THE TENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

ON its reorganization as a national body, in 1894, the AMERICAN MATHEMATICAL SOCIETY immediately adopted the plan of holding an annual summer meeting in addition to the regular meetings which at that time were held monthly from October to May in New York. Coming in the vacation season and thus affording exceptional opportunity for the reunion of the widely scattered members of the Society, these summer meetings have always held the first place in representative character, in attendance, and in scientific interest as well. Since 1897, colloquia, or courses of lectures on special subjects, have contributed additional attraction to several of the meetings. The summer meetings have been held in the following sequence of place: Brooklyn (1894), Springfield, Buffalo, Toronto, Boston, Columbus, New York, Ithaca, Evanston, Boston (1903). Several of the meetings have been held in affiliation with the American association for the advancement of science; in other cases an independent meeting has proved more convenient. The first colloquium was held at Buffalo, in 1896; the second at Cambridge, in connection with the Boston meeting, in 1898; the third at Ithaca, in 1901.

The tenth summer meeting and fourth colloquium of the Society were held at the Massachusetts Institute of Technology during the week August 31–September 5, 1903. At the three sessions of the summer meeting proper, which occupied the first two days of the week, the following forty-seven members were present:

Professor F. H. Bailey, Dr. G. A. Bliss, Professor Maxime Bôcher, Dr. C. L. Bouton, Professor Ellen L. Burrell, Professor Florian Cajori, Mr. Paul Capron, Dr. J. E. Clarke, Professor F. N. Cole, Professor L. L. Conant, Professor E. S. Crawley, Professor E. W. Davis, Mr. H. N. Davis, Professor W. P. Durfee, Mr. W. B. Ford, Dr. A. S. Gale, Miss Alice B. Gould, Professor G. W. Greenwood, Rev. J. G. Hagen, Dr. C. N. Haskins, Professor E. R. Hedrick, Dr. E. V. Huntington, Professor J. I. Hutchinson, Dr. Edward Kasner, Professor F. H. Loud, Professor H. P. Manning, Professor

Frank Morley, Professor G. D. Olds, Professor W. F. Osgood, Professor B. O. Peirce, Dr. A. B. Pierce, Mr. D. L. Pettegrew, Dr. F. H. Safford, Professor I. J. Schwatt, Professor W. E. Story, Miss M. E. Trueblood, Professor H. W. Tyler, Professor E. B. Van Vleck, Dr. Roxana H. Vivian, Professor H. S. White, Dr. E. B. Wilson, Dr. Ruth G. Wood, Professor F. S. Woods, Professor T. W. D. Worthen, Professor R. S. Woodward, Professor J. W. A. Young, Professor Alexander Ziwet.

The Council announced the election of the following persons to membership in the Society: Professor D. P. Bartlett, Massachusetts Institute of Technology; Professor C. E. Comstock, Bradley Polytechnic Institute, Peoria, Ill., Mr. H. N. Davis, Harvard University; Mr. W. J. Graham, New York, N. Y.; Mr. N. J. Lennes, Chicago, Ill.; Mr. T. J. McCormack, La Salle, Ill.; Dr. L. I. Neikirk, University of Pennsylvania; Dr. A. B. Pierce, University of Michigan; Professor W. J. Rusk, Iowa College; Miss M. E. Trueblood, Mt. Holyoke College; Mr. C. B. Upton, Columbia University; Dr. Oswald Veblen, University of Chicago; Mr. R. H. Williams, Columbia University. Seventeen applications for membership in the Society were received.

The committee appointed at the preceding summer meeting to consider the question of definitions of college entrance requirements in mathematics presented a report, which was received and recommended for publication. This report is printed in the present number of the BULLETIN.

A committee consisting of President Thomas S. Fiske, Professor R. S. Woodward, and Professor P. F. Smith was appointed to prepare and submit to the Council at the October meeting a list of nominations of officers of the Society for the coming year.

The following papers were read at this meeting:

(1) Professor I. J. SCHWATT: "On the length of curves."

(2) Professor T. J. I'A. BROMWICH: "Similar conics through three points."

(3) Dr. D. R. CURTISS: "Binary families in a triply connected region, with especial reference to hypergeometric families."

(4) Professor JOHN EIESLAND: "On a certain system of conjugate lines on a surface transformable into asymptotic lines by means of Euler's transformation."

(5) Dr. EDWARD KASNER: "A class of conformal transformations."

(6) Dr. EDWARD KASNER: "Notes on the theory of surfaces."

(7) Professor E. R. HEDRICK: "Note on the existence of a continuous first derivative."

(8) Dr. G. A. BLISS: "Jacobi's condition in the calculus of variations when both end points are variable."

(9) Professor ARNOLD EMCH: "Note on the p-discriminant of ordinary differential equations of the first order."

(10) Professor HELEN A. MERRILL: "On a notable class of linear differential equations of the second order."

(11) Professor FLORIAN CAJORI: "On the circle of convergence of the powers of a power series" (preliminary communication).

(12) Mr. E. T. WHITTAKER: "An expression of certain known functions as generalized hypergeometric functions."

(13) Mr. W. H. YOUNG: "On a test for non-uniform convergence."

(14) Professor J. I. HUTCHINSON: "On the automorphic functions of signature (0, 3; 2, 6, 6)."

(15) Professor B. O. PEIRCE: "On the lines of certain classes of solenoidal or lamellar vectors symmetric with respect to an axis."

