THE JANUARY MEETING OF THE CHICAGO SECTION.

The twelfth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago, on Friday and Saturday, January 2–3, 1903, the first session opening at 10:30 A. M. About thirty persons were in attendance, including the following members of the Society :

Professor Oskar Bolza, Professor D. F. Campbell, Professor E. W. Davis, Professor J. F. Downey, Professor Thomas F. Holgate, Dr. H. G. Keppel, Professor Kurt Laves, Professor H. Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Professor H. B. Newson, Miss Ida M. Schottenfels, Professor J. B. Shaw, Dr. S. E. Slocum, Professor Henry S. White.

In the absence of the President or a Vice-President of the Society, Professor H. B. Newson was elected chairman of the Section. At the first session the Secretary reported that since the organization of the Section in 1897, one hundred and sixty-three papers had been read before it, by sixty-three different persons ; and of these, seventy-two papers had been published. Later, during the meeting, the retiring President of the Society, Professor Moore, favored the Section with the presentation of his presidential address which had been delivered at the annual meeting in New York on December 30.

The following officers of the Section were elected for the ensuing year :

Secretary, Professor Thomas F. Holgate.

Additional members of the programme committee, Professor E. B. Skinner, Dr. S. E. Slocum.

The report of the committee appointed at the last Christmas meeting to devise a scheme of equivalent requirements for candidates proceeding to their second academic degree, with mathematics as their major subject, was taken up and discussed, final action being postponed until the April meeting. The report deals with the undergraduate programme which should be accepted as a basis for graduate work, and outlines in general terms the character of the work which should be demanded during the first year of graduate study. Copies of the report are in the hands of the Secretary for distribution to members of the Society.

The following papers were read:

(1) Dr. SAUL EPSTEEN: "Determination of the group of rationality of a differential equation."

(2) Professor E. W. DAVIS: "A group in logic."

(3) Professor H. B. NEWSON: "On the generation of finite from infinitesimal transformations; a correction."

(4) Professor L. E. DICKSON: "The ternary orthogonal group in a general field."

(5) Professor L. E. DICKSON: "The group defined for a general field by the rotation groups."

(6) Professor A. S. HATHAWAY: "Vector analysis."

(7) Professor J. B. SHAW: "On nilpotent algebras."

(8) Professor D. F. CAMPBELL: "On homogeneous quadratic relations in the solution of a linear differential equation of the fourth order."

(9) Dr. S. E. SLOCUM: "Relation between real and complex groups with respect to their structure and continuity."

(10) Professor ARNOLD EMCH: "On the involution of stresses in a plane."

(11) Mr. R. E. WILSON: "Polar triangles of a conic and certain circumscribed quartic curves" (preliminary communication).

(12) Professor H. S. WHITE: "Orthogonal linear transformations and certain invariant systems of cones" (preliminary communication).

(13) Professor R. E. ALLARDICE: "On the envelope of the axes of similar conics through three fixed points."

Mr. Wilson was introduced by Professor White and Dr. Epsteen by Professor Maschke. In the absence of the authors, Professor Emch's paper and Professor Dickson's two papers were read by title; Professor Allardice's paper was read by Professor White, and Professor Hathaway's by Professor Shaw.

Professor Davis's paper appears in the present number of the BULLETIN. Professor Allardice's paper was published in the January number of the *Transactions*. Abstracts of the other papers are as follows:

1. Dr. Epsteen pointed out that when a special algebraic equation is given, its group can be theoretically determined 1° by constructing a $n!$ valued function and finding the Galois

resolvent, or 2° by applying the characteristic double property of the group, viz.: every rational function of the roots which remains unaltered by all the substitutions of G lies in R; and conversely. Both of these methods are generally impracticable, and in practice some device is resorted to in order to obtain the group. When a single rational function is known the following theorem has been found very useful: "If a rational function $\psi(x_1, \cdots, x_n)$ remains formally unaltered by the substitutions of a group G' and by no other substitutions, and if ψ equals a quantity lying in R and if the conjugates of ψ under $G_{n!}$ are all distinct, then the group of the given equation for the domain R is a subgroup of G' (cf. Dickson, Theory of algebraic equations, which is soon to appear, page 56).

