The necessary and sufficient condition that the equation of the asymptotic lines be factorable is that $d\mu/d\lambda$ be a square, or that three unequal values of μ, as μ_1, μ_2, μ_3, exist for each of which (5) will have a cubic factor. The values μ are the roots of the equation

$$g_2 = 0.$$

Since $\mu_1 \neq \mu_2 \neq \mu_3$ and $g_2 = 0$ is a quadratic in μ, hence $g_2 \equiv 0$ and $\Delta = g_2^3 - 27g_3^2 = -27g_3^2$ is negative for all real values of μ except μ_1, μ_2, μ_3; hence two of the roots λ in (5) are real and two are imaginary except at the pinch points. But the generators through the point (0, 0, 1, w) lie in the planes $x = \lambda y$, $w = \mu z$, hence two of the generators are real and two imaginary.

Conversely, it can be shown that if $g_2 \equiv 0$, the equation of the asymptotic lines is factorable, hence $g_2 \equiv 0$ is the necessary and sufficient condition that the asymptotic lines are reducible.

CORNELL UNIVERSITY,
March 27, 1903.

JOSIAH WILLARD GIBBS, PH.D., LL.D. A SHORT SKETCH AND APPRECIATION OF HIS WORK IN PURE MATHEMATICS.

JOSIAH WILLARD GIBBS, Professor of Mathematical Physics in Yale University, died after a short illness, April 28, 1903. The loss to pure science in America occasioned by his death is difficult to overestimate, and it seems peculiarly fitting that some record should be made in this BULLETIN of the work in pure mathematics of so illustrious a member of the AMERICAN MATHEMATICAL SOCIETY. Moreover, his publications have appeared in a form lacking somewhat in perspicuity and detail, and are rather inaccessible.

No reference will be made in the following to the achievement of Professor Gibbs in his chosen field of Thermodynamics. A complete account of his work in Physics will be found in the *American Journal of Science*, September, 1903.* It is proposed here to give a brief appreciation of the ideas underlying the following papers, which constitute his published work in pure mathematics:

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*Some paragraphs of the present sketch have already appeared in this article.

"Elements of Vector Analysis," arranged for the use of Students in Physics. New Haven. 8°. Pp. 1–36 in 1881, and pp. 37–83 in 1884.

"On Multiple Algebra." Vice-Presidential Address before Section A of the American Association for the Advancement of Science, published in the *Proceedings* of the Association, vol. 33, 1886, pp. 37–66.

"On the Rôle of Quaternions in the Algebra of Vectors." *Nature*, vol. 43, 1891, pp. 511–514.

"Quaternions and the Ausdehnungslehre." *Nature*, vol. 44, 1891, pp. 79–82.

"Quaternions and Vector Analysis." *Nature*, vol. 48, 1893, pp. 364–367.

It is quite apparent from these titles that the interest of Professor Gibbs was most keen in the subject of Multiple Algebra. In fact, for several years he delivered at Yale University a course of lectures on this topic, to which reference will be made below.

The theory of dyadics as developed in the Vector Analysis of 1884 must be regarded as the most important of these contributions. For the Vector Analysis as an *algebra* does not fulfill the definition of the linear associative algebras of Benjamin Peirce, since the scalar product of vectors lies outside the vector domain, nor is it a geometric analysis in the sense of Grassmann, the vector product satisfying the combinatorial law but yielding a vector instead of a magnitude of the second order. While these departures from the systems mentioned testify to the great ingenuity and originality of the author and do not impair the utility of the system as a tool for the use of students in physics, they nevertheless expose the discipline to the criticism of the pure algebraist. Such objection falls to the ground however in the case of the theory of dyadics, for this yields for $n = 3$ a linear associative algebra of nine units, namely nonions, the general nonion satisfying an identical equation of the third degree, the Hamilton-Cayley equation (Cf. Vector Analysis, Wilson, Chapter V).

