THE PRESENT PROBLEMS OF GEOMETRY.

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IN spite of the richness and power of recent geometry, it is noticeable that the geometer himself has become more modest. It was the ambition of Descartes and Leibniz to discover universal methods, applicable to all conceivable questions; later, the Ausdehnungslehre of Grassmann and the quaternion theory of Hamilton were believed by their devotees to be ultimate geometric analyses; and Chasles attributed to the principles of duality and homography the same rôle in the domain of pure space as that of the law of gravitation in celestial mechanics. To-day the mathematician admits the existence and the necessity of many theories, many geometries, each appealing to certain interests, each to be developed by the most appropriate methods; and he realizes that, no matter how large his conceptions and how powerful his methods, they will be replaced before long by others larger and more powerful.

Aside from the conceivability of other spaces with just as self-consistent properties as those of the so-called ordinary space, such diverse theories arise, in the first place, on account of the variety of objects demanding consideration — curves, surfaces, congruences and complexes, correspondences, fields of differential elements, and so on in endless profusion. The totality of configurations is indeed not thinkable in the sense of an ordinary assemblage, since the totality itself would have to be admitted as a configuration, that is, an element of the assemblage.

However, more essential in most respects than the diversity in the material treated, is the diversity in the points of view from which it may be regarded. Even the simplest figure, a triangle or a circle, has an infinity of properties — indeed, recalling the unity of the physical world, the complete study of a single figure would involve its relations to all other figures and thus not be distinguishable from the whole of geometry. For the past three decades the ruling thought in this connection has been the principle (associated with the names of Klein and Lie) that the properties which are deemed of interest in the various geometric theories may be classified according to the group of transformations which leave those properties unchanged. Thus almost all discussions on algebraic curves are connected with the group of displacements (more properly the so-called principal group), or the group of projective transformations, or the group of birational transformations; and the distinction between such theories is more fundamental than the distinction between the theories of curves, of surfaces, and of complexes.

Historically, the advance has been, in general, from small to larger groups of transformations. The change thus produced may be likened to the varying appearance of a painting, at first viewed closely in all its details, then at a distance in its significant features. The analogy also suggests the desirability of viewing an object from several standpoints, of studying geometric configurations with respect to various groups. It is indeed true, though in a necessarily somewhat vague sense, that the more essential properties are those invariant under the more extensive groups; and it is to be expected that such groups will play a predominating rôle in the not far distant future.

The domain of geometry occupies a position, as indicated in the programme of the Congress, intermediate between the domain of analysis on the one hand and of mathematical physics on the other; but in its development it continually encroaches upon these adjacent fields. The concepts of transformation and invariant, the algebraic curve, the space of n dimensions, owe their origin primarily to the suggestions of analysis, while the null-system, the theory of vector fields, the questions connected with the applicability and deformation of surfaces, have their source in mechanics. It is true that some mathematicians regard the discussion of point sets, for example, as belonging exclusively to the theory of functions, and others look upon the composition of displacements as a part of mechanics. While such considerations show the difficulty, if not impossibility, of drawing strict limits about any science, it is to be observed that the consequent lack of definiteness, deplored though it be by the formalist, is more than compensated by the fact that such overlapping is actually the principal means by which the different realms of knowledge are bound together.

If a mathematician of the past, an Archimedes or even a

Descartes, could view the field of geometry in its present condition, the first feature to impress him would be its lack of concreteness. There are whole classes of geometric theories which proceed, not merely without models and diagrams, but without the slightest (apparent) use of the spatial intuition. In the main this is due, of course, to the power of the analytic instruments of investigation as compared with the purely geometric. The formulas move in advance of thought, while the intuition often lags behind; in the oft-quoted words of d'Alembert, "L'algèbre est généreuse, elle donne souvent plus qu'on lui demande." As the field of research widens, as we proceed from the simple and definite to the more refined and general, we naturally cease to picture our processes and even our results. It is often necessary to close our eyes and go forward blindly if we wish to advance at all. But admitting the inevitableness of such a change in the spirit of any science, one may still question the attitude of the geometer who rests content with his blindness, who does not at least strive to intensify and enlarge the intuition. Has not such an intensification and enlargement been the main contribution of geometry to the race, its very raison d'être as a separate part of mathematics, and is there any ground for regarding this service as completed?

From the point of view here referred to, a problem is not to be regarded as completely solved until we are in position to construct a model of the solution, or at least to conceive of such a construction. This requires the interpretation, not merely of the results of a geometric investigation, but also, as far as possible, of the intermediate processes — an attitude illustrated most strikingly in the works of Lie. This duty of the geometer, to make the ground won by means of analysis really geometric, and as far as possible concretely intuitive, is the source of many problems of to-day, a few of which will be referred to in the course of this address.

The tendency to generalization, so characteristic of modern geometry, is counteracted in many cases by this desire for the concrete, in others by the desire for the exact, the rigorous (not to be confused with the rigid). The great mathematicians have acted on the principle "Devinez avant de démontrer," and it is certainly true that almost all important discoveries are made in this fashion. But while the demonstration comes after the discovery, it cannot therefore be disregarded. The spirit of rigor, which tended at first to the arithmetization of all mathematics and now tends to its exhibition in terms of pure logic, has always been more prominent in analysis than in geometry. Absolute rigor may be unattainable, but it cannot be denied that much remains to be done by the geometers, judging even by elementary standards. We need refer only to the loose proofs based uopn the invaluable, but insufficient, enumeration of constants, the so-called principle of the conservation of number, and the discussions which confine themselves to the "general case." Examples abound in every field of geometry. The theorem announced by Chasles concerning the number of conics satisfying five arbitrary conditions was proved by such masters as Clebsch and Halphen before examples invalidating the result were devised. Picard recently called attention to the need of a new proof of Noether's theorem that upon the general algebraic surface of degree greater than three, every algebraic curve is a complete intersection with another algebraic surface. The considerations given by Noether render the result highly probable, but do not constitute a complete proof; while the exact meaning of the term general can be determined only from the context.