The situation is exactly the same when we deal with linear differential equations. When a special linear differential equation is given, its group may be found: 1° by constructing the Picard resolvent, or 2° by applying the Picard-Vessiot characteristic double property of the group, viz.: every rational differential function (of a fundamental system) of the integrals which remains unaltered as a function of x, by the most general transformation of G lies in R and conversely. In this case these methods are even more impracticable than for the algebraic equations, and in order to determine the group a device should be resorted to. With this end in view the author has proved the following theorem: "If a rational differential function (of a fundamental system) of the integrals

$$\psi(y_1, \cdots, y_n; y_1', \cdots, y_n'; \cdots; y_1^{(n-1)}, \cdots, y_n^{(n-1)})$$

remains formally unaltered by the transformations of a complex r-parameter linear homogeneous group G_r and if, when ψ is transformed by the most general linear homogeneous transformation, the resulting function contains $n^2 - r$ essential parameters, then the group of the given equation G is a subgroup of G_r."

4. The first paper by Professor Dickson is a study of the structure of the ternary orthogonal group in an arbitrary realm of rationality. The problem had been solved in two cases: for continuous fields by Weber (Algebra, II, second edition, pages 244–54); for finite fields by the writer (Linear Groups, page 164). The group of all ternary orthogonal transformations of

determinant unity in a field F is designated $O(3, F)$. If $i = \sqrt{-1}$ belongs to F, $O(3, F)$ is simply isomorphic with the group of all linear fractional transformations in F. The latter has as an invariant subgroup the simple group of all linear fractional transformations in F of determinant unity. Simple generators of $O(3, F)$ are given. If i extends F to a larger field $F(i)$, then $O(3, F)$ is simply isomorphic with the group of all transformations

$$z' = \frac{\alpha z + \beta}{-\sigma\bar{\beta}z + \sigma\bar{\alpha}} \qquad \left(\begin{array}{l}\alpha, \beta \text{ in } F(i), \\ \sigma\bar{\sigma} = 1.\end{array}\right).$$

The transformations with $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$ form a subgroup, the fractional binary hyperorthogonal group (compare Linear Groups, page 132). Certain theorems on this group are established.

5. The second paper by Professor Dickson is a contribution to the theory of group matrices and group determinants, investigated by Frobenius and further developed by Burnside for continuous fields and by the writer for an arbitrary field. The paper employs only very elementary methods and is quite independent of the papers cited. For the cyclic group g_k, the dihedron group g_{2k}, the alternating and symmetric groups on four letters, and the alternating group on five letters, the group matrix is reduced to a simple canonical form of the type shown to exist by Frobenius, the reducing transformation being exhibited explicitly. In particular, the irreducible factors of the group determinant are given. Both papers (printed October 1) are to appear in the *University of Chicago Decennial Publications*, Volume 9.

6. Professor Hathaway contends that if a vector analysis short of quaternions is desired, then all attempts at bringing in conceptions of products of vectors should be avoided as necessarily incomplete and uselessly metaphysical. Any quantity whose value depends upon and is determined by the values of given arguments **xyz** is simply a function of those arguments, and the ordinary form of notation f**xyz** should suffice.

The requisite analysis is afforded by the linear functions, defined with respect to any argument **x**, by

$$f(\mathbf{x} + \mathbf{x}')\mathbf{yz} = f\mathbf{xyz} + f\mathbf{x}'\mathbf{yz}.$$

In consequence, a scalar factor of an argument of a linear function is simply a factor of the function; and we have general expansions of the type

$$f(\mathbf{x} + \mathbf{x}')(\mathbf{y} + \mathbf{y}')\mathbf{z} = f\mathbf{xyz} + f\mathbf{xy}'\mathbf{z} + f\mathbf{x}'\mathbf{yz} + f\mathbf{x}'\mathbf{y}'\mathbf{z}.$$

The similarity of this to distributive multiplication is evident; and any symmetric or alternate properties of the function concerned impose corresponding commutative laws. Only three linear functions (a *scalar* and *vector* alternate, and a *scalar* symmetric) need be permanently defined:

$S\mathbf{abc}$ = volume of parallelepiped whose edges are **abc** in order.
$V\mathbf{ab}$ = vector of the area of the parallelogram whose edges are **ab** in order.
$S\mathbf{ab}$ = volume of parallelepiped of edge **a** and vector base area **b**.

