32. The paper of Professor Heffter, which will appear in the *Transactions*, extends to space of m dimensions considerations concerning curvilinear integrals previously developed by him in the *Göttinger Nachrichten* of February 8, 1902, for the case $m = 2$ of the plane. As the curve C of integration is taken the general rectifiable curve; in case the curve C lies within a closed continuous region of continuity of the integrand function, the usual definition of the integral receives a certain modification, in that the range of values of the limitand sum is enlarged; this modification is seen to simplify the development of the theory as regards the so-called fundamental theorem of the integral calculus, the theory of the curvilinear integral as function of the upper limit in case it is independent of the path C, and the theorem of Cauchy.

EDWARD KASNER,
Assistant Secretary.

COLUMBIA UNIVERSITY.

THE MEETING OF SECTION A OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, PITTSBURGH, PA., JUNE 28–JULY 3, 1902.

BOTH in attendance and in the number and character of the papers presented, the Pittsburgh meeting of Section A of the American Association for the Advancement of Science, was very successful. The Secretary has not preserved a memorandum of the members who were present, but he estimates that forty or more attended the different sessions. The Section met on Monday afternoon, June 30, the Vice-President, Professor G. W. Hough, of Northwestern University, presiding, to hear the address of the retiring Vice-President, Professor James McMahon, of Cornell University. In the absence of Professor McMahon, his address, the subject of which was "Some recent applications of function theory to physical problems," was read by Professor R. S. Woodward, of Columbia University. This address was published in *Science* for July 25.

Amongst the papers on the programme of the Section were three important reports: I, "A report on quaternions," by Professor Alexander Macfarlane, which will be published in the *Proceedings* of the Association. II, "A second report on recent progress in the theory of groups of finite order," by Professor G. A. Miller, Stanford University, which was presented in abstract by Dr. W. B. Fite, Cornell, in Professor Miller's absence. This report appears in the present number of the BULLETIN. III, "A report on the theory of collineations," by Professor H. B. Newson, the University of Kansas, which was read by title, in Professor Newson's absence. This report will be published in the *Proceedings* of the Association. Titles and abstracts of the other papers follow below.

Professor David P. Todd, Amherst College Observatory, presented two papers :

I. " On the adaptability of the glycerine clock to the diurnal motion of astronomical instruments, particularly those used in photographing solar eclipses."

The glycerine clock is an accurately constructed cylinder, about four inches in diameter, in which travels a piston, the rate of flow of the glycerine being controlled by a series of needle valves. By attaching a mirror or objective to a frame, equatorially mounted, and setting the clock under one of its arms at any convenient distance from the axis, the requisite rate for counteracting the diurnal motion of the sun can be given by means of the needle valves. This permits very heavy weights to be thrown on the piston and thereby the vibration of the instrument by wind can be precluded. When the run of the piston is finished, the glycerine is pumped out of the top of the cylinder and forced back into the bottom, and the run is commenced over again.

II. "On a convenient type of finder for very large equatorials."

In equatorials above twenty inches in aperture, the ordinary finder is necessarily mounted so far away from the axis of the great telescope that its use occasions much inconvenience, on account of the distance of its eye piece from that of the great tube. To obviate this difficulty Professor Todd proposes to construct the finder with a pair of reflectors, either planes or prisms, set at 45°, and to mount its tube in rings or bearings. By turning the tube in these, the finder's eye piece can be brought as near the eye piece of the great tube as is desired, or pushed away from it to admit attachment or adjustment of subsidiary apparatus.

" Series whose product is absolutely convergent," Professor Florian Cajori, Colorado College.

This paper is a continuation of the subject as developed by the author in his previous papers (*Transactions of the American Mathematical Society*, volume 2, pages 25–36 (1901); *Science*, volume 14, page 395 (1901), BULLETIN, volume 8, pages 231–236 (1902), and in the article of Alfred Pringsheim (*Transactions of the American Mathematical Society*, volume 2, pages 404–412 (1901). Some of the results previously obtained, relating to absolutely convergent products of two or more series, are generalized and the method of treatment is simplified. The construction of pairs of divergent series with real or complex terms is given, such that the product of the two series is not only absolutely convergent, but equal to any desired value, including zero.