(16) Professor H. T. EDDY: "The multiplication of complex numbers and of vectors compared."

(17) Professor J. N. VAN DER VRIES: "On monoids."

(18) Professor JACOB WESTLUND: "On the congruence $x^{\phi(P)} \equiv 1$, mod. P^n."

(19) Professor ALFRED LOEWY: "Zur Gruppentheorie mit Anwendungen auf die Theorie der linearen homogenen Differentialgleichungen."

(20) Dr. SAUL EPSTEEN: "Semireducible hypercomplex number systems."

(21) Professor L. E. DICKSON: "On the subgroups of order a power of p in the quaternary abelian group in the Galois field of order p^n."

(22) Professor L. E. DICKSON: "The subgroups of order a power of 2 of the simple quinary orthogonal group in the Galois field of order $p^n = 8l \pm 3$."

(23) Professor L. E. DICKSON: "Determination of all groups of binary linear substitutions with integral coefficients taken modulo 3 and of determinant unity."

(24) Professor L. E. Dickson: "Determination of all the subgroups of the known simple group of order 25920."

(25) Professor L. E. Dickson: "The systems of subgroups of the quaternary abelian group in a general Galois field."

(26) Dr. C. N. Haskins: "On the invariants of quadratic differential forms."

(27) Professor Frank Morley: "On projective coördinates."

(28) Professor Frank Morley: "On a skew quadrangle covariant with six points of space" (preliminary communication).

(29) Dr. E. B. Wilson: "The projective definition of area."

(30) Professor R. S. Woodward: "On the values of the stretches and the slides in the theory of strain."

(31) Professor R. S. Woodward: "The radial compressibility of the earth compatible with the Laplacian law of density distribution."

(32) Professor E. O. Lovett: "Periodic solutions of the problem of four bodies."

(33) Professor E. O. Lovett: "Central conservative systems with prescribed trajectories."

(34) Professor S. E. Slocum: "Rational formulas for the strength of concrete-steel beams."

(35) Professor A. S. Chessin: "On a class of linear differential equations."

(36) Dr. C. M. Mason: "On certain systems of differential equations: generalization of Green's functions, analytic character of the solutions."

(37) Dr. E. V. Huntington: "A set of independent postulates for the algebra of logic."

The papers of Professor Loewy and Dr. Epsteen were communicated to the Society through Professor E. H. Moore. Professor Merrill was introduced by Professor M. B. Porter, Dr. Mason by Professor H. W. Tyler. In the absence of the authors Professor Chessin's paper was read by Professor Schwatt, and the papers of Professor Bromwich, Dr. Curtiss, Professor Eiesland, Professor Emch, Mr. Whittaker, Mr. Young, Professor Peirce, Professor Eddy, Professor Van der Vries, Professor Westlund, Professor Loewy, Dr. Epsteen, Professor Dickson, Professor Lovett and Professor Slocum were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The definitions of the length of curves as given by Duhamel and Scheefer are applied by Professor Schwatt to the finding of the length of curves of functions with an infinite number of discontinuities within a finite interval, and also of functions with discontinuities at primary places which form an infinite denumerable assemblage.

2. The note of Professor Bromwich treats the envelope of the principal axes of a system of similar conics circumscribing a triangle by the use of the conjugate complex coördinates. It will be published in the *Transactions*.

3. Riemann's celebrated paper of 1857 on hypergeometric functions (Werke, page 67), which first emphasized the importance of the Monodromiegruppe for such functions, omitted from consideration three important cases. Two restrictions are explicitly introduced: first, that the exponents of no singular point differ by an integer, and second that the exponent sum is 1. As Klein pointed out in his winter semester lectures of 1893–94, Riemann also fails to discuss the possibility that the sum of three exponents, one from each pair, is an integer, a case which presents exceptions to his theorem that functions whose corresponding exponents differ by integers have the same group. Moreover, many of the ideas involved admit an extension to binary families in general, *i. e.*, families composed of the solutions of homogeneous linear differential equations of the second order. With these extensions of Riemann's idea Dr. Curtiss's paper is occupied.

In Part I, following Riemann's course, a definition is given from which the properties of the general binary family is developed, the group being the central object of study. The properties of kindred functions are deduced, and a criterion is given by means of which we may in all cases find whether two functions have the same group. This criterion is then applied to a case of especial interest to which hypergeometric families belong, namely, the case of binary families analytic in a triply connected region. A classification is introduced such that if we know the multipliers of two families and their places in that classification we can at once answer the question as to whether they are kindred.

In Part II hypergeometric families, with Riemann's restrictions removed, exclusively occupy the attention. Again starting from a minimum of definitions, the main properties of these families are developed, including their differential equations. The criteria for kindred families given in Part I are expressed in terms of the exponents except in certain doubtful cases, where additional data are required. For these cases also a method is indicated by means of which kindred families may be recognized. Following this discussion the relations between kindred functions are considered, especial attention being paid to the expression of a general hypergeometric function in terms of functions without apparently singular points. In conclusion, treatments of families whose exponent sums are 1 and 0 respectively are appended as corollaries of the more general results previously obtained.

