methods, an inexhaustible creative imagination, the fearless introduction and employment of ideal elements, and an appreciation for a refined and logical development of all its parts.

We who stand on the threshold of a new century can look back on an era of unparalleled progress. Looking into the future an equally bright prospect greets our eyes; on all sides fruitful fields of research invite our labor and promise easy and rich returns. Surely this is the golden age of mathematics!

OUTER ISLAND,
September, 1904.

DE SÉGUIER'S THEORY OF ABSTRACT GROUPS.

Eléments de la Théorie des Groupes Abstraits. By J.-A. DE SÉGUIER. Paris, Gauthier-Villars, 1904. ii + 176 pp.

THE title for the complete treatise is Théorie des groupes finis. The present first volume deals with the theory as far as it demands no concrete representation. The second volume is to be entitled Compléments.

The Eléments gives a remarkably compact presentation of purely abstract group theory, including the most recent results. The attempt has been made to extend as far as possible the general theorems to infinite groups. The broader view thus gained more than compensates for the increased abstruseness. It appeals particularly to the reviewer who has given much attention to the coördination of the various branches of analytic group theory into a comprehensive theory of analytic groups in an arbitrary field. The inclusion of infinite groups, moreover, gives the author the means of a natural presentation of negative and rational numbers, Galois's imaginaries, and algebraic numbers as elements of certain groups. The author is therefore justified in giving (pages 27–51) a very compact, but practically complete, account of Galois fields (champ, corps de Galois). Relative to a first mode of composition, called addition, C_N is an additive group; relative to a second mode of composition C_N, with zero omitted, is a multiplicative group, and one may set $0x = x0 = 0$ by definition; a final postulate makes multiplication distributive with respect to addition.

The opening six pages on Cantor's assemblages establish his distinction between finite and infinite sets, but make no classification of the latter. Throughout the text the term *corps* is used to designate a set of elements whose law of composition is associative. It is stated on page 7 that an abelian assemblage is always associative; but this is contradicted by the assemblage of all positive rational members with the law of composition $a \circ b = 1/ab$.

A group is defined by Huntington's three independent postulates (BULLETIN, volume 8, page 296); it being a theorem that the product of any two elements lies in the set. Most readers, I think, would find it more natural to have this property as a postulate, as is the case in Moore's definitions.*

The author introduces (on page 8) a *semigroup* G in connection with any subset S containing a system of generators of G. The postulates defining G are: (1) associativity; (2) for any α in S and any b in G, there is at most one solution [x in G] of $\alpha x = b$; (3) similarly for $x\alpha = b$. The author states that it follows that $ax = ax'$ requires $x = x'$ for any a in G. If so, the definition itself might better read: for any two elements α and b in G, there is at most one solution x in G of $\alpha x = b$. The conclusion was presumably reached about as follows: Express a in terms of the generators, say $a = \alpha_1\alpha_2\alpha_3$. Then $\alpha_1(\alpha_2\alpha_3 x) = \alpha_1(\alpha_2\alpha_3 x')$ would require $\alpha_2(\alpha_3 x) = \alpha_2(\alpha_3 x')$, whence $\alpha_3 x = \alpha_3 x'$, $x = x'$. However, this argument assumes that $\alpha_2\alpha_3 x$ belongs to G. But it does not follow from the postulates that the product of every two elements of G belongs to G, as stated explicitly at the bottom of page 58. Take as an example the set of any finite number of line translations to the right; the three postulates hold, but not every product occurs in the set. To include the properties desired by de Séguier, I suggest that the name semigroup be given to the very important assemblage and rule of combination defined by the following four postulates: †

(1) If a and b belong to the set, then $a \circ b$ belongs to the set.

(2) $(a \circ b) \circ c = a \circ (b \circ c)$, whenever a, b, c, $a \circ b$, $b \circ c$ $(a \circ b) \circ c$ and $a \circ (b \circ c)$ belong to the set.

(3) and [4] For every two elements a and b of the set, there exists at most one element x in the set such that $a \circ x = b$ [$x \circ a = b$].

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* *Transactions*, vol. 3 (1903), p. 485. Professor Moore has observed that his postulate ($3_r''$) is redundant, so that his second definition becomes simpler. A simplification of his first definition may be made by changing $a_l'a$ into aa_l' in (4_l) and dropping postulate (3_l). The new definitions, which are ideally simple, will be given by Moore and the reviewer in the *Transactions* for April, 1905.