Professor George Bruce Halsted, University of Texas, presented two papers:

I. "A new treatment of volume."

By a sect calculus, the product of two sects is fixed. The area of a triangle is defined as half the product of base and altitude, which is proved independently of the side chosen as base. Then the volume of a tetrahedron is defined as one third the product of base and altitude, which is proved independently of the side chosen as base. Then is proven the theorem: The volume of any tetrahedron is equal to the sum of the volumes of all the tetrahedra which arise from the first by making successively a set of transversal partitions. Then the theorem: However a tetrahedron is cut by a plane, this partition can be obtained in a set of transversal partitions using not more than two other planes. Then is proven as a theorem: If a tetrahedron T is in any way cut by any planes into a certain finite number of tetrahedra T_k, then is always the volume of the tetrahedron T equal to the sum of the volumes of all the tetrahedra T_k. Then the volume of any polyhedron is defined as the sum of the volumes of any set of tetrahedra into which it is cut, and this is proved independently of the set taken. A corollary is that if a polyhedron be cut into polyhedra, the sum of their volumes equals its volume. Then a prismatoid is defined, and by cutting it into tetrahedra its volume is proved to be

$$V = \frac{a}{4}(B + 3S) \text{ [see Halsted's Elements].}$$

All the forms of elementary geometry follow as special cases of this. For example, the volume of the cube with edge 1 is thus *proved* to be 1. Thus volume is completely treated without ratio, without continuity, without limits, without infinity or any infinite process.

II. "A new founding of spherics."

This founds two-dimensional spherics and deduces all its theorems by assuming three sets of postulates. It makes no use of euclidean space, no use of straight line, center, radius, no use of any theorem of solid geometry. Thus it is an establishing of two-dimensional Riemannian geometry. The second group of assumptions, the "betweenness" assumptions, are entirely new. There is no assumption that the shortest path between two points is on a straightest line or great circle. It is rigorously proved that two sides of a spherical triangle are together greater than the third. Thus is eliminated the objectionable assumption analogous to "the straight line is the shortest distance between two points."

"A new solar attachment," Professor Herbert A. Howe, Chamberlin Observatory, University Park, Colorado.

The purpose of this paper is to call the attention of teachers of astronomy to a new solar attachment, the "Shattuck solar," which is capable of illustrating the uses of a number of astronomical instruments. When it is attached to an ordinary engineer's transit, a teacher may give his students a good deal of practice with it in some of the fundamental processes of practical astronomy, without having recourse to expensive astronomical instruments. The instruments illustrated are the prism transit in the meridian, the zenith telescope, the prime vertical transit, the sextant, and the equatorial coudé.

"On the periodic solutions of the problem of three bodies," Professor E. O. Lovett, Princeton University.

Lagrange found five exact solutions of the problem of three bodies in each of which the bodies preserve an unvarying configuration which revolves with a uniform velocity. When the third body is of infinitesimal mass compared with the other two, it can describe small periodic orbits in the vicinity of the points where exact solutions exist. The latter points were called centers of libration by Gyldén, and Darwin calls the infinitely small body an oscillating satellite. Hill pointed out the fertility of the notion and made a splendid application of it in his lunar theory. Poincaré elaborated the mathematical theory in his celebrated researches, and we owe to Darwin an extended collection of examples of periodic orbits.

One of the most recent investigations of such orbits is a suggestive paper by Charlier, in No. 18 of the *Meddelanden från Lunds Astronomiska Observatorium.* In the *Monthly Notices of the Royal Astronomical Society* for November, 1901, Plummer has discussed some of Charlier's results in a more general manner.

It is the object of Professor Lovett's paper to determine the imaginary centers of libration and their corresponding orbits, and thus complete the analytical solution proposed by Charlier. The results cannot be expected to fit the sky, but they may be of some interest to mathematical astronomers. It appears that there are real periodic orbits corresponding to imaginary centers of libration.