The paper will be published in the *Memoirs of the American Academy of Arts and Sciences.*

4. In a paper read at the summer meeting of the Society in 1902, Professor Eiesland proved a number of theorems concerning curves and two-dimensional surfaces belonging to a so-called asymptotic complex, that is, a complex whose lines satisfy the differential equations

$$dx_5 + x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 = 0,$$
$$dx_1dx_2 + dx_3dx_4 = 0.$$

If (u) and (v) are the coördinate lines on a surface belonging to such a complex, and if we make use of Lie's transformation

$$x_1 = \tfrac{1}{2}P_1,\ x_2 = X_1,\ x_3 = \tfrac{1}{2}P_2,\ X_4 = X_2,\ x_5 + x_1x_2 + x_3x_4 = X_3,$$

where $X_1, X_2, X_3, P_1, P_2, -1$, are the coördinates of a surface element in ordinary space, there is obtained a surface considered as an ensemble of surface elements, or an element M_2, on which the lines (u) and (v) are asymptotic lines.

It is the purpose of the present paper further to complete the theory of asymptotic complexes and especially to point out the important rôle that a certain contact transformation known as Euler's transformation plays in the theory of surfaces. This transformation is

$$X_1 = \tfrac{1}{2}\bar{P}_2,\quad X_2 = \bar{X}_1,\quad \tfrac{1}{2}P_1 = \bar{X}_2,\quad P_2 = -\bar{P}_1,$$
$$X_3 = \bar{P}_2\bar{X}_2 - \bar{X}_3.$$

It is shown that Euler's transformation transforms the asymptotic lines into a system of conjugate lines with a well defined geometric property which characterizes the system. These lines have been called Euler's lines, since they are inseparably bound up with the transformation which bears his name.

In the case of a certain class of surfaces defined by a partial differential equation of the second order with equal invariants Euler's lines are also lines of curvature, so that for these surfaces Euler's transformation partakes to some extent of the property of a large class of contact transformations, the classic one of Lie's which transforms asymptotic lines into lines of curvature. It must be noticed, however, that Euler's transformation does not transform lines into spheres as does Lie's. The differential equation of Euler's lines has also been developed and its relation to the so-called conjugate complex in M_5 is shown.

5. In a paper published in the July BULLETIN, Dr. Kasner showed that if a point transformation converts more than three systems of straight lines into straight lines, then it is necessarily a collineation. It follows that a conformal transformation cannot possess more than one (real) system of straight lines with this property, unless it is merely a similarity. The object of the first part of the paper is to determine all the conformal transformations possessing such a system of lines. It is proved that they may be reduced, by similarity transformations, to one of three types:

$$Z = z^k, \quad Z = \log z, \quad Z = e^z.$$

The first converts a pencil into a pencil, the second a pencil into a set of parallels, the third a set of parallels into a pencil. The second part of the paper deals with certain interesting isothermal systems of curves closely related to the preceding transformations. In particular all the isothermal systems which allow either a translation or a rotation group are obtained.

6. Dr. Kasner's second paper includes three notes dealing with distinct topics in the infinitesimal geometry of surfaces. The first note considers "Surfaces characterized by their level and slope curves." The curves mentioned are those which are of particular interest in a topographical survey. The level curves of a surface $z = f(x, y)$ are cut out by the planes $z =$ const., while the orthogonal trajectories are the slope

curves. Among the questions considered is the determination of the surfaces for which either one or both of the systems mentioned have a special property. For example, the only surface upon which both the level and slope curves are asymptotic, is the skew helicoid. Of especial interest is the case where the slope lines are geodesic, the corresponding surfaces being defined by a partial differential equation of the second order.

The second note is entitled "The orthogonal projections of asymptotic lines and lines of curvature." The orthogonal projections of the asymptotic lines of a surface $z = f(x, y)$, into the plane $z = 0$, are given by $rdx^2 + 2sdxdy + tdy^2 = 0$, where r, s, t are the partial derivatives of the second order of f. The double system of plane curves so obtained is not of general character. The question arises then, when can a given double system $Adx^2 + 2Bdxdy + Cdy^2 = 0$, where A, B, C are functions of x, y, be regarded as the projection of the asymptotic lines of some surface. The required condition is obtained, and the result appears to be of interest as a fundamental formula in analysis. The discussion of this leads to the result that, if the double system is orthogonal it is necessarily isothermal. Similar questions arise in connection with the lines of curvature; for example, if the projections of the two systems of lines of curvature are orthogonal, then one system projects into straight lines and the other into parallel curves.

The subject of the last note is "Isothermal systems of geodesics." The result obtained leads to the following classification of surfaces: 1°. Upon the surfaces of constant gaussian curvature there exist a double infinity of isothermal systems of geodesics; 2°. Upon all surfaces deformable into surfaces of revolution, excluding 1°, there exists just one such system; 3°. Upon all the remaining surfaces no system exists. It follows then that the surfaces of constant curvature are completely characterized by the existence of more than one isothermal system of geodesics.

7. In Professor Hedrick's paper attention was called to an easily derivable necessary and sufficient condition for the existence of a continuous first derivative of a function of a single real variable. That this condition is necessary is noticed, for instance, by Goursat, in his Cours d'analyse, volume I, page 11. But the author of the paper is not aware that it has been

noticed that the same condition is also sufficient. The condition may be stated as follows: The necessary and sufficient condition that a function $f(x)$ should have a continuous first derivative, at a point $x = c$, is that the quotient

$$\frac{f(b) - f(a)}{b - a}$$

should approach one and the same limit, as b and a both approach c, independently of each other, and in any manner whatever. It should be noticed that this condition covers the existence, as well as the continuity, of the derivative in question; and it is at once extensible to a whole interval. The proof consists of a *reductio ad absurdum*, making use of a certain method of approach along a chosen assemblage of points.

