present number of the BULLETIN. Reports of the lectures of Professors Van Vleck and Woods will appear in later numbers.

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LINEAR SYSTEMS OF CURVES UPON ALGEBRAIC SURFACES.

AN ABSTRACT OF THREE LECTURES DELIVERED AT THE BOSTON COLLOQUIUM, SEPTEMBER 2–5, 1903.

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THE notion of equivalence as formulated in projective geometry has simplified greatly the study of algebraic curves and surfaces, particularly those of low order. The next step toward a wider survey is the admission of all birational transformations of the plane, or of space of three or more dimensions. In the plane, the theory of Cremona transformations is no longer new, and the elements are familiar to all students of geometry. Not so, however, in space of more than two dimensions; probably for the reason that nothing is known analogous to the theorem that a plane Cremona transformation is resolvable into a succession of quadric transformations and collineations. And even in plane geometry the intricacies of the transformations themselves have kept most students from the matter of higher importance, the properties of figures that remain invariant under all transformations of the group. Yet there does exist a body of doctrine under the accepted title of "Geometry upon an algebraic curve," and a fair beginning has been made upon a similar theory, the "Geometry upon an algebraic surface." * These titles are intended to cover only such properties of a curve or surface as appertain to the entire class of curves or surfaces that can be related birationally to the fundamental form.

* Consult, for an outline of the geometry upon an algebraic curve, Pascal's Repertorium der höheren Mathematik, Part II, Chapter V, § 4; or the more extended articles: C. Segre, "Introduzione alla geometria sopra un ente algebrico semplicemente infinito"; E. Bertini, "La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico,"—both in *Annali di Matematica*, series II, vol. 22 (1894). For the corresponding theories regarding surfaces, the best reference is to the comprehensive summary by Castelnuovo and Enriques: "Sur quelques récents résultats dans la théorie des surfaces algébriques," *Math. Annalen*, vol. 48 (1896).

A plane algebraic curve may have its order changed by a Cremona transformation, but not its deficiency (Genre, Geschlecht). As to sets of points on the curve, two sets which together make up a complete intersection of a second curve with the first do not lose that property by birational transformation, if we exclude from consideration fundamental points introduced by the transformation itself. * Mutually residual set of points, and coresidual sets, preserve their relation. Hence the group of sets of points coresidual with any given set becomes of importance. If a given set of D points lies on a curve of deficiency p, and if a coresidual set can be found containing k arbitrary points, then these numbers are connected by the relation constituting the Riemann-Roch theorem

$$k = D - p + \rho,$$

where ρ is zero if $D > 2p - 2$.

The totality of all sets of D points coresidual to any one set is termed a group or *series*, and is denoted by a symbol g_D^k. Such a series is called *complete*. If by any algebraic restrictions a series is separated out from it, of course that would be called incomplete or partial. For example, on a plane nodal cubic a series g_3^2 is cut out by all straight lines, incomplete because any three arbitrary points of the curve are coresidual to any other three. Every series g_D^k can be cut out upon the fundamental curve by a linear system of auxiliary curves whose equation may be written, with k parameters :

$$F_0 + l_1F_1 + l_2F_2 + \cdots + l_kF_k = 0.$$

As on a single curve sets of points, so in a plane, linear systems of curves are studied. By every birational transformation, linear systems are carried over into linear systems. A complete linear system is defined most easily by specifying the multiplicity that a curve of the system must have in each point of a fundamental set, and by prescribing the *order* of the curves. Thus $\binom{a_1 \, a_2 \cdots}{s_1 \, s_2 \cdots}$ can indicate that in a_1 every curve is to have a multiple point of order at least s_1, etc. If the base points alone, with their respective multiplicities, determine a system under consideration, that system is termed *complete*. If the base points actually impose, for curves of order m, fewer conditions than would be expected from their several multiplic-

* Or if we employ no auxiliary curves except such as are adjoint to that containing the point sets.

ities, the system is *special;* otherwise it is *regular.* It is an important theorem that no set of r base points can be so *located* as to produce an $(r+1)$th variable multiple point on the curves of the system; *i. e.*, the multiple points of the generic curve of a plane linear system lie all in the base points of the system.

Adjoint curves of a linear system are familiar to the student of function theory; they have in every multiple point of order s for the given system a multiplicity of order at least $s-1$. The adjoints of order lower by 3 than the original system are important from the fact that they transform always into the corresponding system of adjoints to the transformed curves. On this account the term adjoint, as used ordinarily, implies a curve of order $m-3$ unless differently specified. *Second adjoints* are adjoint to adjoints of the system, etc. The employment of successive adjoint systems as a means of investigation is due to S. Kantor and to G. Castelnuovo, the latter acknowledging the priority of the former. * On every curve its adjoints cut out a unique complete series g^{p-1}_{2p-2}, called the *canonical series.* The deficiency of the first or second adjoints of a linear system is denoted by P_1 or P_2, and may be termed first, or second, canonical deficiency. Aside from the canonical series upon curves of a system, the most important are the *characteristic series* of the system, that is the totality of sets of points in which two curves of the system intersect. If a plane linear system is complete, then the *characteristic series* on each curve is a *complete series* upon that curve. So far the definitions and propositions refer to curves in a plane; the question is in order, whether they can be transferred to systems of curves lying upon curved surfaces.