"The rate of the Riefler sidereal clock, No. 56," Professor Charles S. Howe, Case School of Applied Sciences.

In order to obtain the best rate from a clock, it is necessary to have it under constant pressure and constant temperature. Constant temperature may be obtained in a clock room. Dr. Riefler, of Munich, is the only manufacturer in the world who makes a clock in an air-tight glass case. There are three of these clocks in the United States, one of which is at the Case Observatory. The mean daily rate of this clock for a trifle over three months was .116 of a second. The average daily variation from this mean daily rate was .015 of a second, and the maximum variation was .022 of a second. It is believed that these rates are the best ever published.

"A representation of the coördinates of the moon in power series which are proved to converge for a finite interval of time," Dr. F. R. Moulton, University of Chicago.

In the construction of a lunar theory three conditions are very desirable, if not absolutely essential, viz.: (*a*) that every step and the general process shall be proved to be logically valid, at least under certain conditions which are fulfilled by the moon; (*b*) that the labor of constructing the series shall not be so great as to be impracticable, and (*c*) that the arbitrary constants shall enter in such a way that they may be determined from observations with a high degree of precision.

The first condition has not been insisted upon heretofore. The nearest approach is in Delaunay's theory in which every one of the finite number of steps taken is separately valid, but in which the infinite sequence which is involved has not been shown to converge. Dr. Moulton imposes everywhere condition (*a*), and also (*b*) and (*c*) so far as possible.

By making use of Cauchy's existence theorem for the solutions of total differential equations and Mittag-Leffler's recent splendid researches on the representation of analytic function in extended areas by generalizations of power series, and by defining conveniently a certain anchor ring, Dr. Moulton arrives at a method of obtaining expressions for the coördinates of the moon which are valid for any length of time chosen in advance; and, moreover, it is possible to determine in advance a number of terms which will give an arbitrary degree of accuracy. These very general results fulfill condition (*a*), but unfortunately do not fulfill (*b*) and (*c*).

Another method is given depending upon the expansion of the expressions for the coördinates as power series in certain appropriately chosen parameters, the coefficients in the case of the linear variables being trigonometric functions of the time, and in the case of the angular variables, linear and trigonometric functions of the time. It is proved that these series converge for all values of the time not too remote from the initial epoch. When a sufficient number of terms have been taken so that the error committed shall be less than an arbitrary amount they may be rearranged as a Fourier series so that the expressions will be of the form given in the standard methods. This method fulfills all three of the conditions, but the second not so well as that first developed by Hill in his remarkable researches.

"The mass of the rings of Saturn," Professor Asaph Hall, South Norfolk, Conn.

The mass of these rings was first determined by Bessel in 1831 from the motion of the apsides of the orbit of Titan. This motion is about half a degree in a year. But the action of the figure of the planet, and the attractions of the other satellites were neglected; and, as Bessel pointed out, the resulting mass of the rings was too great. This mass is $1/118$, the mass of Saturn being taken as the unit.

In this paper an equation was formed containing two indeterminate quantities depending on the figure of the planet, the mass of the rings, and the masses of the three brighter satellites, Rhea, Dione and Tethys. The small resulting action of the other satellites was estimated. The coefficients of these six indeterminate quantities can be computed with sufficient accuracy. The uncertainty in finding the mass of the rings arises chiefly from the lack of good values of the masses of the satellites. These masses must be found from the mutual perturbations of the satellites. Substituting the values of the masses of the satellites determined by Professor H. Struve, the principal coefficient depending on the figure of the planet was assumed to be 0.0222. The mass of the rings is 1/7092. It is probable that Struve's masses of the satellites are too small, and the above mass of the rings would be too great.

Saturn will soon return to our northern skies, and it is hoped that further observations and their discussion will give good values of the constants of this interesting system.

