A FUNDAMENTAL THEOREM WITH RESPECT TO TRANSITIVE SUBSTITUTION GROUPS.

BY PROFESSOR G. A. MILLER.

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LET G represent any transitive group of degree n and of order $g = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$; $p_1, p_2, \cdots, p_m$ being distinct primes. According to Sylow's theorem G contains at least one subgroup of each of the orders $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_m^{\alpha_m}$. It will be convenient to speak of these subgroups as the *Sylow subgroups of* G. Since G is transitive, all the prime factors of n are included among the factors of g. The theorem in question may now be stated as follows:

THEOREM. *If p_1^β is the highest power of p_1 which divides n, each Sylow subgroup of order $p_1^{\alpha_1}$ in G has a transitive constituent of degree p_1^β and all its other transitive constituents are of degree $p_1^{\beta+\gamma}$, $\gamma \geqq 0$.*

COROLLARY I. *If $n = 2p^\alpha$, p being any odd prime, each of the Sylow subgroups whose order is a power of p has just two transitive constituents, each being of degree p^α.*

COROLLARY II. *If n is a power of a prime, a Sylow subgroup of G whose order is a power of the same prime must be transitive.*

The proof of the theorem results from the following elementary considerations: Let G_1 represent the subgroup of G which is composed of all its substitutions which omit a given letter. Every Sylow subgroup of G_1 is found in some Sylow subgroup of G. Whenever the orders of these subgroups are different the former is composed of all the substitutions of the latter which omit one letter.*

Since p_1 is any prime divisor of g we may confine our attention to the Sylow subgroups of G_1 and G whose orders are $p_1^{\alpha_1-\beta}$ and $p_1^{\alpha_1}$ respectively. At least one transitive constituent of the latter must be such that the order of its largest subgroups of lower degree than its own may be obtained by dividing its own order by p_1^β; *i. e.*, one of its transitive consti-

* Cf. Burnside, Theory of groups of finite order, 1897, p. 94.

tuents must be of degree p_1^β. If another transitive constituent were of degree p_1^δ, $\delta < \beta$, G_1 would contain a subgroup of order $p_1^{a_1-\delta} > p_1^{a_1-\beta}$. As this is impossible, a Sylow subgroup of order $p_1^{a_1}$ contains at least one constituent of degree p_1^β but it contains no constituent of lower degree.

This proof remains true even if $\beta = 0$, for in this case a Sylow subgroup of order $p_1^{a_1}$ is of degree less than n and hence may be supposed to have a constituent of degree 1. In all other cases the degree of this Sylow group is evidently equal to n. That the Sylow subgroup of order $p_1^{a_1}$ may have constituents whose degrees exceed p_1^β follows directly from the three transitive groups of degree 6 and order 24. In two of these the Sylow subgroups of order 8 involve just two transitive constituents, of degrees 4, 2 respectively, while our theorem merely proves that there is a constituent of degree 2 and that there cannot be one of a lower degree.

As an application of this theorem we may prove the following: *If the degree of a simply transitive primitive group is* np^a, p *being any prime which does not divide* $n > 1$, *the maximal subgroup of degree* $np^a - 1$ *cannot involve a transitive constituent of degree* $(n-1)p^a$. If $a = 0$ this theorem requires no proof. If $a > 0$, the Sylow subgroup of order p^λ in G would contain two transitive constituents, of degrees $(n-1)p^a$, p^a, respectively. Hence G would involve a substitution not found in G_1 which would not connect all the transitive constituents of G_1. This is impossible since G_1 is maximal in a primitive group. For instance, in a simply transitive primitive group of degree 18, the maximal subgroup of degree 17 cannot involve a transitive constituent of degree 9.

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