The reaction against such loose methods is represented by Study * in algebraic geometry, and Hilbert in differential geometry. The tendency of a considerable portion of recent work is towards the exhaustive treatment of definite questions, including the consideration of the special or degenerate cases ordinarily passed over as unimportant. Another aspect of the same tendency is the discussion of converses of familiar problems, with the object of obtaining conditions at once necessary and sufficient, that is, completely characteristic results. †

Another set of problems is suggested by the relation of geometry to physics. It is the duty of the geometer to abstract from the physical sciences those domains which may be expressed in terms of pure space, to study the geometric founda-

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* "[Es ist eine] tief eingewurzelte Gewohnheit vieler Geometer, Sätze zu formulieren, die 'im allgemeinen' gelten sollen, d. h. einen klaren Sinn *überhaupt nicht* haben, zudem noch häufig als *allgemein* gültig hingestellt oder mangelhaft begründet werden. [Dies Verfahren wird], trotz etwanigen Verweisungen auf Träger sehr berühmter Namen, späteren Geschlechtern sicher als ganz unzulässig erscheinen, scheint aber in unserem 'kritischen' Zeitalter von vielen als eine *berechtigte Eigentümlichkeit der Geometrie* betrachtet zu werden . . ." *Jahr. Deut. Math.-Ver.*, vol. 11 (1902), p. 100.

† As an example may be mentioned the theorem of Malus and Dupin, known for almost a century, that the rays emanating from a point are converted, by any refraction, into a normal congruence. Quite recently, Levi-Civita succeeded in showing that this property is characteristic; that is, any normal congruence may be refracted into a bundle.

tions (or, as some would put it, the skeletons) of the various branches of mechanics and physics. Most of the actual advance, it is true, has hitherto come from the physicists themselves, but undoubtedly the time has arrived for more systematic discussions by the mathematicians. In addition to the importance which is due to possible applications of such work, it is to be noticed that we meet, in this way, configurations as interesting and remarkable as those created by the geometer's imagination. Even in this field, one is tempted to remark, truth is stranger than fiction.

We have now considered, briefly and inadequately, some of the leading ideals and influences which are at work towards both the widening and the deepening of geometry in general; and turn to our proper topic, a survey of the leading problems or groups of problems in certain selected (but it is hoped representative) fields of contemporaneous investigation.

Foundations.

The most striking development of geometry during the past decade relates to the critical revision of its foundations, more precisely, its *logical* foundations. There are, of course, other points of view, for example, the physical, the physiological, the psychological, the metaphysical, but the interest of mathematicians has been confined to the purely logical aspect. The main results in this direction are due to Peano and his coworkers; but the whole field was first brought prominently to the attention of the mathematical world by the appearance, five years ago, of Hilbert's elegant Festschrift.

The central problem is to lay down a system of primitive (undefined) concepts or symbols and primitive (unproved) propositions or postulates, from which the whole body of geometry (that is, the geometry considered) shall follow by purely deductive processes. No appeal to intuition is then necessary. "We might put the axioms into a reasoning apparatus like the logical machine of Stanley Jevons, and see all geometry come out of it" (Poincaré). Such a system of concepts and postulates may be obtained in a great (probably endless) variety of ways: the main question, at present, concerns the comparison of various systems, and the possibility of imposing limitations so as to obtain a unique and perhaps simplest basis.

The first requirement of a system is that it shall be consistent. The postulates must be compatible with one another. No one has yet deduced contradictory results from the axioms of Euclid, but what is our guarantee that this will not happen in the future? The only method of answering this question which has suggested itself is the exhibition of some object (whose existence is admitted) which fulfills the conditions imposed by the postulates. Hilbert succeeded in constructing such an ideal object out of numbers; but remarks that the difficulty is merely transferred to the field of arithmetic. The most far-reaching result is the definition of number in terms of logical classes as given by Pieri and Russell; but no general agreement is yet to be expected in these discussions. Will the ultimate conclusion be the impossibility of a direct proof of compatibility?

More accessible is the question concerning the independence of postulates (and the analogous question of the irreducibility of concepts). Most of the work of the last few years has been concentrated on this point. In Hilbert's original system the various groups of axioms (relating respectively to combination, order, parallels, congruence and continuity) are shown to be independent, but the discussion is not carried out completely for the individual axioms. In Dr. Veblen's recently published system of twelve postulates, each is proved independent of the remaining eleven.* This marks an advance, but, of course, it does not terminate the problem. In what respect does a group of propositions differ from what is termed a single proposition? Is it possible to define the notion of an absolutely simple postulate? The statement that any two points determine a straight line involves an infinity of statements, and its fulfillment for certain pairs of points may necessitate its fulfillment for all pairs. If in Euclid's system the postulate of parallels is replaced by the postulate concerning the sum of the angles of a triangle, a well known example of such a reduction is obtained; for it is sufficient to assume the new postulate for a single triangle, the general result being then deducible. As other examples we may mention Peano's reduction of the euclidean definition of the plane; and the definition of a collineation which demands, instead of the conversion of *all* straight lines into straight lines, the existence of four simply infinite systems of such straight lines.†

* *Trans. Amer. Math. Soc.*, vol. 5 (1904).