Professor John A. Eiesland, Thiel College, presented two papers:

I. "On a class of real functions to which Taylor's theorem does not apply."

As an example of a function to which Taylor's expansion does not apply Cauchy gave $f(x) = e^{-1/x^2}$. Pringsheim has the following series

$$f(x) = \sum \frac{(-1)^\nu}{\nu\,!} \frac{1}{a^\nu x + 1}, \qquad a > 1 \text{ and real,}$$

which expanded gives rise to the series

$$\phi(x) = e^{-1} - e^{-a}x + e^{-a^2}x^2 - \cdots \pm e^{-a^n}x^m \mp \cdots$$

This series is convergent throughout the finite portion of the plane, while $f(x)$ in the neighborhood of $x = 0$ has an infinite number of poles $x = 0$. There is a class of functions of this kind which are distinguished by the additional property that they give rise to identically the same expansion. Such a function is

$$f(x) = \frac{\sum (-1)^\nu \left(\frac{\pi}{2}\right)^{2\nu+1}}{(2\nu+1)\,!} \cdot \frac{1}{1 + a^{2\nu+1}x},$$

which expanded becomes

$$\phi(x) = \sin\frac{\pi}{2} - \sin\frac{a\pi}{2}x + \sin\frac{a^2\pi}{2}x^2 - \cdots \qquad (a > 1).$$

If a is any even integer,

$$\phi_1(x) = 1,$$

if a is an odd integer of the form $4n + 1$,

$$\phi_2(x) = 1 - x + x^2 - x^3 + \cdots = \frac{1}{x+1},$$

and if a is an odd integer of the form $4n - 1$,

$$\phi_3(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Hence all functions $f(x)$ having *different* poles may give rise to the same expansion in Taylor's series. The reason for this singular phenomenon must be sought in the periodicity of the coefficients in the expanded series. Another class of functions all having the *same poles* but differing in the values assigned to the coefficients present the same feature. Functions of the type

$$f(x) = \sum \frac{c_\nu}{1 + a_\nu x} \qquad (\lim a_\nu = \infty)$$

may be expanded in a power series $A_0 + A_1x + A_2x^2 + \cdots$ in which

$$A_0 = \sum_0^\infty c_\nu, \quad A_1 = \sum_0^\infty c_\nu a_\nu, \cdots, A_n = \sum_0^\infty c_\nu a_\nu^n.$$

Now this system of equations may be solved for the unknown c_ν by a method employed by Poincaré;* the system of values obtained is not unique, in fact as Poincaré shows

$$c_0a^0, c_1a, c_2a^2, \cdots, c_\nu a^\nu, \cdots, -1$$

also form a system of solutions, and more generally

$$c_0a^{0P}, c_1a^P, c_2a^{2P}, \cdots, c_\nu a^{\nu P}, \cdots, (-1)^P$$

are solutions of the system. But to all the different functions $f(x)$ that may thus be constructed there corresponds only one

* *Bull. de la Soc. Math. de France*, vol. 8.

expansion $\phi(x)$ which of course does not represent any of the functions.

II. "On a class of transcendental functions with line singularities."

Borel has given the function

$$\phi(z) = \Sigma b_\nu x^{c\nu} \qquad \left(\frac{c_\nu - c_{\nu-1}}{c_\nu} > N\right)$$

as an example of a class of functions which with all their derivatives remain finite and continuous within the unit circle as well as on it. To this class of transcendentals the following type may be added:

$$F(z) = \prod \frac{z - (1 + a_\nu)e^{2\pi i\nu a}}{z - (1 + b_\nu)e^{2\pi i\nu a}},$$

where $\lim a_\nu = 0$, $\lim b_\nu = 0$, and $a =$ an incommensurable number. The circle of radius unity is an essential singular line for all such functions, and this with respect to zeros as well as poles which are situated outside the circle, this curve being the point limit of poles and zeros. A more general type is

$$\Phi(z) = \prod \frac{z - (1 + a_\nu)e^{\nu i a}}{z - (1 + b_\nu)e^{\nu i a}} e^{\psi_\nu(z)},$$

where

$$\psi_\nu(z) = \frac{(a_\nu - b_\nu)e^{\nu i a}}{z - (1 + b_\nu)e^{\nu i a}} + \frac{1}{2}\left[\frac{(a_\nu - b_\nu)e^{\nu i a}}{z - (1 + b_\nu)e^{\nu i a}}\right]^2 + \cdots + \frac{1}{\nu - 1}\left[\frac{(a_\nu - b_\nu)e^{\nu i a}}{z - (1 + b_\nu)e^{\nu i a}}\right]^{\nu-1}.$$

All these functions can be proved finite and continuous within as well as on the unit circle together with all their derivatives up to any order as high as we please. They admit of no analytical expansion beyond this circle; the zeros and poles approach the unit circle in a narrowing spiral, but none of these singularities are situated on the curve.