Attention was also called to a modification of the condition, suggested to the author by Professor Porter, of the University of Texas. This consists in requiring that the function $\xi = \phi(\Delta, x)$, which occurs in the statement of the law of the mean, should be a continuous function of the single variable Δ. This function does not seem to have been investigated extensively.

8. Dr. Bliss considered the problem of the calculus of variations where a curve is sought which joins two given fixed curves A and B and minimizes the integral

$$I = \int F(x, y, x', y')dt.$$

It is well known that the solution of the problem must satisfy Euler's differential equation, and cut the two given curves "transversally." If a curve E has been found satisfying these conditions, then a further condition is necessary which corresponds to Jacobi's condition when the end points are fixed. To find this condition was the object of the paper. It turns out that the curvature of the two fixed curves A and B in their intersections a and b with E, must make a certain bilinear expression greater than or equal to zero. This is equivalent to saying that the intersections a and b and the "critical points" a' and b' of the curves A and B on the extremal E, must lie in the order $a\ b\ b'\ a'$. The exceptional case when a' and b' coincide was also investigated. Finally sufficient conditions for a minimum were determined.

9. It is well known that the equation $h(x, y) = 0$ resulting from the elimination of $p = dy/dx$ between the differential equations

$$\phi(x, y, p) = 0, \tag{1}$$

and

$$\frac{\partial \phi}{\partial x} + p \frac{\partial \phi}{\partial y} = 0, \tag{2}$$

where ϕ is a polynomial in x, y and p, in general represents the locus of the points of inflexion of the integral curves.

Applying to (1) and (2) a transformation by reciprocal polars

$$x = -\frac{p_1}{y_1 - x_1 p_1}, \quad y = \frac{1}{y_1 - x_1 p_1}, \quad p = -\frac{x_1}{y_1}, \tag{4}$$

where $p_1 = dy_1/dx_1$, Professor Emch proves that the configuration [(1), (2)] is transformed to

$$\phi_1(x_1, y_1, p_1) = 0, \tag{5}$$

$$\frac{\partial \phi_1}{\partial p_1} = 0. \tag{6}$$

Eliminating p_1 from (5) and (6) an equation $g(x_1, y_1) = 0$ is obtained, which clearly is the transformed of $h(x, y) = 0$ by reciprocal polars. Thus, Darboux's theorem is proved that, in general, the p-discriminant of a differential equation of the first order represents the cusp locus of the integral curves.

10. Miss Merrill's paper deals with a class of homogeneous linear differential equations, of which the equation considered by Sturm in his celebrated memoir in the first volume of *Liouville's Journal* is a special case. The method used is presumably the first (unpublished) finite difference method of Sturm which, in the present state of analysis, not only admits of a rigorous handling, but is capable of interesting generalizations.

11. In Professor Cajori's paper it was shown: that, when a power series $P(x)$ has infinite points on its circle of convergence R, the circle of convergence R' of the square, or of any positive integral power, of $P(x)$ is equal to R; that there are divergent series $P(x)$ whose squares are absolutely convergent,

$R' > R$, $P(x)$ being absolutely convergent for all points $|x| = R = 1$; that there are series $P(x)$ which are conditionally convergent for all points $|x| = R = 1$, but whose squares are absolutely convergent; that R' is never less than R. Illustrative examples were given.

12. It has long been known that many functions (*e. g.*, the logarithm and the Legendre functions) are particular cases of the hypergeometric function, and that other functions (*e. g.*, the exponential function and the Bessel functions) can be derived from the hypergeometric function by limit processes. The object of Mr. Whittaker's paper is to show that the latter class of functions is larger than has hitherto been supposed; that in fact it includes (in addition to those functions already known to belong to it) the following five types of known functions, namely: 1° The functions which arise in harmonic analysis in connection with the parabolic cylinder; 2° the error function; 3° the incomplete gamma function; 4° the logarithm integral; 5° the cosine integral. It is shown that all these functions, as well as the Bessel functions, can be derived by specialization and transformation from a simple new function, which can itself be derived from the hypergeometric function by a limit process (making some of the exponents infinite in a certain way); and this new function possesses many properties which are not possessed by the parent hypergeometric function.

14. The paper by Professor Hutchinson discusses the group of transformations arising from a monodromy of the branch points of the Riemann surface $y^6 = (x - a)^2(x - b)^2(x - c)^3 (x - d)^5$. The fundamental region for the group is a triangle with angles $\pi/6$, $\pi/6$, $\pi/6$. The arithmetic character of the group is determined, and the functions belonging to the group are expressed in terms of the theta constants by means of the transformation theory of the theta functions.

15. Professor Peirce's paper is in abstract as follows: When a vector is symmetric about an axis, every one of its lines lies in a plane which passes through the axis, and the whole field of the vector may be studied by examining its lines in any such plane. Given a family of curves $u = k$ in the xy plane, without multiple points, or points of intersection with each

other or with the x axis, it is possible to form an infinite number of vectors, symmetric with respect to the x axis, which shall have as lines such of the u curves as lie on one side of the x axis. Of these vectors an infinite number are lamellar and an infinite number solenoidal, but no one can be both lamellar and solenoidal unless u satisfies Lamé's condition in columnar coördinates for isothermic surfaces. This paper discusses the conditions which must be satisfied by the function u, or by an orthogonal function v, if the u curves are to be the lines of several of the classes of vectors commonly occurring in books on mathematical physics.