† Together with certain continuity assumptions. Cf. BULLETIN, vol. 9 (1903), p. 545.

These examples illustrate the difficulty, if not the impossibility, of formulating a really fundamental, that is absolute, standard of independence and irreducibility. It is probable that the guiding ideas will be obtained in the discussion of simpler deductive theories, in particular, the systems for numbers and groups.

Two features are especially prominent in the actual development of the body of geometry from its fundamental system. First, the consideration of what may be termed the collateral geometries, which arise by replacing one of the original postulates by its opposite, or otherwise varying the system. Such theories serve to show the limitation of that point of view which restricts the term general geometry (pangeometry) to the euclidean and non-euclidean geometries. The variety of possible abstract geometries is, of course, inexhaustible; this is the central fact brought to light by the exhibition of such systems as the non-archimedean and the non-arguesian. In the second place, much valuable work is being done in discussing the various methods by which the same theorem may be deduced from the postulates, the ideal being to use as few of the postulates as possible. Here again the question of *simplicity* (simplest proof), though it baffles analysis, forces itself upon the attention.

Among the minor problems in this field, it is sufficient to consider that concerning the relation of the theory of volume to the axiom of continuity. This axiom need not be used in establishing the theory of areas of polygons; but after Dehn and others had proved the existence of polyhedra having the same volume though not decomposable into mutually congruent parts (even after the addition of congruent polyhedra), it was stated by Hilbert, and deemed evident generally, that reference to continuity could not be avoided in three dimensions. In a recent announcement* of Vahlen's forthcoming Abstrakte Geometrie this conclusion is declared unsound. It seems probable, however, that the difference is merely one concerning the interpretation to be given to the term continuity.

The work on logical foundations has been confined almost entirely to the euclidean and projective geometries. It is desirable, however, that other geometric theories should be treated in a similar deductive fashion. In particular, it is to be hoped that we shall soon have a really systematic foundation for the

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* *Jahr. Deut. Math. Ver.*, vol. 13 (1904), p. 395.

so-called inversion geometry, dealing with properties invariant under circular transformations. This theory is of interest, not only for its own sake and for its applications in function theory, but also because its study serves to free the mind from what is apt to become, without some check, slavery to the projective point of view.

The Curve Concept — Analysis Situs.

Although curves and surfaces have constituted the almost exclusive material of the geometric investigation of the thirty centuries of which we have record, it can hardly be claimed that the concepts themselves have received their final analysis. Certain vague notions are suggested by the naive intuition. It is the duty of mathematicians to create perfectly precise concepts which agree more or less closely with such intuitions, and at the same time, by the reaction of the concepts, to refine the intuition. The problem, evidently, is not at all determinate. It would be of interest to trace the evolution which has actually produced several distinct curve concepts defining more or less extensive classes of curves, agreeing in little beyond the possession of an infinite number of points.

The more familiar special concepts or classes of curves are defined in terms of the corresponding equation $y = f(x)$ or function $f(x)$. Such are, for example: (1) algebraic curves, (2) analytic curves, (3) graphs of functions possessing derivatives of all orders, (4) the curves considered in the usual discussions of infinitesimal geometry, in which the existence of first and second derivatives is assumed, (5) the so-called regular curves with a continuously turning tangent (except for a finite number of corners), (6) the so-called ordinary curves possessing a tangent and having only a finite number of oscillations (maxima and minima) in any finite interval, (7) curves with tangents, (8) the graphs of continuous functions.

How far are such distinctions accessible to the intuition? Of course there are limitations. For over two centuries, from Descartes to the publication of Weierstrass's classic example, the intuition of mathematicians declared the classes (7) and (8) to be identical. Still later it was found that such extraordinary (pathological or crinkly) curves may present themselves in class (7). However, even here partially successful attempts to connect with intuition have been made by Wiener, Hilbert, Schoenflies, Moore, and others.

Let us consider a simpler extension in the field of ordinary curves. If the function $f(x)$ is continuous except for a certain value of x where there is an ordinary discontinuity, this is indicated by a break in the graph; if f is continuous, but the derivative f' has such a discontinuity, this shows itself by a sharp turn in the curve; if the discontinuity is only in the second derivative, there is a sudden change in the radius of curvature, which is, however, relatively difficult to observe from the figure; finally, if the third derivative is discontinuous, the effect upon the curve is no longer apparent. Does this mean that it is impossible to picture it? Does it not rather indicate a limitation in the usual geometric training which goes only as far as relations expressible in terms of tangency and curvature? For the interpretation of the third derivative it is necessary to consider say the pair of osculating parabolas at each point of a curve: in the case referred to, as we pass over the critical point, the tangent line and osculating circle change continuously, but there is a sudden change in the osculating parabolas. If in fact our intuition were trained to picture osculating algebraic curves of all orders, it would detect a discontinuity in a derivative of any order. A partial equivalent would be the ability to picture the successive evolutes of a given curve; a complete equivalent would be the picturing of the successive slope curves $y = f'(x)$, $y = f''(x)$, etc. All this requires, evidently, only an increase in the intensity of our intuition, not a change in its nature.