It is easy to make clear the precise point of view adopted by Professor Gibbs in this matter. This is well expounded in the Vice-Presidential Address, and also in his warm defence of Grassmann's priority rights as against Hamilton, in the first article in *Nature*. He points out that the key to all matricular algebras is to be found in the open (or indeterminate) product, (*i. e.*, products in which no equations subsist between the factors), and, after calling attention to the brief development of this product in Grassmann's work of 1844, affirms that Sylvester's assignment of the date 1858 to the "second birth of algebra" (this being the year of Cayley's Memoir on Matrices), must be changed to 1844. Grassmann, however, ascribes very little importance to the open product, regarding it as affording no important applications. On the contrary, Professor Gibbs assigns to it the very first place in the three kinds of multiplication considered in the Ausdehnungslehre, since from it may be derived the algebraic and the combinatorial product, and shows in fact that both of these may be expressed in terms of indeterminate products. Thus the multiplication rejected by Grassmann becomes, from the standpoint of Professor Gibbs, the key to all others. The originality of the latter's treatment of the algebra of dyadics, as contrasted with the methods of other authors in the allied theory of matrices, consists exactly in this, that Professor Gibbs regards a matrix of order n as a multiple quantity in n^2 units, each of which is an indeterminate product of two factors. On the other hand, C. S. Peirce, who was the first to recognize (1870) the quadrate linear associative algebras identical with matrices, uses for the units a *letter pair*, but does not regard this combination as a product. In addition, Professor Gibbs, following the spirit of Grassmann's system, does not confine himself to *one* kind of multiplication of dyadics, as do Hamilton and Peirce, but considers two sorts, both originating with Grassmann. Thus it may be said that quadrate or matricular algebras are brought entirely within the wonderful system expounded by Grassmann in 1844.

As already remarked, the exposition of the theory of dyadics given in the Vector Analysis is not in accord with Grassmann's system. In a footnote of his Vice-Presidential Address Professor Gibbs shows the slight modification necessary for this purpose, while the subject has been treated in detail and in all generality in his Lectures on Multiple Algebra.

In order to bring the algebra of dyadics within the system invented by Grassmann it is only necessary to replace the dyadic of Vector Analysis, that is, the indeterminate product of two vectors, by the indeterminate product of two magnitudes of Grassmann's system the sum of whose orders (in general different) is always equal to n, the order of the system itself.

The details are briefly as follows: Suppose that

$$e_1, e_2, \cdots, e_n$$

are the fundamental units of the system; then *derived* units are formed from these by taking all possible products according to the combinatorial law

$$e_i e_k = - e_k e_i.$$

Thus we obtain

$n(n-1)/1 \cdot 2$ units of the second order, of the type $e_i e_k$;
$n(n-1)(n-2)/1 \cdot 2 \cdot 3$ of the third order, of the type $e_i e_k e_l$;
etc.,
n of the $(n-1)$th order, of the type $e_1 e_2 \cdots e_{n-1}$;

and one of the nth order, $e_1 e_2 e_3 \cdots e_n$, which for brevity we set equal to unity. The n units of order $n-1$ can be chosen and designated

$$\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n$$

such that $\bar{e}_j e_j = 1$, but $\bar{e}_i e_k = 0$, $i \neq k$.

From these units we form the fundamental dyads, by multiplying any unit of order r into any other of order $n-r$, using the sign / to indicate the indeterminate product. This gives for example,

n^2 fundamental dyads $e_i / \bar{e}_k$,
$[n(n-1)/1 \cdot 2]^2$ " " $e_i e_j / \bar{e}_k \bar{e}_l$,

since, by Grassmann's regressive law of multiplication any unit of order $n-2$ may be expressed in the form $\bar{e}_k \bar{e}_l$.

From the fundamental dyads are formed as in the Vector Analysis the *dyadics*

$$\Phi_1 = \sum_{i,k} a_{i,k} e_i / \bar{e}_k, \qquad \Phi_2 = \sum_{i,j,k,l} a_{ij,kl} e_i e_j / \bar{e}_k \bar{e}_l, \qquad \text{etc.,}$$

the a's being ordinary scalars.