16. The multiplication of complex numbers being arithmetical and consequently commutative, it appears from physical considerations that vector multiplication of a pair of vectors is necessarily non-commutative, and the more important differences which appear in Professor Eddy's paper due to this fact are these:

1°. The position angle of a vector product of two constant vectors is the difference instead of the sum of the component position angles as is the case with complex numbers.

2°. The position and magnitude of the vector product of two constant vectors rotating uniformly with a given angular velocity is invariable, instead of rotating with twice the given velocity as the arithmetical product does.

3°. The product of two vectors, one of which rotates uniformly in a given period while the other varies harmonically in equal period, may be regarded as the sum of a fixed vector and a vector of equal absolute magnitude rotating with twice the given speed. The position angle of the fixed vector is opposite in sign to the corresponding part of the arithmetical product.

4°. The average value of the harmonic product of two vectors mentioned in 3° differs from the average value of the corresponding harmonic product of arithmetical factor in the sign of the position angle only.

5°. A complex vector product, being an entity distinct in significance and different in dimensions from its component vectors, may itself be a vector or not and should be considered to lie in a separate plane of its own with an initial line determined by existing physical conditions, which statements do not hold true in case of a complex arithmetical product.

17. In the investigation of "The multiple points on twisted curves" (see *Proceedings of the American Academy of Arts and Sciences*, volume 38, number 17, January, 1903) Professor Van der Vries made use of the method, introduced by Cayley, of considering the twisted curve as the partial intersection of a cone and a monoid. In the present paper, the general monoid has been considered and the quartic monoids have been classified in detail. Multiple lines are found to be of four kinds according to their relative multiplicities on the two cones of the monoid. In the different quartic monoids, the relative positions of the different lines and points were determined. Some interesting theorems on the general monoid were also developed, *e. g.*, "If a monoid has its maximum number of double points, these all lie in one plane," and one easily deduced from this, "If a monoid having a second vertex has its maximum number of double points, these double points lie on one line."

18. Professor Westlund's paper appears in full in the present number of the BULLETIN.

19. If a group G of linear homogeneous transformations in n variables is given and if r denotes a positive integer > 1, then three species of groups isomorphic with G can be set up (A. Hurwitz: "Zur Invariantentheorie," *Mathematische Annalen*, volume 45, page 381, et seq.): 1°. the group $\Pi_r G$ of rth product transformations; 2°, the group $P_r G$ of rth power transformations; 3°. the group $C_r G$ of rth determinant transformations, the latter existing only for $r \leqq n$.

Professor Loewy shows that the group $\Pi_r G$ is always reducible, $P_r G$ and $C_r G$ being two of its irreducible constituents. In some investigations, a separate study of $P_r G$ and $C_r G$ is therefore not necessary. This theorem is a special case of a more general theorem of reducibility, from which it also follows that $\Pi_r G$ is similar to a decomposable group.

The theorems concerning groups are then applied to linear homogeneous differential equations, and serve to connect with a given equation with rational coefficients corresponding to the operations Π_r, P_r, C_r three infinite systems of such equations; all these equations, embracing in particular the well known "associated" equations of Forsyth, bear to one another relations characterized by the concepts of species (*Art*) and reducibility. The paper will be published in the *Transactions*.

20. Scheffers calls a hypercomplex number system E reducible when its n units $e_1, \cdots, e_m, e_{m+1}, \cdots, e_n \equiv E_1E_2$ satisfy the conditions

$$\begin{array}{llll} A_1) & E_1 \text{ forms a system by itself,} & B_1) & e_je_k = 0, \\ A_2) & E_2 \text{ forms a system by itself,} & B_2) & e_ke_j = 0, \end{array}$$
$$(j = 1, \cdots, m;\ k = m + 1, \cdots, n).$$

Dr. Epsteen finds that under conditions A_1, A_2, B_1, B_2 the group of the system has the form

$$\begin{array}{c|c} G_{11} & 0 \\ \hline 0 & G_{22} \end{array}.$$

If, however, the system satisfies A_1, A_2 and the less exacting conditions (C_1, C_2) that no product e_je_k, e_ke_j involve the units e_j, the system is said to be semireducible of the first kind. A semireducible system of the second kind satisfies the conditions A_1, A_2, C_1, B_2. In any case its group has the reducible form

$$\begin{array}{c|c} G_{11} & 0 \\ \hline G_{21} & G_{22} \end{array}$$

Here G_{11} is the group of the system E_1, and in all the cases calculated G_{22} is the group of a system Q, the so-called quotient system $E|E_1$. If E is semireducible of the second kind, then E_2 itself is the quotient system $E|E_1$.

A system with irreducible group is called absolutely irreducible. If a system E is, according to two choices of the system of units, decomposable once into the sequence of p absolutely irreducible systems $E_1, \cdots, E_p$ where the system $E_1, \cdots, E_h, E_{h+1}$ is semireducible of the second kind ($h = 1, \cdots, p - 1$), and again similarly into the sequence of the q systems $\bar{E}_1, \cdots, \bar{E}_q$, then $q = p$ and the subsystems $\bar{E}_1, \cdots, \bar{E}_p$ are similar in some order to the systems $E_1, \cdots, E_p$.

The paper will appear in the *Transactions*.

21. The first paper by Professor Dickson relates to the problem of the p-section of the periods of hyperelliptic functions of four periods. For $p = 3$, the problem relates also to the 27 lines on a general cubic surface as well as to the reduction of a binary sextic to Cayley's canonical form $T^2 - U^3$.