This however would not apply to all questions. There are functions which while possessing derivatives of all orders (then necessarily continuous) are not analytic (*i. e.*, not expressible by power series). What is it that distinguishes the analytic curves among this larger class? Is it possible to put the distinction in a form capable of assimilation by an idealized intuition? In short, what is the really geometric definition of an analytic curve?*

Much recent work in function theory has had for its point of departure a more general basis than the theory of curves, namely, the theory of sets or assemblages of points, with special reference to the notions of derived set and the various contents or areas. The geometry of point sets must indeed be regarded

* One method of attack would be the interpretation of Pringsheim's conditions; this requires not merely the curves $y = f^{(n)}(x)$, but the limit of the system.

as one of the most important and promising in the whole field of mathematics. It receives its distinctive character, as compared with the general abstract theory of assemblages (Mengenlehre), from the fact that it operates not with all one-to-one correspondences, but with the group of analysis situs, the group of *continuous* one-to-one correspondences. From the point of view of the larger group, there is no distinction between a one-dimensional and a two- or many-dimensional continuum (Cantor). This is still the case if the correspondence is continuous but not one-to-one (Peano, 1890). In the domain of continuous one-to-one correspondence, however, spaces of different dimensions are not equivalent (Jürgens, 1899).

An important class of curves, much more general than those referred to above, consists of those point sets which are equivalent (in the sense of analysis situs) to the straight line or segment of a straight line. This is Hurwitz's simple and elegant geometric formulation of the concept originally treated analytically by Jordan, the most fundamental curve concept of today. The closed Jordan curves are defined in analogous fashion as equivalent to the perimeter of a square (or the circumference of a circle).

A curve of this kind divides the remaining points of the plane into two simply connected continua, an inside and an outside. The necessity for proof of this seemingly obvious result is seen from the fact that the Jordan class includes such extraordinary types as the curve with positive content constructed recently by Osgood.* Such a separation of the plane may, however, be brought about by other than Jordan curves: the concept of the boundary of a connected region gives perhaps the most extensive class of point sets which deserve to be called curve. Schoenflies proposes a definition for the idea of a simple closed curve which makes it appear as the natural extension, in a certain sense, of the polygon: a perfect set of points P which separates the plane into an exterior region E and an interior region I such that any E point can be connected with any I point by a path (Polygonstrecke) having only one point in common with P. This is in effect a converse of Jordan's theorem, and shows precisely how the Jordan curve is distinguished from other types of boundaries of connected regions.

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* *Trans. Amer. Math. Soc.*, vol. 4 (1903), p. 107.

These discussions are mentioned here simply as aspects of a really fundamental problem: the revision of the concepts and results of that division of geometry which has been variously termed analysis situs, theory of connection, topology, geometry of situation — a revision to be carried out in the light of the theory of assemblages.*

Algebraic Surfaces and Birational Transformations.

After the demonstration of the power of the methods based upon projective transformation — the chief contribution due to the geometers of the first half of the nineteenth century — attempts were made to introduce other types of one-to-one correspondence or transformation into algebraic geometry; in particular the inversion of William Thomson and Liouville, and the quadratic transformation of Magnus. The general theory of such Cremona transformations was inaugurated by the Italian geometer in his memoir "Sulle trasformazioni geometriche delle figure piane," published in 1863. Within a few years, Clifford, Noether, and Rosanes, working independently, established the remarkable result that every Cremona transformation in a plane can be decomposed into a succession of quadratic transformations, thus bringing to light the fact that there are at bottom only two types of algebraic one-to-one correspondence, the homographic and the quadratic.†

The development of a corresponding theory in space has been one of the chief aims of the geometers of Italy, Germany and England for the last thirty years, but the essential question of decomposition still remains unanswered. Is it possible to reduce the general Cremona transformation of space to a finite number of fundamental types?

In its application to the study of the properties of algebraic curves and surfaces, the theory of the Cremona transformation is usually merged in the more general theory of the birational transformation. By means of the latter, a correspondence is established which is one-to-one for the points of the particular figure considered and the transformed figure, but not for all the points of space. In the plane theory an important result is that a curve with the most complicated singularities can, by

* Cf. Schoenflies, *Math. Annalen*, vols. 58, 59 (1903, 1904).

† Segre recently called attention to a case where the usual methods of discussion fail to apply; the proof has been completed by Castelnuovo. Cf. *Atti di Torino*, vol. 36 (1901).

means of Cremona transformations, be converted into a curve whose only singularities are multiple points with distinct tangents (Noether); furthermore, by means of birational transformations, the singularities may be reduced to the very simplest type, ordinary double points (Bertini). The known theory of space curves is also, in this aspect, quite complete. The analogous problem of the reduction of higher singularities of a surface has been considered by Noether, Del Pezzo, Segre, Kobb, and others, but no ultimate conclusion has yet been obtained.

One principal source of difficulty is that, while in case of two birationally equivalent curves the correspondence is one-to-one without exception, on the other hand, in the case of two surfaces, there may be isolated points which correspond to curves, and just such irregular phenomena escape the ordinary methods. Again, not only singular points require consideration, as is the case in the plane theory, but also singular lines, and the points may be isolated or superimposed on the lines. Most success is to be expected from further application of the method of projection from a higher space due to Clifford and Veronese. In this direction the most important result hitherto obtained is the theorem, of Picard and Simart, that any algebraic surface (in ordinary space) can be regarded as the projection of a surface free from singularities situated in five-dimensional space.

A question which awaits solution even in the case of the plane is that relating to the invariants of the group of Cremona transformations proper. The genus and the moduli of a curve are unaltered by all birational transformations, but the problem arises: Are there properties of curves which remain unchanged by Cremona, although not by other birational transformations? From the fact that birationally equivalent curves need not be equivalent under the Cremona group, it would seem that such invariants — Cremona invariants proper — do exist, but no actual examples have yet been obtained. The problem may be restated in the form: What are the necessary and sufficient conditions which must be fulfilled by two curves if they are to be equivalent with respect to Cremona transformations? Equality of genera and moduli, as already remarked, is necessary but not sufficient.

