1, a_2, b_1, b_2, and must therefore be satisfied identically. Hence equations (12) will still persist identically in virtue of equations (4) if we assign definite values $\bar{b}_1$, $\bar{b}_2$ to b_1, b_2. For this purpose let $\bar{b}_1 = 1$, and denote the functions $\Psi(a,\ \bar{b})$ and $\Phi(x',\ \bar{b})$ by $\psi(a)$ and $\xi(x')$ respectively. Then we have

$$(13) \qquad \frac{\partial x_i'}{\partial a_k} = \psi_{1k}(a)\xi_{1i}(x') + \psi_{2k}(a)\xi_{2i}(x')$$
$$(i = 1,\ 2\ ;\ k = 1,\ 2),$$

where

$$\xi_{11} = 0, \quad \xi_{12} = -1, \quad \xi_{21} = -1, \quad \xi_{22} = -x_2' - 1,$$
$$\psi_{11} = -e^{a_2}, \quad \psi_{12} = 1, \quad \psi_{21} = 0, \quad \psi_{22} = -1.$$

The determinant of the ψ's, namely

$$\begin{vmatrix} \psi_{11}, & \psi_{12} \\ \psi_{21}, & \psi_{22} \end{vmatrix} = \begin{vmatrix} -e^{a_2}, & 1 \\ 0, & -1 \end{vmatrix},$$

not being identically zero, equations (13) may be solved for the ξ's, giving

$$\xi_{ji}(x_1',\ x_2') = \alpha_{j1}(a)\frac{\partial x_i'}{\partial a_1} + \alpha_{j2}(a)\frac{\partial x_i'}{\partial a_2}$$

$$(i=1,\ 2\ ;\ j=1,\ 2),$$

where

$$\alpha_{11} = -e^{-a_2}, \quad \alpha_{12} = 0, \quad \alpha_{12} = -e^{-a_2}, \quad \alpha_{22} = -1.$$

In order to obtain the system of functions $a_k = M_k(\mu_1, \mu_2)$ $(k = 1, 2)$ which, when introduced into the finite equations (4) of the group, will transform these equations into their canonical form, it is necessary to integrate the simultaneous system

$$\frac{da_k}{dt} = \sum_1^2 {}_j\lambda_j a_{jk}(a_1, a_2) \qquad (k = 1, 2)$$

subject to the condition that $a_k = a_k^0$ ($k = 1, 2$) for $t = 0$, that is, we must integrate the simultaneous system

$$\frac{da_1}{dt} = -\lambda_1 e^{-a_2} - \lambda_2 e^{-a_2}, \qquad \frac{da_2}{dt} = -\lambda_2,$$

subject to the condition that $a_1 = a_2 = 0$ for $t = 0$. The integrals of these equations are

$$a_1 = -\frac{\lambda_1 t + \lambda_2 t}{\lambda_2 t}(e^{\lambda_2 t} - 1) \equiv M_1(\lambda_1 t, \lambda_2 t),$$

$$a_2 = -\lambda_2 t \equiv M_2(\lambda_1 t, \lambda_2 t),$$

or, if we let $\mu_1 = -(\lambda_1 + \lambda_2)t$, $\mu_2 = -\lambda_2 t$,

$$(14) \qquad \begin{aligned} a_1 &= -\frac{\mu_1}{\mu_2}(e^{-\mu_2} - 1) \equiv M_1(\mu_1, \mu_2), \\ a_2 &= \mu_2 \equiv M_2(\mu_1, \mu_2). \end{aligned}$$

Inserting these values, equations (4) become

$$x_1' = x_1 + \mu_2,$$

$$x_2' = x_2 e^{\mu_2} + \frac{\mu_1}{\mu_2}(e^{\mu_2} - 1),$$

which is the canonical form of the equations defining a transformation T_μ of the group. These equations are of precisely the same form as the equations obtained by summation of the series

$$x_i' = x_i + \sum_1^2 {}_j\mu_j X_j x_i + \frac{1}{2\,!}\sum_1^2 {}_j \sum_1^2 {}_k \mu_j\mu_k X_j X_k x_i + \cdots \quad (i = 1, 2),$$

where

$$X_1 \equiv \frac{\partial}{\partial x_2}, \quad X_2 \equiv x_2\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}.$$