7. In Professor Shaw's paper it was shown that the general form of any number of a nilpotent algebra is

$$\phi = \sum a_{ijk}\lambda_{ijk},$$

where $\lambda_{ijk}\lambda_{jlk'} = \lambda_{ilk+k'}$, $\lambda_{ijk}\lambda_{j'lk'} = 0 (j \neq j')$, and in any case λ_{rst} does not exist unless t is positive or zero, and $\mu_r - \mu_s - 1 < t < \mu_r$. The numbers μ_r, μ_s, etc., are certain "multiplicities" in the expressions. The units of the algebra may be expressed in the form

$$\begin{array}{l} \phi_1, \phi_2, \cdots, \phi_m, \phi_{m+1}, \phi_1\phi_{m+1}, \phi_1\phi_{m+1}^2, \cdots \\ \qquad\qquad\qquad\qquad \phi_2\phi_{m+1}, \phi_2\phi_{m+1}^2, \cdots \\ \qquad\qquad\qquad\qquad \cdots\cdots \end{array}$$

where

$$\phi_r\phi_s = a\phi_{s+1} + b\phi_{s+2} + \cdots + f\phi_{m+1} + g\phi_1\phi_{m+1} + \cdots$$

Certain typical cases are given as examples.

8. Professor Campbell's paper is in abstract as follows: Let

$$\sum_{i=1}^{i=4}\sum_{j=1}^{j=4} a_{ij}y_iy_j \qquad (a_{ij} = a_{ji})$$

be a homogeneous quadratic form in the solutions y_1, y_2, y_3, y_4

of the linear differential equation $y^{\text{iv}} + 6p_2y'' + 4p_3y' + p_4y = 0$. It can be written symbolically as

$$\left(\sum_{i=1}^{i=4} \alpha_i y_i\right)^2.$$

If successive derivatives be taken of this form and use be made of the fact that $y_i^{\text{iv}} = -6p_2y_i'' - 4p_3y_i' - p_4y_i$ $(i = 1, 2, 3, 4)$, any derivative will be composed of terms, each of which consists of a constant or a rational integral function of the coefficients of the given differential equation, and their derivatives, multiplied by the symbolic factors

$$\left(\sum_{i=1}^{i=4} \alpha_i y_i\right)^{r_0} \left(\sum_{i=1}^{i=4} \alpha_i y_i'\right)^{r_1} \left(\sum_{i=1}^{i=4} \alpha_i y_i''\right)^{r_2} \left(\sum_{i=1}^{i=4} \alpha_i y_i'''\right)^{r_3}$$

where $r_0 + r_1 + r_2 + r_3 = 2$. These factors may be kept in the order

$$\left(\sum_{i=1}^{i=4} \alpha_i y_i^{(j)}\right) \qquad (j = 1, 2, 3),$$

and indicated by the exponents $[r_1, r_2, r_3]$. Then the derivative of the symbolic part of each term of any derivative will be given by the formula

$$\frac{d\,[r_1r_2r_3]}{dx} = r_1[r_1 - 1, r_2 + 1, r_3] + r_2[r_1, r_2 - 1, r_3 + 1]$$
$$+ (2 - r_1 - r_2 - r_3)\,[r_1 + 1, r_2, r_3] - 6p_2r_3[r_1, r_2 + 1, r_3 - 1]$$
$$- 4p_3r_3[r_1 + 1, r_2, r_3 - 1] - p_4r_3[r_1, r_2, r_3 - 1].$$

The following theorems are used in the investigations:

First. If there are g functions U_i, $i = 1, 2, \cdots g$ of an independent variable among which ν and only ν linearly independent relations exist, then

$$u = \sum_{i=1}^{i=g} a_i U_i,$$

a_i arbitrary constants, is the general solution of a linear differential equation of order $g - \nu$.