The invariant theory of birational transformations has for its principal object the study of the linear systems of point groups on a given algebraic curve, that is, the point groups cut out by linear systems of curves. Its foundations were implicitly laid by Riemann in his discussion of the equivalent theory of algebraic functions on a Riemann surface, though the actual application to curves is due to Clebsch. Most of the later work has proceeded along the algebraic-geometric lines developed by Brill and Noether, the promising purely geometric treatment inaugurated by Segre being rather neglected.

The extension of this type of geometry to space, that is, the development of a systematic geometry on a fundamental algebraic surface (especially as regards the linear systems of curves situated thereon), is one of the main tasks of recent mathematics. The geometric treatment is given in the memoirs of Enriques and Castelnuovo, while the corresponding functional aspect is the subject of the treatise of Picard and Simart on algebraic functions of two variables, at present in course of publication.

The most interesting feature of the investigations belonging in this field is the often unexpected light which they throw on the interrelations of distinct fields of mathematics, and the advantage derived from such relations. For example, Picard (as he himself relates on presenting the second volume of his treatise to the Paris Academy a few months ago) * for a long time was unable to prove directly that the integrals of algebraic total differentials can be reduced, in general, to algebraic-logarithmic combinations, until finally a method for deciding the matter was suggested by a theorem on surfaces which Noether had stated some twenty years earlier. Again, in the enumeration of the double integrals of the second species, Picard arrived at a certain result, which was soon noticed to be essentially equivalent to one obtained by Castelnuovo in his investigations on linear systems; and thus there was established a connection between the so-called numerical and linear genera of a surface, and the number of distinct double integrals.†

A closely related set of investigations, originating with Clebsch's theorems on intersections and Liouville's on confocal quadrics, may be termed the "geometry of Abel's theorem." As later applications we can merely mention Humbert's memoirs on certain metric properties of curves, and Lie's determination of surfaces of translation.

Investigations in analysis have often suggested the introduc-

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* *Comptes Rendus*, Feb. 1, 1904.

† *Ibid.*, Feb. 22, 1904.

tion of new types of configurations into geometry. The field of algebraic surfaces is especially fruitful in this respect. Thus, while in the case of curves (excluding the rational) there always exist integrals everywhere finite, this holds for only a restricted class of surfaces; their determination depends on the solution of a partial differential equation which has been discussed in a few special cases.

In addition to such relations between analysis and geometry, important relations arise between various fields of geometry. Just as an algebraic function of one variable is pictured by either a plane curve or a Riemann surface (according as the independent and dependent variables are taken to be real or complex), so an algebraic function of two independent variables may be represented by either a surface in ordinary space or a Riemannian four-dimensional manifold in space of five dimensions. In the case of one variable, the single invariant number (deficiency or genus p) which arises is capable of definition in terms of the characteristics of the curve or the connectivity of the Riemann surface. In passing to two variables, however, it is necessary to consider several arithmetical invariants — just how many is an unsettled question. For the algebraic surface we have, for instance, the geometric genus of Clebsch, the numerical genus of Cayley, and the so-called second genus, each of which may be regarded as a generalization, from a certain point of view, of the single genus of a curve; all are invariant with respect to birational transformation.

The other geometric interpretation, by means of a Riemannian manifold, has rendered necessary the study of the analysis situs of higher spaces. The connection of such a manifold is no longer expressed by a single number as in the case of an ordinary surface, but by a set of two or more, the so-called numbers of Betti and Riemann. The detailed theory of these connectivities, difficult and delicate because it must be derived with little aid from the intuition, has been made the subject of an extensive series of memoirs by Poincaré.

From the point of view of analysis, the chief interest in these investigations is the fact that the connectivities are related to the number of integrals of certain types. The chief problem for the geometer, however, is the discovery of the precise relations between the connectivities of the Riemann manifold and the various genera of the algebraic surface. That relations do exist between such diverse geometries — the one operating with all *continuous*, the other with the *algebraic*, one-to-one correspondence — is one of the most striking results of recent mathematics.

Geometry of Multiple Forms.

For some time after its origin, the linear invariant theory of Boole, Cayley, and Sylvester confined itself to forms containing a single set of variables. The needs of both analysis and geometry, however, have emphasized the importance and the necessity of further development of the theory of forms containing two or more sets of variables (of the same or different type), so-called multiple forms.

In the plane we have both point coordinates (x) and line coordinates (u). A form in x corresponds to a point curve (locus), a form in u to a line curve (envelope), and a form involving both x and u to a connex. The latter was introduced into geometry, some thirty years ago, by Clebsch, the suggestion coming from the fact that, even in the study of a simple form in x, covariants in x and u present themselves, so that it seemed desirable to deal with such forms ab initio.

Passing to space, we meet three simple elements, the point (x), the plane (u), and the line (p). Forms in a single set of variables represent, respectively, a surface as point locus, a surface as plane envelope, and a complex of lines. The compound elements composed of two simple elements are the point-plane, the point-line, and the plane-line. The first type, leading to point-plane connexes, has been studied extensively during the past few years; the second to a more limited degree; the third is merely the dual of the second. To complete the series, the case of the point-line-plane as element, or forms involving x, u and p, requires investigation.

In the corresponding n-dimensional theory it is necessary to take account of n simple elements and the various compound elements formed by their combinations.

The importance of such work is twofold: First, on account of connection with the algebra of invariants. A fundamental theorem of Clebsch states that, in the investigation of complete systems of comitants, it is sufficient to consider forms involving not more than one set of variables of each type: if in the given forms the types are involved in any manner, it is possible to find an equivalent reduced system of the kind described. On the other hand, it is impossible to further reduce the system, so that the introduction of the n types of variables is necessary for the algebraically complete discussion. Geometry must accordingly extend itself to accommodate the configurations defined by the new elements.