The paper will appear in the October number of the *Transactions*.

22. The second paper by Professor Dickson establishes the result that, independent of the values of p and n (such that p^n is of the form $8l \pm 3$), the simple quinary orthogonal group O of order $\Omega = \frac{1}{2}p^{4n}(p^{4n} - 1)(p^{2n} - 1)$ contains the same number of distinct sets of conjugate subgroups of order each power of 2, one set of representative groups serving for every O_Ω. Moreover, except for the subgroups of the orders 2 and 4 and certain of the types of order 8, the order of the largest subgroup of O_Ω in which a group of order a power of 2 is self-conjugate is independent of p and n. The paper will appear in the *Transactions*, January, 1904.

23. The third paper by Professor Dickson determines all the subgroups of the group Γ_{24} of binary linear substitutions with integral coefficients modulo 3 of determinant unity. There result self-conjugate groups of orders 2 and 8, a single set of conjugate cyclic groups of each of the orders 3, 4 and 6, but no further subgroups. We may represent Γ_{24} as a substitution group on 8 letters, but not on fewer. The paper will appear in *Annals of Mathematics*, volume 5 (1903–4).

24. The fourth paper by Professor Dickson makes a complete determination of the subgroups of the known simple group of order 25920, occurring in the problem of the 27 lines on a general cubic surface and in the trisection of the periods of hyperelliptic functions of four periods. Incidentally a proof by pure group theory is obtained of the Jordan theorem that the above problems have no resolvent of degree < 27. The following more general theorem, not heretofore stated, is established: *All the maximal subgroups of* G_{25920} *are conjugate with* G_{960}, G_{720}, G_{648}, H_{648}, or G_{576}. Under different notations, these 5 groups appear in the geometric and function-theoretic work of Witting and Burkhardt.

The group is particularly rich in subgroups, although none are of index < 27. All the 5 possible abstract groups of order 12 are represented, as also all substitution groups on 6 or fewer letters. The orders of the 114 distinct types of non-conjugate subgroups, other than itself and identity are as follows, the number in parenthesis indicating the number of types of the given order: 2 (2), 3 (3), 4 (6), 5, 6 (7), 8 (9), 9 (4), 10, 12 (7), 16 (6), 18 (7), 20, 24 (11), 27 (3), 32 (3), 36 (4), 48 (5), 54 (3), 60 (2), 64, 72 (2), 80, 81, 96 (4), 108 (4), 120 (2), 160, 162, 192 (2), 216 (2), 288, 324, 360, 576, 648 (2), 720, 960.

The number of conjugates to the various groups is given by one of the 23 numbers: 27, 36, 40, 45, 90, 120, 135, 160, 162, 216, 240, 270, 320, 360, 405, 480, 540, 720, 810, 960, 1080, 1296, 1620.

The paper will appear in the *Transactions* during the year 1904.

26. Dr. Haskins's paper is an extension of an earlier one bearing the same title. The number of differential parameters for the general quadratic differential form in any number $n > 2$ of variables is exactly determined.

27. Professor Morley's first paper is in abstract as follows: With two points x, y and two lines ξ, η of a plane are connected two double ratios — *e. g.*, the ratio of the ratios in which the segment $\overline{xy}$ is cut by ξ and by η. This notion of the double ratios of a point pair and a line pair, though rarely if ever emphasized, is one of great convenience in starting projective geometry. In the memoir, which has appeared in the *Transactions* of the Society (July, 1903) it is utilized to define coördinates explicitly as double ratios *without* a "factor of proportionality." Passing to "supernumerary homogeneous coördinates" x_j, ξ_j $(j = 0, \cdots, 3)$ where $\Sigma x_j \xi_j = 0$, the geometric significance of what usually appears as an analytic artifice is ascertained. The pedagogic ground so gained is then utilized in a brief study of collineations.

28. Professor Morley's second paper deals with a geometric aspect of some of the algebraic results of Hilbert's Memoir (*Mathematische Annalen*, volume 28). Replacing a binary form of even order by $2n$ points on a norm-curve R_n, we consider especially the case $n = 3$, when the points determine the curve. The irrational covariant cubics give four points of space, or rather two pairs of points forming a skew quadrangle or 4-gon. By means of this 4-gon, which is uniquely given when a general 6-point is given, some covariants of the sextic are geometrically interpreted. Thus the covariant $C_{2,4}$ is represented by those 4 lines of the cubic R_3 which meet the diagonals of the 4-gon. The 4-gon stands to the 6-point much as the diagonal triangle of a plane 4-point to the 4-point.

29. In a memoir presented last summer at the Karlsbad meeting of the Deutscher Mathematiker-Vereinigung Dr. Wilson showed how the concept of volume (n-dimensional extent) and the measure of volume in n-dimensional space were independent of the concept and measure of length. The memoir is now being published in the *Jahresbericht* of the Vereinigung The object of the present paper is to develop and apply the generalized concept of extent (area) in the case of the plane. The paper closes with the result that the necessary and sufficient condition that a collineation in the plane be resolvable into two projective reflections (involutory collineations) is that areas in the plane are left invariant when measured in reference to some one of the fixed lines of the collineation. The paper will appear in the *Annals of Mathematics.*

30. The object of Professor Woodward's first paper is to explain a method of deriving the stretch (or dilatation, or elongation) of any line, and the slide (or change in mutual inclination) of any two lines to terms of any order; and to call attention to the fact that the coördinate stretches given in treatises on elasticity published during the past thirty years are commonly erroneous in terms of the second order.