Second. The necessary and sufficient condition that the n functions of x, U_i, $(i = 1, 2, \cdots, n)$ are linearly independent is that the determinant

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}$$

vanishes.

Third. In any homogeneous quadratic relation $u = [000]$ that may exist in the solutions of the given differential equation, the expression $[200]$ cannot vanish.

Fourth. The relations $[000] = 0$, $[\overline{000}] = 0$, $\cdots$ in the solutions of the given differential equation are linearly independent if $[200]$, $[\overline{200}]$, $\cdots$ are linearly independent.

By successive differentiation of $u = [000]$ there result ten equations in the expressions $[r_1 r_2 r_3]$ whose coefficients are constant or rational integral functions of the coefficients of the differential equation and their derivatives, and whose right hand members are $u, u', u'', \cdots, u^{\text{ix}}$. Arrange the terms so that the expressions appear in the order $[000]$, $[100]$, $[010]$, $[200]$, $[110]$, $[101]$, $[020]$, $[011]$, $[001]$, $[002]$. Form the determinant Δ of the coefficients. Denote any determinant formed from the matrix derived from Δ by omitting the last n rows, $n = 1, 2, 3$, by the numbers indicating the columns chosen for the determinant. It is shown that:

The necessary and sufficient conditions that one and only one homogeneous quadratic relation exists in the solutions of the given differential equation are that Δ vanishes and (1235678910) does not vanish. The necessary and sufficient conditions that two and only two linearly independent homogeneous quadratic relations exist in the solutions of the given differential equation are that (1235678910) vanishes and not all (123478910), (123578910) and (123678910) vanish. The necessary and sufficient conditions that three linearly independent homogeneous quadratic relations exist in the solutions of the given differential equation are that (123478910), (123578910) and (123678910) vanish. There cannot be more than three linearly independent homogeneous quadratic relations in the solutions of the given differential equation.

9. The first part of Dr. Slocum's paper finds the form of the infinitesimal transformation U_a by which any given finite transformation T_a of a group with continuous parameters is generated when the equations defining this finite transformation are in their *non-canonical* form. For certain values of the parameters the transformation U_a may not be infinitesimal, in which case the corresponding finite transformation T_a is not generated by an infinitesimal transformation of the group.

Associated with each r-parameter structure is a certain determinant the constituents of which are functions of the r-parameters. If any system of values of the parameters can be found for which this determinant vanishes, the parameter group belonging to that type of structure is discontinuous.

A group may be continuous as a real group but discontinuous as a complex group. Consequently the idea, as developed by Lie, of the relation between the transitivity of the real and complex groups which have the same symbols of infinitesimal transformation, requires modification.

There being more types of structure possible for real groups than for complex groups, if we have given two structures which are of the same type, A, for complex groups, but which constitute distinct types, B and C, for real groups, it is shown by means of examples that one of the following cases may occur with respect to the continuity of the real groups of types B and C: 1° All real groups of both types B and C are continuous. 2° All real groups of both types B and C are discontinuous. 3° All real groups of one type, B, are continuous, and one or more, but not all, real groups of the other type, C, are discontinuous. 4° All real groups of one type, B, are continuous, and all real groups of the other type, C, are discontinuous.

10. Through Culmann's investigations in graphical statics it is known that the directions corresponding to the sections and the resultant stresses upon these sections at every point of a thin strained plate, form an involutoric pencil. This involution is hyperbolic or elliptic according as the plate is either subject to tensions and compressions, or to tensions or compressions only. Professor Emch briefly discusses these cases, establishes the equation of the stress ellipse and the differential equation of the tension and compression curves, and finally applies the results to the linear infinitesimal deformation of the plane.