Second, on account of connection with the theory of differential equations. The ordinary plane connex in x, u assigns to each point of the plane a certain number of directions (represented by the tangents drawn to the corresponding curve), and thus gives rise to an (algebraic) differential equation of the first order in two variables; the point-plane connex in space, associating with each point a single infinity of incident planes, defines a partial differential equation of the first order; the point-line connex yields a Monge equation. The point-line-plane case has not yet been interpreted from this point of view.

One special problem in this field deserves mention on account of its many applications. This is the study of the system composed of a quadric form in any number of variables and a bilinear form in contragredient variables, that is, a quadric manifold and an arbitrary (not merely automorphic) collineation in n-space. For $n = 6$, for example, this corresponds to the general linear transformation of line or sphere coordinates.

In addition to forms containing variables of different types, the forms involving several sets of variables of the same type require consideration. Forms in two sets of line coordinates present themselves in connection with the pfaffian problem of differential systems. The main interest attaches, however, to forms in sets of point coordinates, since it is these which occur in the theory of contact transformations and of multiple correspondences. For example, while the ordinary homography in a line is represented by a bilinear form in binary variables the trilinear form in similar variables gives rise to a new geometric variety, the so-called homography of the second class (associating with any two points a unique third point), which has applications to the generation of cubic surfaces and to the constructions at the basis of photogrammetry. The theory of multilinear forms in general deserves more attention than it has yet received.

Other important problems, connected with the geometric phases of linear invariant theory, can merely be mentioned: (1) The general geometric interpretation of what appears algebraically as the simplest projective relation, namely, apolarity. (2) The invariant discussion of the simpler discontinuous varieties, for example, the polygon considered as n-point or as n-line.* (3) The establishment of a system of forms corresponding to the general space curve. (4) The study of the properties and the groups of the configurations corresponding in hyperspace to the simpler systems of invariants. (5) Complete systems of orthogonal or metric invariants for the simpler curves.†

Transcendental Curves.

To reduce to systematic order the chaos of non-algebraic curves has been the aspiration of many a mathematician; but, despite all efforts, we have no theory comparable with that of algebraic curves. The very vagueness and apparent hopelessness of the question is apt to repel the modern mathematician, to cause him to return to the more familiar field. The resulting concentration has led to the powerful methods, already referred to, for studying algebraic varieties. In the transcendental domain, on the other hand, we have a multitude of interesting but particular geometric forms, — some suggested by mechanics and physics, others derived from their relation to algebraic curves, or by the interpretation of analytic results — a few thousands of which have been considered of sufficient importance to deserve specific names.‡ The problem at issue is then a practical one (comparable with corresponding discussions in natural history): to formulate a principle of classification which will apply, not to all possible curves, but to as many as possible of the usual important transcendental curves.

The most fruitful suggestion hitherto applied has come from the consideration of differential equations: almost all the important transcendental curves satisfy algebraic differential equations, and these in the great majority of cases are of the first order. Hence the need of a systematic discussion of the curves defined by any equation $F(x, y, y') = 0$, the so-called *panalgebraic* curves of Loria. If F is of degree n in y' and of degree ν in x, y, the curve is said to belong to a system with

* Cf. F. Morley "On the geometry whose element is the 3-point of a plane," *Trans. Amer. Math. Soc.*, vol. 5 (1904). E. Study in his Geometrie der Dynamen develops a new foundation for kinematics by employing as element the Soma or trirectangular trihedron.

† Here would belong in particular the theory of algebraic curves based on linkages. Little advance has been made beyond the existence theorems of Kempe and Koenigs. An important unsolved problem is the determination of the linkage with minimum number of pieces by which a given curve can be described.

‡ Cf. Loria, Spezielle Kurven, Leipzig, 1902.

the characteristics (n, ν), and we thus have an important basis for classification. Closely related is the theory of the Clebsch connex; this figure, it is true, is considered as belonging to algebraic geometry, but it defines (by means of its principal coincidence) a system of usually transcendental panalgebraic curves.

Both points of view appear to characterize certain systems of curves rather than individual curves. The following interpretation may serve as a simple geometric definition of the curves considered.

With any plane curve C we may associate a space curve in this way: at each point of C erect a perpendicular to the plane whose length represents the slope of the curve at that point; the locus of the end points of these perpendiculars is the associated space curve C'. Not every space curve is obtained in this way, but only those whose tangents belong to a certain linear complex. If C is algebraic, so is C', and then an infinite number of algebraic surfaces may be passed through the latter. If C is transcendental, so is C', and usually no algebraic surface can be passed through it. Sometimes, however, one such algebraic surface F exists. (If there were two, C' and C would be algebraic). It is precisely in this case that the curve C is panalgebraic in the sense of Loria's theory. That such a curve belongs to a definite system is seen from the fact that while the surface F is unique, it contains a singly infinite number of curves whose tangents belong to the linear complex mentioned, and the orthogonal projections of these curves constitute the required system.

The principal problems in this field which require treatment are: first, the exhaustive discussion of the simplest systems, corresponding to small values of the characteristics n and ν; second, the study of the general case in connection with (1) algebraic differential equations, (2) connexes, and (3) algebraic surfaces and linear complexes.

Natural or Intrinsic Geometry.