First, it is noticed that by means of a linear system of curves the plane may be related point for point to a surface in space of three or more dimensions.† If the system is k-fold infinite, $k+2$ members of the system can be related arbitrarily to $k+2$ hyperplanes in space of k dimensions. Take $k=3$ for ease; then a curve of the system

$$u_1f_1 + u_2f_2 + u_3f_3 + u_4f_4 = 0$$

may be assigned to a plane $(u_1 : u_2 : u_3 : u_4)$ in ordinary space.

* See *Math. Annalen*, vol. 44, p. 127.

† Exceptional cases are discussed by Enriques: "Ricerche di geometria sulle superficie algebriche," *Torino Memorie*, series II, vol. 44 (1893), p. 178.

Curves through one point become then planes through one point, and the ∞^2 points of the plane become the ∞^2 points of some algebraic surface F. All such surfaces are called *rational*. Similarly a linear family of curves triply infinite upon any surface relate that surface point for point to another surface in threefold space, linear systems of curves in one giving rise to linear systems upon the other, and *the transformed system will lack fundamental or base points*. The value of such projectively related pictures of a linear system of curves was first emphasized by C. Segre.

If on any surface, rational or not, there exists a system of curves doubly infinite, such that two arbitrary points determine one and only one curve containing them, that may be termed a linear system upon the surface in question; and Enriques proves that the ∞^2 curves of such a system can be *projectively* related to the straight lines of a plane. If the series is ∞^3, then its curves are referable projectively to the planes of three-space, etc. Only *simply* infinite systems escape this far-reaching theorem, and thus give rise to a most interesting unsettled question, indicated by Castelnuovo.*

Definitions of residual and corresidual curves upon a surface are those which any one could formulate at once from the use of these terms for sets of points upon a curve; their significance upon a twisted curve is the same as upon its plane projection. So of complete systems, both of curves and of surfaces, the latter admitting of course multiple curves as well as base points. For a surface of order m, the adjoints invariantively related are of order $m - 4$, containing as $(s - 1)$-fold curve every s-fold curve of the given surface. If these first adjoint surfaces form a k-fold infinite linear system, the number k is an invariant of the surface, and is termed its *geometric deficiency* (p_g). Attempting to express this number in terms of the order m of the surface, the order d and deficiency π of its double curve (if any), and of the number t of triple points on this double curve, one would find a second number

$$p_n = \tfrac{1}{6}(m-1)(m-2)(m-3) - d(m-4) + 2t + \pi - 1,$$

called the *numerical deficiency* of the surface. This number also is an invariant of the surface, as Noether first proved, and may

* Castelnuovo: "Alcuni risultati sui sistemi lineari di curve appartenenti ad una superficie algebrica." *Memorie di matematica e di fisica della Società Italiana delle Scienze*, Series 3°, vol. 10 (1896), pp. 82–102. See especially the close of his preface.

be *either equal to or less than* p_g, but never greater. Rational surfaces have $p_g = p_n = 0$; ruled surfaces have p_n negative. If $p_g = p_n$, then the above mentioned theorem of Enriques concerning linearity holds true also for systems which are only simply infinite. Surfaces of the first adjoint system cut out upon a given surface a system of curves, each of deficiency $p^{(1)}$ or less. This invariant number $p^{(1)}$ we may call the *canonical deficiency* of the surface; the curves form an unique complete linear system, just as do the point sets of the canonical series on a plane curve.

The definitions here given are but a part of those found useful in this fascinating branch of geometry. The true way to learn something of the subject is not to master first all its definitions and distinctions, but to study the proofs of some few leading theorems. Such are Enriques's proof of the equivalence of two geometrical definitions of the linearity of a system (mentioned above), and the following less elementary propositions:

1. Surfaces whose plane or hyperplane sections are irreducible unicursal curves are either ruled or rational (Noether).*

2. So also surfaces whose plane or hyperplane sections are irreducible elliptic curves (Castelnuovo),† or hyperelliptic of any deficiency π (Enriques).‡ For plane sections, not hyperelliptic, of deficiency $\pi > 2$, the corresponding theorem is not yet fully known.

3. Upon any algebraic surface $f(x, y, z, t) = 0$ a linear differential of first kind is said to exist (Picard), if an expression

$$\int [P_1(x, y, z, t) \cdot dx + P_2 \cdot dy + P_3 \cdot dz + P_4 \cdot dt]$$

is finite and determinate, independent of the path of integration, when taken upon the surface between any two arbitrary points. If the surface $f = 0$ is a *cone*, such differentials exist, for they are the abelian differentials of first kind upon its plane sections. Picard proves§ that if the surface $f = 0$ have no multiple points or curves, then no such differential can exist upon it. The proof of this theorem concluded these lectures.

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* Noether's theorem is more general. See *Math. Annalen*, vol. 3: "Ueber Flächen, welche Schaaren rationaler Curven besitzen."

† "Sulle superficie algebriche," etc., *Lincei Rendiconti*, January, 1894.

‡ "Sui sistemi lineari," etc., *Math. Annalen*, vol. 46 (1895), pp. 179–199.

§ Picard et Simart: Théorie des fonctions algébriques de deux variables indépendantes, vol. 1 (1897), pp. 119–120.