For every finite system of values of the μ's, the a's are finite. Consequently every transformation of the family

T_μ belongs to the family T_a, and is generated by an infinitesimal transformation of the group. Since $M_1(\mu)$, $M_2(\mu)$ in equations (14) are independent functions of the μ's, Lie assumes that every transformation of the family T_a belongs to the family T_μ. But although the functions a_1, a_2, defined by equations (14), are independent of the μ's since the Jacobian

$$\frac{\partial(a_1, a_2)}{\partial(\mu_1, \mu_2)} \equiv \frac{e^{\mu_2} - 1}{\mu_2 e^{\mu_2}}$$

is not identically zero, nevertheless for certain values of the a's the μ's are infinite. Thus, solving equations (14), we have

$$\mu_1 = \frac{a_1 a_2 e^{a_2}}{e^{a_2} - 1} \equiv N_1(a_1, a_2), \qquad \mu_2 = a_2 \equiv N_2(a_1, a_2).$$

For $a_1 \neq 0$ and a_2 an even multiple of $\pi\sqrt{-1}$, μ_1 becomes infinite, and this transformation of the family T_a is distinct from any transformation of the family T_a for which the μ's are finite. But infinite values of the μ's are excluded from consideration, for $\mu_1 = -(\lambda_1 + \lambda_2)t$, $\mu_2 = -\lambda_2 t$, and since t cannot be infinite if μ_1 or μ_2 is infinite, λ_1 or λ_2 must be infinite, and by supposition the λ's are arbitrary but definite constants. Consequently we cannot assume that every transformation of the family T_a belongs to the family T_μ.

This necessitates a restriction upon the criterion for the continuity of a group. For a system of values of the a's for which one or both of the functions $N_1(a)$, $N_2(a)$ are infinite there is no equivalent transformation of the family T_μ, and consequently such a transformation T_a cannot be generated by an infinitesimal transformation of the group. For example, the transformation considered above, for which $a_1 \neq 0$, $a_2 =$ an even multiple of $\pi\sqrt{-1}$, cannot be generated by an infinitesimal transformation of the group. Such a transformation is termed a *singular* transformation, and a group which contains a singular transformation is said to be discontinuous.

A group is said to be continuous if it contains no singular transformation. In other words, an r-parameter group is said to be continuous if the composition of two arbitrary transformations T_a and T_b of the group, generated by the infinitesimal transformations

$$a_1 X_1 + \cdots + a_r X_r, \quad b_1 X_1 + \cdots + b_r X_r$$

respectively, is equivalent to a transformation T_c of the group, generated by the infinitesimal transformation

$$c_1X_1 + \cdots + c_rX_r,$$

with finite parameters $c_1, \cdots, c_r$; that is to say, if a system of finite values of the c's can be found to satisfy the symbolic equation $T_b\, T_a = T_c$. On page 282 we saw that the composition of the two arbitrary transformations T_a and T_b of the family defined by equations (4) was equivalent to a transformation T_c of the family, with finite parameters c. But equations (4) were not in their canonical form, and therefore it did not necessarily follow that the transformation T_c could be generated by an infinitesimal transformation of the group, as shown above. Consequently, if the finite equations of a group are not in their canonical form, the condition that for every finite system of values of the a's and b's a finite system of the c's can be found to satisfy the symbolic equation $T_b\, T_a = T_c$ is a necessary but not a sufficient condition for the continuity of the group.

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SOME APPLICATIONS OF GREEN'S THEOREM IN ONE DIMENSION.

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GREEN'S theorem ordinarily has reference to Laplace's equation in either two or three dimensions. It has been generalized however in the case of two dimensions by replacing Laplace's equation by the general homogeneous linear differential equation of the second order. In the generalized form the theorem relates not only to the given differential equation, but also to its adjoint differential equation.* A further extension of the theorem is possible by considering a differential equation of the nth order in two or more independent variables, and its corresponding adjoint.†

* Cf. Encyklopädie, II, A. 7 c., p. 513.

† Cf. Darboux, Théorie des Surfaces, vol. 2, pp. 72, 74, for the case of two independent variables.