In spite of the immediate triumph of the cartesian system at the time of its introduction into mathematics, rebellion against what may be termed the tyranny of extraneous coördinates, first expressed in the Characteristica geometrica of Leibniz, has been an ever present though often subdued influence in the development of geometry. Why should the properties of a curve be expressed in terms of x's and y's which are defined not by the curve itself, but by its relation to certain arbitrary elements of reference? The same curve in different positions may have unlike equations, so that it is not a simple matter to decide whether given equations represent really distinct or merely congruent curves. The idea of the so-called natural or intrinsic coordinates had its birth during the early years of the nineteenth century, but it is only the systematic treatment of recent years which has created a new field of geometry.

For a plane curve there is at each point the arc s measured from some fixed point on the curve, and the radius of curvature ρ; these intrinsic coordinates are connected by a relation $\rho = f(s)$ which is precisely characteristic of the curve, that is, the curves corresponding to the equation differ only in position. There is, however, still something arbitrary in the point taken as origin. This is eliminated by taking as coordinates ρ and its derivative δ taken with respect to the arc; so that the final intrinsic equation is of the form $\delta = F(\rho)$. There is no difficulty in extending the method to space curves. The two natural equations necessary are here $\tau = \phi(\rho)$, $\delta = \psi(\rho)$, where ρ and τ are the radii of first and second curvature and δ is the arc derivative of ρ.

The application to surfaces is not so evident. Thus, in Cesàro's standard work, while the discussion of curves is consistently intrinsic, this is true to only a slight extent in the treatment of surfaces. The natural geometry of surfaces is in fact only in process of formation. Bianchi proposes as intrinsic the familiar representation by means of the two fundamental quadratic differential forms; but, although it is true that the surfaces corresponding to a given pair of forms are necessarily congruent, there is the disadvantage, arising from the presence of arbitrary parameters, that the same surface may be represented by distinct pairs of forms. One way of overcoming this difficulty is to introduce the common feature of all pairs corresponding to a surface, *i. e.*, the invariants of the forms: in this direction we may cite Ricci's principle of covariant differentiation and Maschke's recent application of symbolic methods.

The basis of natural geometry is, essentially, the theory of differential invariants. Under the group of motions, a given configuration assumes ∞^r positions, where r is in general six but may be smaller in certain cases. The r parameters which thus enter in the analytic representation may be eliminated by the formation of differential equations. The aim of natural geometry is to express these differential equations in terms of the simplest geometric elements of the given configuration.

The beginning of such a discussion of surfaces was given by Sophus Lie in 1896 and his work has been somewhat simplified by Scheffers. As natural coordinates we may take the principal radii of curvature R_1, R_2 at a point of the surface, and their derivatives

$$\delta_{11} = \frac{dR_1}{ds_1}, \quad \delta_{12} = \frac{dR_1}{ds_2}, \quad \delta_{21} = \frac{dR_2}{ds_1}, \quad \delta_{22} = \frac{dR_2}{ds_2},$$

taken in the principal directions. For a given surface (excluding the Weingarten class) the radii are independent, and there are four relations of the form

$$\delta_{11} = f_{11}(R_1, R_2), \quad \delta_{12} = f_{12}(R_1, R_2), \quad \delta_{21} = f_{21}(R_1, R_2),$$
$$\delta_{22} = f_{22}(R_1, R_2).$$

Conversely, these equations are not satisfied by any surfaces except those congruent or symmetric to the given surface.

It is to be noticed that four equations thus appear to be necessary to define a surface, although two are sufficient for a twisted curve. If a single equation in the above mentioned natural coordinates is considered, it is not, as in the case of ordinary coordinates, characteristic: surfaces not congruent or symmetric to the given surface would satisfy the equation. The apparent inconsistency which arises is removed, however, by the fact that the four natural equations are dependent.* It is just this that makes the subject difficult as compared with the theory of curves, in which the defining equations are entirely arbitrary. The questions demanding treatment fall under these two headings: first, the derivation of the natural equations of the familiar types of surfaces, and second, the study of the new types that correspond to equations of simple form. The natural geometry of the Weingarten class of surfaces requires a distinct basis.

The fact that intrinsic coordinates are, at bottom, differential invariants with respect to the group of motions, suggests the extension of the same idea to the other groups. Thus in the

* The three relations connecting the functions f_{11}, f_{12}, f_{21}, f_{22} have been worked out recently by S. Heller, *Math. Annalen*, vol. 58 (1904).

projective geometry of arbitrary (algebraic or transcendental) curves, coordinates are required which, unlike the distances and angles ordinarily used, are invariant under projection. These might, for example, be introduced as follows. At each point of the general curve C, there is a unique osculating cubic and a unique osculating W (self-projective) curve. Connected with each of these osculating curves is an absolute projective invariant defined as an anharmonic ratio. These ratios may then be taken as natural projective coordinates γ and ω, and the natural equation on the curve is of the form $\gamma=f(\omega)$. The principal advantage of such a representation is that the necessary and sufficient condition for the equivalence of two curves under projective transformations is simply the identity of the corresponding equations.

Returning to the theory of surfaces, natural coordinates may be introduced so as to fit into the so-called geometry of a flexible but inextensible surface, originated by Gauss, in which the criterion of equivalence is applicability or, according to the more accurate phraseology of Voss, isometry. Intrinsic coordinates must then be invariant with respect to bending (Biegungsinvariante). This property is fulfilled, for example, by the gaussian curvature κ and the differential parameters connected with it $\lambda = \Delta(\kappa, \kappa)$, $\mu=\Delta(\kappa, \lambda)$, $\nu = \Delta(\lambda, \lambda)$, all capable of simple geometric interpretation. The intrinsic equations are then of the form $\mu = \phi(\kappa, \lambda)$, $\nu = \psi(\kappa, \lambda)$.