11. Following a plan outlined by Professor White, in a paper read before the Society in September, 1902, an attempt is made by Mr. Wilson to investigate the number of independent relations that must exist among the coefficients in order that a quartic curve shall contain an infinity of triangles self-conjugate with respect to a given conic.

The binary quartic and conic were taken in the form

$$f_x^4 = A_x^2 a_x^2 \equiv 0, \quad \phi_x^2 = \alpha_x^2 \equiv 0.$$

The condition that two roots of the quartic shall separate two roots of the conic harmonically is

$$\{(a\alpha)^2(A\alpha)^2\}\{(AB)^2(ab)^2(\alpha\beta)^4 - (\alpha\beta)^2(A\alpha)^4 + 4(\alpha\beta)^2(a\alpha)^2(A\alpha)^2(Aa)^2 - 4(a\alpha)^4(A\alpha)^4\} = 0$$

(see Clebsch, Binäre Formen.) Reducing and expressing in terms of invariants of $f_x^4 = 0$ and $\phi_x^2 = 0$, this becomes

$$\bar{A}^3 - \tfrac{1}{3}j\Delta^3 - \tfrac{1}{2}iA\Delta^2 \equiv 0,$$

where i and j are invariants of $f_x^4 = 0$ alone, Δ of $\phi_x^2 = 0$, and $\bar{A}$ a simultaneous invariant of f and ϕ.

By bordering, the condition is obtained that a line $\mu_x = 0$ (ternary) shall satisfy the harmonic division. By considering the locus of the pole of $\mu_x = 0$, a 12th-ic is obtained which breaks down to a sextic $C_x \equiv 0$. The general sextic contains 28 coefficients, and hence the maximum number of conditions is 28.

The curves

$$f_x^4 = 4ax_2^3x_3 + 4bx_3^3x_1 + 4cx_1^3x_2 = 0, \quad \phi_x^2 = x_1^2 + x_2^2 + x_3^2 = 0$$

have been examined and show that the quartic above cannot contain infinitely many inscribed triangles self-conjugate with respect to a conic which has the triangle $x_1x_2x_3 = 0$ as a self-conjugate triangle.

The forms

$$f_x^4 = 6\sum a_1x_1^2x_3^2 + 12\sum b_1x_1^2x_2x_3 = 0, \quad \phi_x^2 = x_1^2 + x_2^2 + x_3^2 = 0$$

have also been examined. Although the result is not in complete form, it has disclosed properties that promise to lend interest to investigations other than the one in hand.

12. The formulas of Euler (and Rodrigues), which give rationally in terms of three parameters the nine coefficients of an orthogonal transformation in space, are available for expressing the vertices of any polar triangle of the conic

$$x_1^2 + x_2^2 + x_3^2 = 0.$$

This gives a means of discussing curves of the second and third orders, at least, which contain infinitely many inscribed polar triangles of a conic. Professor White's preliminary communication exhibited the method as applied to conics. The further question was raised, what sort of curve is the locus of points whose coördinates are the eulerian parameters of polar triangles inscribed in a single conic, or of orthogonal transformations which rotate the axes through the surface of an orthogonal quadric cone.

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SOME GROUPS IN LOGIC.

BY PROFESSOR E. W. DAVIS.

(Read before the Chicago Section of the American Mathematical Society, January 2, 1903.)

DE MORGAN has pointed out * that his eight forms of proposition identical with the A, E, I, O, and their contranominals of the older logic, can be derived from any one by the three operations of reversing the subject, reversing the predicate, denying the copula. If, in fact, we denote the operations in question by s, p, and f respectively, we have

$$Ap = E, \quad Af = O, \quad Afp = I;$$

while sp changes any proposition to its contranominal $X \leqslant Y$ to $Y \leqslant X$; or, what is the same thing to $\bar{X} \leqslant \bar{Y}$. Here $\leqslant$ is the sign of implication and the bar written over a letter or sym-

* Formal Logic, p. 63 et seq.