"On a general method of subdividing the surface of a sphere into congruent parts:" Mr. Harold C. Goddard, Amherst College.

The problem was incidental to the practical problem of constructing a steel sphere one hundred feet in diameter, in connection with a new method of mounting a telescope, as outlined in an article in the *American Journal of Science* for June, 1902, by Professor David P. Todd.

If a regular dodecaedron be inscribed in a sphere, planes determined by the center and each edge of the dodecaedron cut out on the sphere twelve equal regular spherical pentagons. If the vertices of each pentagon be connected with its center by arcs of great circles the surface of the sphere is divided into sixty congruent isosceles spherical triangles, whose angles are determined as 60°, 60° and 72°.

Professor J. Burkitt Webb, Stevens Institute of Technology, presented two papers:

I. "A possible new law in the theory of elasticity."

II. "Displacement polygons."

The first of these papers was presented by abstract, and the second by title. The law referred to in the first is: "If the forcible change of the distance between two points in an elastic system changes the distance of two other points by a certain amount, then the same force applied to alter the distance of the two other points will change the distance of the first two points by the same amount."

"On extracting roots of numbers by subtraction," Dr. Artemas Martin, Washington, D. C.

This paper is a development and extension of an article, published in the *Philosophical Magazine* for September, 1880, by the Rev. F. H. Hummell. Dr. Martin shows how to determine any root of any perfect power by subtracting from the given power a certain series of subtrahends. If the given number is not a perfect power that fact is revealed at a close of the operation. The last remainder is always equal to the root sought, if the given number is a perfect power, and the number of subtractions is less by unity than the root sought. The general formula for the nth subtrahend of the mth root is

$$S_n = (n+1)^m - (n^m + 1),$$

or

$$S_n = S_{n-1} + (n+1)^m + (n-1)^m - 2n^m.$$

The paper will be published in the *Mathematical Magazine*.

"On the positions of the northern circumpolar stars:" Professor Milton Updegraff, Washington, D. C.

The questions discussed by Professor Updegraff were:

(1) The importance of accurate knowledge of the positions of the fixed stars, and the greater attention, comparatively speaking, which has hitherto been paid to the places of the equatorial stars.

(2) The fundamental work on the circumpolar stars of the Greenwich and Pulkowa observatories. Differential work done by Groombridge, Carrington and Schwerd. Later work done at Greenwich and Pulkowa, and in America. Observations of circumpolar stars requested by Auwers.

(3) By observing the circumpolar stars at both upper and lower culminations on the same day, their places can be determined so that the results are practically fundamental. The fainter stars can be observed at both culminations on the same day, during the fall and winter months in middle latitudes, by observing a certain list of stars in the early morning and again early in the evening. Instrumental errors are largely eliminated, as also are errors of personal equation and of the refraction tables. The periodic variation of the position of the pole may be determined and also eliminated from the observed places. The formula for reducing the observations is the following well known modification of Bessel's formula for the reduction of observations in right ascension,

$$a = T + \Delta T + m + (n + c) \tan \delta + c (\sec \delta - \tan \delta).$$

(4) The desirability was pointed out of an extended series of fundamental observations of both equatorial and circumpolar stars for increasing, for example, our knowledge of the motions and distribution of the stars in space. Such a series of observations would be useful in constructing a general catalogue of fundamental stars, numbering several thousand, and selected with reference to their suitability to serve as standard points of reference.