31. Professor Woodward's second paper shows how to compute the elastic change in length of the earth's radius due to a change in surface pressure, like that of the atmosphere for example, on the supposition that the mass of the earth conforms to the law of Laplace. The investigation will be published in full in the *Astronomical Journal.*

32. In the problem of three bodies we owe to G. W. Hill the introduction of the fertile notion of periodic solutions; to Poincaré the perfecting of their mathematical theory; to G. H. Darwin a splendid collection of examples of such orbits. In the problem of four bodies few examples of periodic orbits seem to have been given. Lehmann-Filhés in 1891 (*Astronomische Nachrichten,* No. 3033) found the exact solutions of the problem of four bodies corresponding to those of Lagrange in that of three; the collinear solutions of Lehmann-Filhés are periodic. Moulton in 1900 (*Transactions* of the AMERICAN MATHEMATICAL SOCIETY, volume I, pages 17–29) determined the exact solutions of the particular problem of four bodies when three are finite in motion according to one or the other

of the Lagrangian solutions, and the fourth infinitesimal. The above seem to be the only constructions published of periodic solutions of the problem of four bodies. To be sure, extensions of Lagrange's solutions had been given previous to the paper of Lehmann-Filhés, *e. g.*, those of Veltmann (1875), Sloudsky (1878) and Hoppe (1879); Veltmann's has been subjected to criticism, and the generalizations of Sloudsky and Hoppe are more special than those of Lehmann-Filhés. See Whittaker's Report on the problem of three bodies, British Association Report, 1899.

The paper of Professor Lovett is one of the by-products of a course in celestial mechanics given at Princeton University during the last academic year. It suggests several questions for later discussion. It was itself suggested while reading Chartier's recent paper on periodic solutions of the problem of three bodies in which he reconstructs analytically certain of Darwin's orbits. Given any three finite bodies in motion according to one or the other of the Lagrangian solutions of the problem of three bodies, and a fourth infinitesimal body acted on by the finite bodies according to Newton's law but itself incapable of affecting their motion; the immediate problem is to locate the centers of libration, determine the roots of the characteristic equation, and construct periodic orbits in the immediate vicinity of those centers of libration about which such orbits are possible. The force function appears in a symmetric form expressed in terms of the three masses and their distances from the fourth particle; this is analogous to Darwin's form in the three body problem. As to centers of libration, the analytic conditions determining their positions possess solutions both without and within the plane of the finite bodies; the existence of the former can be accounted for by ascribing to one of the finite bodies properties analogous to those attributed to the sun of our system to explain cometary phenomena while the comet is passing its perihelion. Periodic orbits exist about these remarkable centers. Numerical application is made to the case of three unit masses in all cases. In conclusion certain generalizations of these problems in process of solution are formulated. A postscript presents one of these, namely that of four bodies of arbitrary masses, or in fact of n bodies, on a straight line, analogous to the collinear solutions of Lagrange, studied under a method which Oppenheim used in the problem of three bodies in a memoir inserted in the fourth volume of the publications of the von Kuffner observatory. Relative to another of the questions enumerated, it may perhaps be remarked with propriety that in the meantime the author has found that a method employed by Levi-Civita recently in the restricted problem of three bodies may be applied effectively in the discussion of conditions for collisions in the problems of four bodies considered in the first part of the present paper.

33. Tisserand in the first volume of his classic treatise on celestial mechanics studies the problem of determining the force under which a particle, whatever be its initial position and velocity, always describes a conic section whose equation is taken in its most general form. Bertrand (*Comptes Rendus* of the Paris academy of sciences, volume 84, pages 671 and 731) was the first to set this problem, which finds its most interesting application in the theory of double stars. Bertrand solved it, and later Darboux and Halphen in more complete form (*Comptes Rendus*, volume 84, pages 760 and 939). Battaglini (*Giornale di Matematiche*, volume 17, page 43) determined the components of the forces in the case of motion in a conic, and Dainelli in the eighteenth volume of *Battaglini's Journal* treated the more general one of finding the components of the forces acting on the particle as functions of its coördinates when the trajectory is a general curve. All these discussions have been concerned with the motion of a single material point. Similar problems may be proposed for material systems. It is the purpose of Professor Lovett's paper on central conservative systems with prescribed trajectories to consider certain questions of this kind and of considerable generality. A careful search of the bibliography has failed to reveal more than one previous paper occupied with more than one particle; this is an article by Oppenheim in the third volume of the publications of the von Kuffner observatory, in which the author studies the forces under which those bodies describe given coplanar curves; he constructs the analytic machinery for the case of three arbitrary coplanar orbits, but the only particular solutions realized were previously known. Of the present article the first section is occupied with the determination of the forces capable of maintaining the motion of a particle on an arbitrarily given curve in space of any dimensions independently of the initial conditions of the motion; in the second, the forces of a central

conservative system capable of maintaining a system of m particles on as many prescribed but arbitrary orbits in a space of n dimensions are investigated; in the succeeding sections applications are made to the study of a system of three bodies describing any given trajectories in ordinary space under central conservative forces and independently of initial conditions. Of special interest perhaps is a system of differential equations which the functions defining the orbits must satisfy if the velocities and accelerations are to be determinate, certain of these equations vanishing identically when the motions take place in a single plane.