Professor Gibbs next asks the question: What distributive products of dyads or dyadics are simple and useful? Of these, he finds two, corresponding precisely to the *double dot* and *double cross products* of Dr. Wilson's Vector Analysis, page 306. For example, the first sort gives

$$e_i/\bar{e}_k : e_j/\bar{e}_l = 0 \quad \text{or} \quad e_i/\bar{e}_l$$

according as $k \neq j$ or $k = j$.

This law of multiplication is exactly that of C. S. Peirce's vids, and accordingly the algebra of dyadics based upon the double-dot law of multiplication is precisely the matricular algebra of the author mentioned.

The *double-cross product* is defined by the law

$$e_i/\bar{e}_k \overset{*}{\underset{*}{}} e_j/\bar{e}_l \equiv e_i e_j/\bar{e}_k \bar{e}_l,$$

i. e., gives a dyad of different type. In the Vector Analysis, however, no new type of dyad results, and we have, in contradiction to Grassmann's system for $n = 3$, one type of dyad only.

This second type of product is a distinct contribution to the algebra of dyadics, as the development on pages 306–331 of the Vector Analysis of Dr. Wilson abundantly proves.

For geometric application this comes out in the following very interesting way. Suppose $n = 4$; then quantities of the first order may be represented as points in ordinary space, and the general dyadic Φ_1 gives a general collineation T. Now the dyadic

$$\Phi_2 = \frac{\Phi_1 \overset{*}{\underset{*}{}} \Phi_1}{1 \cdot 2}$$

is T expressed in line coördinates, *i. e.*, is the corresponding collineation of Plücker's line geometry; moreover

$$\Phi_3 = \frac{\Phi_1 \overset{*}{\underset{*}{}} \Phi_1 \overset{*}{\underset{*}{}} \Phi_1}{1 \cdot 2 \cdot 3}$$

is T expressed in tangential or plane coördinates, while Φ_4 is simply the determinant of T.

This brief sketch will show, therefore, that Professor Gibbs's exposition of the algebra of dyadics treats of these multiple quantities not only as linear operators on quantities of the first order, to which the investigations of all authors on the related topic of matricular algebras are limited, but also *as operators within the entire derived domain of Grassmann.* The lectures contain a very simple and elegant treatment of various questions in the theory of matrices, including Sylvester's laws of nullity and latency, the proof of the identical equation and the reduction of a dyadic to a canonical form both in the general and special cases.

It is interesting and gratifying to note that the interest aroused in the general subject of Multiple Algebra among his pupils, stimulated and directed by Professor Pierpont through his lectures on Hypercomplex Numbers, has resulted in several noteworthy publications by the late Professor Starkweather and by Dr. Hawkes, which exhibit the method and spirit of Benjamin Peirce's memoir on Linear Associative Algebras in a new and remarkably fruitful light.

PERCEY F. SMITH.

SHEFFIELD SCIENTIFIC SCHOOL,
July, 1903.

NOTES.

AT the forty-second annual convention of the National educational association, held in Boston, July 6–10, 1903, a conference on mathematics was conducted by Professor D. E. SMITH. The topic considered was the organization and the work of associations of teachers of mathematics. Over two hundred teachers of mathematics from various parts of the country were present at the conference, and the following papers were read: "On the work of the Central association of science and mathematics teachers," by Mr. C. E. COMSTOCK; "On the New England Association," by Mr. E. H. NICHOLS; "On the proposed association for the middle states and Maryland," by Dr. J. S. FRENCH; "On the investigation being made in New England as to geometry in the grammar school," by Mr. W. T. CAMPBELL; "On the report of the committee appointed by the AMERICAN MATHEMATICAL SOCIETY, on college entrance requirements," by Professor H. W. TYLER; "The relation of these associations to the AMERICAN MATHEMATICAL SOCIETY," by Professor W. F. OSGOOD. From the inquiries made at the conference it is probable that several associations will be formed during the coming year.