† For an assemblage satisfying postulates (1) and (2) I suggest the name algebra.

These four postulates are seen to be independent and consistent. For a semigroup, either $ax = ax'$ or $xa = xa'$ requires that $x = x'$. An example of a semigroup is the totality of the powers of an operator a of no finite period. An infinite cyclic group requires two generators a and a^{-1}.

The author's discussion of a complete set of generational relations is excellent. A point not usually noted is that arbitrarily given relations between the elements regarded as generators are never incompatible; they may of course require that certain of the generators be identical or reduce to identity. On the other hand, a multiplication table cannot be assigned arbitrarily, in view of the associative law.

The luminous account of the chief results in the abstract Galois field theory runs quite parallel to that of the reviewer's Linear Groups, the fundamental concepts being due to Galois and Moore. It is therefore not easy to understand how the author could overlook the chapters in Linear Groups which give a complete account of the theory of quadratic forms in a Galois field.* Instead, the source cited is Jordan's course of lectures at the Collège de France in 1904. I take this opportunity to state that Jordan arrived at the chief results of the theory independently.†

Passing from finite to infinite fields, a brief account is given of the most elementary properties of algebraic fields. The case of an infinite field C with a finite modulus p is disposed of in a single line (page 51) by saying "on obtient C en formant des C_{p^n} [Galois field of order p^n] où n croît indéfiniment." Unfortunately the matter is not so simple as the author thus indicates. Quite a number of years ago Professor Moore and I noticed independently the existence of the infinite field C modulo p given as the aggregate of all $GF[p^n]$, $n = 1, 2, 3, \cdots$. Replying to my query as to possible properties of an arbitrary infinite field of modulus p, he indicated the existence of fields other than C—citing the example C^* defined as the aggregate of all $GF[p^n]$, $n = m_1, m_2, \cdots$, where m_1 divides m_2, m_2 divides $m_3, \cdots$. I was later led to consider the field $C^{(p^n)}$ of all

* My original paper appeared in the *American Journal of Mathematics*, vol. 21 (1899), pp. 193–256.

† In a letter to me dated February 21, 1904, M. Jordan says: "J'avais préparé il y a trois ou quatre ans un travail sur les groupes linéaires à invariant quadratique et je me suis aperçu après coup que tous mes résultats se trouvaient déjà exposés dans votre bel ouvrage sur les groupes linéaires, de sorte que je n'ai eu qu'à jeter au feu tout ce que j'avais écrit."

rational functions of an arbitrary variable with coefficients in the $GF[p^n]$ and analogous infinite fields whose elements are rational functions in the $GF[p^n]$ of any number of independent variables. These types of fields have very different properties. Thus C has the very interesting property, noted by Moore, that in it every equation is completely solvable.

For two isomorphic finite groups A and B, the set A_0 of the elements of A which correspond to the identity of B form a group. For infinite groups the result is different and the author is in error. The converse statement (page 66, line 9) cannot be proved. I have constructed examples in which neither A_0 nor B_0 is a group. The correct theorem is that A_0 and B_0 are semi-groups. If either is a group the other is also, so that, for the ordinary case in which $B_0 = 1$, A_0 is a group. I will discuss this fundamental question in detail in the *Transactions*.

In the one hundred pages devoted to finite groups in the usual sense, the author has given the statement of theorems and their proofs in unusually condensed form. This is due partly to the use of Frobenius' notations throughout, partly to the use of symbols such as A_e or $A_{(e)}$ for an operator of period equal or a divisor of e respectively, and partly to the use of numerous new terms such as *central* of G for the subgroup of all the invariant operators of G. We find the new terms principal group, dicyclic, commutant, normalisant, rank, special, figure, as well as new or alternative designations in place of those in current use. It is to be hoped that a complete index will appear in the second volume.

Note I gives Jordan's work on groups of movements, now of practical interest in crystallography. Note II considers matrices, Elementartheiler, systems of linear equations and congruences.

The only new errata noted were b for b_1 on page 13, line 12; $\leqq$ for $\geqq$ on page 51, line 17. A phrase on page 62 might mislead students of Lie's theory, since "groupe continu de points $(x_1, \cdots, x_n)$" does not refer to a group of point transformations in n variables, but refers to ∞^n operators in Lie's symbolic sense.

The reader of the present volume will be impressed with the author's complete mastery of his subject and will find in it a useful compact summary of the results to date in the purely abstract part of finite group theory.

L. E. DICKSON.