34. The subject of concrete-steel construction has recently developed with great rapidity, but is as yet upon a very unsatisfactory basis as regards structural dimensioning. In Dr. Slocum's paper the fundamental assumptions underlying the theory of beams are first considered, and it is shown how these assumptions may be simplified for the purpose of practical analysis. The general theory of beams is then briefly developed for the purpose of calling special attention to certain tensile stresses which are of utmost importance in concrete construction, but upon which little emphasis has as yet been laid. To make the paper of practical value, a form of concrete-steel beam of recent design is then chosen, and the theory applied to it in detail. By a simple, rational analysis the internal stresses in such a beam are determined, and formulas for structural dimensioning obtained. The causes of various forms of failure of loaded concrete-steel beams are also pointed out, and means suggested as a remedy for such failure. The paper appeared in the *Engineering News*, July 30.

35. Professor Chessin generalized the results obtained by him in a previous paper on differential equations reducible to Bessel's equation with second member. If

$$D^{(n)} \equiv A_0 \frac{d^n}{dx^n} + A_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + A_n$$

and $y_k \equiv D^{(n)} y_{k-1}$, then the integration of $y_m + a_1 y_{m-1} + \cdots + a_m y = f(x)$ is reducible to the integration of $D^{(n)} y = f(x)$.

36. Green's theorem and the concept of Green's function, which have been extended to the linear differential equation

of the elliptic type, were developed in the paper of Dr. Mason for a system of equations of the form

$$(1)\quad L_i(u_1, \cdots, u_n) \equiv \Delta u_i + \sum_{j=1}^{n}\left(a_{ij}\frac{\partial u_j}{\partial x} + b_{ij}\frac{\partial u_j}{\partial y} + c_{ij}u_j\right) = 0$$

$(i = 1, 2, \cdots, n)$. The Green's functions $G^{ik}(x, y, \xi, \eta)$ of (1) for a region Ω consist of n sets of n functions each. Each set forms a solution of (1) within Ω, each function vanishes on the boundary of Ω, and all except the functions G^{ii} are continuous, these having the singularity of $\log r$ for $x = \xi$, $y = \eta$. The set of solutions of $L_i = f_i$ admit of integral representation in terms of f_i, their boundary values $\bar{u}_i$ and $H^{ik}(x, y, \xi, \eta)$, the Green's functions of the "adjoint system" to (1). The functions G^{ik} and H^{ik} obey the law of reciprocity: $G^{ik}(x, y, \xi, \eta) = H^{ki}(\xi, \eta, x, y)$ $(i, k = 1, 2, \cdots, n)$. In the second part of the paper it was proved that system $L_i = f_i$, where a_{ij}, b_{ij}, c_{ij} are real analytic functions, can have only analytic solutions, the same result holding for the solutions of

$$\Delta^{(n)}u + e_1\Delta^{(n-1)}u + \cdots + e_{n-1}\Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = f.$$

37. In Dr. Huntington's paper the number of postulates required for the algebra of symbolic logic * is reduced to the following nine, which are shown to be independent:

1*a*. $a + b$ is in the set whenever a and b are in the set.

1*b*. $a \cdot b$ is in the set whenever a and b are in the set.

2*a*. There is an element 0 such that $a + 0 = a$ for every element a.

2*b*. There is an element i such that $a \cdot i = a$ for every element a.

3*a*. $a + b = b + a$ whenever a, b, $a + b$, and $b + a$ are in the set.

3*b*. $a \cdot b = b \cdot a$ whenever a, b, $a \cdot b$, and $b \cdot a$ are in the set.

4*a*. $a + (b \cdot c) = (a + b) \cdot (a + c)$ whenever a, b, $a + b$, etc., are in the set.

4*b*. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ whenever a, b, $a \cdot b$, etc., are in the set.

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* See A. N. Whitehead, Universal Algebra, vol. 1, pp. 35-37.

5. If the elements 0 and i in postulates $2a$, $2b$ are uniquely determined, then for every element a there is an element a' such that

$$a + a' = i \quad \text{and} \quad a \cdot a' = 0.$$

The associative laws do not appear among the postulates, but are deduced as theorems.

COLUMBIA UNIVERSITY. F. N. COLE.

REPORT OF THE COMMITTEE OF THE AMERICAN MATHEMATICAL SOCIETY ON DEFINITIONS OF COLLEGE ENTRANCE REQUIREMENTS IN MATHEMATICS.

AT the summer meeting of the AMERICAN MATHEMATICAL SOCIETY in September, 1902, a special committee was appointed to prepare standard formulations of college entrance requirements in mathematics, in coöperation with committees already appointed by the Society for the Promotion of Engineering Education and the National Educational Association. The following report has been prepared by the committee of the Mathematical Society, taking due account, on the one hand, of previous work along similar lines, as represented for example in the mathematical definitions of the College Entrance Examination Board and the Commission of Colleges in New England, and, on the other hand, of existing conditions in the mathematical instruction of colleges and schools.

The membership of the committee represents various forms of higher education only, but advice of value has been sought and obtained from other members of the Mathematical Society and from secondary teachers.

In its selection of topics the committee has aimed to emphasize fundamental matters of principle, and to omit complicated processes and subjects not capable of rigorous treatment in the secondary school.

By the selection of subjects it is not implied that all should be required by any one college, or be taught in any one school.

The committee understands its duties in the following sense:

First: To specify those mathematical subjects which are generally recognized as appropriate requirements for admission to colleges and scientific schools.