(5) In conclusion certain advantages in the location of the Naval Observatory in Washington for fundamental work on the fixed stars were stated. Washington is twelve and one half degrees in latitude south of Greenwich, twenty degrees south of Pulkowa, and seven and one half degrees south of the new branch of the Pulkowa Observatory at Odessa. It is however far enough north so that the northern circumpolar stars can be advantageously and accurately observed. For these reasons it may be said that perhaps no better location, as

regards latitude, could be found for supplementing the fundamental work done at Greenwich, Pulkowa and the Cape of Good Hope. The climate at Washington is very good for work of this kind. The local conditions are also favorable and are likely to remain so.

"The definite determination of the causes of variation in level and azimuth of large meridian instruments," Professor G. W. Hough, Dearborn Observatory, Evanston, Ill.

Professor Hough went into an elaborate discussion of several series of observations upon the causes and effects of this variation with different styles of mounting. His conclusion was that stone piers give the best results. The paper gave rise to some spirited discussion.

"Some theorems on ordinary continued fractions," Professor Thomas E. McKinney, Marietta College.

Let D be any positive integer not a perfect square and let its square root be represented by an ordinary continued fraction. Professor McKinney determines the form of D so that the continued fraction representing its square root may have a period with one, two, three or four elements and applies the results to the determination of the number of reduced forms in the class to which the indefinite quadratic form $(1, 0, D)$ belongs.

"On the forms of sextic scrolls of genus one," Dr. Virgil Snyder, Cornell University.

Dr. Snyder applied the same methods to find the forms of sextic scrolls of genus 1 as he previously employed in the study of the unicursal surfaces. Thirty-three types are found, ten of which have a multiple conic. The same method is being applied to sextic scrolls of genus greater than 1. The complete paper will be published in the *American Journal of Mathematics*.

"Transformation of the hypergeometric series," Professor Edgar Frisby, U. S. Naval Observatory.

If in the differential equation of the second order connecting the elements of the hypergeometric series

$$P = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha\beta(\alpha+1)(\beta+1)}{1\cdot 2\cdot\gamma\cdot(\gamma+1)}x^2 + \text{etc.}$$

$x^\mu P'$ be substituted for P, new relations are obtained in which

P' takes the place of P, and the new elements are functions of the original elements. μ is determined from the condition that the new series must be of the same general form as the old. If, in addition, x be replaced by $1/x$ another series is obtained. From these two new series, by proper substitution of the new derived elements, are obtained, almost by inspection, the twenty-four different series ordinarily given in works on differential equations.

EDWIN S. CRAWLEY,
Secretary.

UNIVERSITY OF PENNSYLVANIA.

SECOND REPORT ON RECENT PROGRESS IN THE THEORY OF GROUPS OF FINITE ORDER.

BY PROFESSOR G. A. MILLER.

(Read before Section A of the American Association for the Advancement of Science, Pittsburg, July 2, 1902.)

THE main extensive treatments of this theory which have appeared during the four years since my first report was presented before this Section are: The articles in volume I of the Encyklopädie der mathematischen Wissenschaften on "Endliche discrete Gruppen," "Galois'sche Theorie mit Anwendung," and "Endliche Gruppen linearer Substitutionen," by Burkhardt, Hölder, and Wiman respectively; Weber, Lehrbuch der Algebra, second edition, volume 2, 1899; Bianchi, Lezioni sulla teoria dei gruppi di sostituzioni, 1899; * Echegaray, Lecciones sobre resolucion de ecuaciones y teoria de ecuaciones, 1899; † Netto, Vorlesungen über Algebra, volume 2, 1900; Pierpont, "Galois theory of algebraic equations," ‡ 1900; Dickson, Linear groups, 1901; Burnside and Panton, Theory of equations, volume 2, 1901.

In the present, as in my first report, it is my intention to avoid, as far as practicable, the consideration of those recent advances which have received considerable attention in these

* Printed edition of the work lithographed in 1897.

† Reviewed in *L'Enseignement Mathématique*, vol. 2 (1900), p. 227.

‡ *Annals of Math.*, 2d series, vols. 1 and 2.