A pair of equations of this kind thus represent, not so much a single surface S, as the totality of all surfaces applicable on S (or into which S may be bent) — a totality which is termed a complete group G since no additional surfaces are obtained when the same process is applied to any member of the totality. The discussion of such groups is ordinarily based on the first fundamental form (representing the squared element of length), since this is the same for isometric surfaces; though of course it changes on the introduction of new parameters.

The simplest example of a complete isometric group is the group typified by the plane, consisting of all the developable surfaces. In this case the equations of the group may be obtained explicitly, in terms of eliminations, differentiations and quadratures. This is, however, quite exceptional; thus, even in the case of the surfaces applicable on the unit sphere (surfaces of constant gaussian curvature $+1$), the differential equation of the group has not been integrated explicitly.

In fact, until the year 1866, not a single case analogous to that of the developable surfaces was discovered. Weingarten, by means of his theory of evolutes, then succeeded in determining the complete group of the catenoid and of the paraboloid of revolution, and, some twenty years later, a fourth group defined in terms of minimal surfaces.

During the past decade, the French geometers have concentrated their efforts in this field mainly on the arbitrary paraboloid (and to some extent on the arbitrary quadric). The difficulties even in this extremely restricted and apparently simple case are great, and are only gradually being conquered by the use of almost the whole wealth of modern analysis and the invention of new methods which undoubtedly have wider fields of application. The results obtained exhibit, for example, connections with the theories of surfaces of constant curvature, isometric surfaces, Backlund transformations, and motions with two degrees of freedom. The principal workers are Darboux, Goursat, Bianchi, Thybaut, Cosserat, Servant, Guichard, and Raffy.

Geometry im Grossen.

The questions we have just been considering, in common with almost all the developments of general or infinitesimal geometry, deal with the properties of the figure studied im Kleinen, that is, in the sufficiently small neighborhood of a given point. Algebraic geometry, on the other hand, deals with curves and surfaces in their entirety. This distinction, however, is not inherent in the subject matter, but is rather a subjective one due to the limitations of our analysis : our results being obtained by the use of power series are valid only in the region of convergence. The properties of a curve or surface (assumed analytic) considered as a whole are represented not by means of function elements but by means of the entire functions obtained say by analytic continuation.

Only the merest traces of such a transcendental geometry im Grossen are in existence, but the interest of many investigators is undoubtedly tending in this direction. The difficulty of the problems which arise (in spite of their simple and natural character) and the delicacy of method necessary in their treatment may be compared to the corresponding problems and methods of celestial mechanics. The calculation of the ephemeris of a planet for a limited time is a problem im Kleinen, while the discovery of periodic orbits and the theory of the stability of the solar system are typical problems im Grossen.

The principal problems in this field of geometry are connected with closed curves and surfaces. Of special importance are the investigations relating to the closed geodesic lines which can be drawn on a given surface, since these are apt to lead to the invention of methods applicable to the wider field of dynamics. Geodesics may in fact be defined dynamically as trajectories of a particle constrained to the surface and acted upon either by no force or by a force due to a force function U whose first differential parameter is expressible in terms of U. The few general theorems known in this connection are due in the main to Hadamard (*Journal de Mathématiques*, 1897, 1898). Thus, on a closed surface whose curvature is everywhere positive, a point describing a geodesic must cross any existing closed geodesic an infinite number of times, so that, in particular, two closed geodesics necessarily intersect.* On a surface of negative curvature, under certain restrictions, there exist closed geodesics of various topological types, as well as geodesics which approach these asymptotically.

As regards surfaces all of whose geodesics are closed, the investigations have been confined entirely to the case of surfaces of revolution, the method employed being that suggested by Darboux in the Cours de Mécanique of Despeyrous. Last year Zoll † succeeded in determining such a surface (beyond the obvious sphere) which differs from the other known solutions in not having any singularities. Analogous problems in connection with closed lines of curvature and asymptotic lines will probably soon secure the consideration they deserve.

A problem of different type is the determination of applicability criteria valid for entire surfaces. The ordinary conditions (in terms of differential parameters) assert, for example, the applicability of any surface of constant positive curvature upon a sphere ; but the bending is actually possible only for a sufficiently small portion of the surface. A spherical surface as a whole cannot be applied on any other surface, that is, cannot be bent without extension or tearing. This result is analogous to the theorem known to Euclid, although first proved by

* In a paper read before the St. Louis meeting of the American Mathematical Society, Poincaré stated reasons which make very probable the existence of at least three closed geodesics on a surface of this kind.

† *Math. Annalen*, vol. 57 (1903).

Cauchy, that a closed convex polyhedral surface is necessarily rigid. Lagrange, Minding, and Jellet stated the result for all closed convex surfaces, but the complete discussion is due to H. Liebmann.* The theory of the deformation of concave surfaces is far more complicated, and awaits solution even in the case of polyhedral surfaces.

Beltrami's visualization of Lobachevsky's geometry by picturing the straight lines of the Lobachevsky plane as geodesics on a surface of constant negative curvature is well known. However, since the known surfaces of this kind, like the pseudosphere, have singular lines, this method really depicts only part of the plane. In fact Hilbert (*Transactions of the American Mathematical Society* for 1900), by very refined considerations, has shown that an analytic surface of constant negative curvature which is everywhere regular does not exist, so that the entire Lobachevsky plane cannot be depicted by any analytic surface.† There remains undecided the possibility of a complete representation by means of a non-analytic surface. The partial differential equation of the surfaces of negative constant curvature is of the hyperbolic type and hence does admit non-analytic solutions.‡ (This is not true for surfaces of positive curvature, since the equation is then of elliptic type). The discussion of non-analytic curves and surfaces will perhaps be one of the really new features of future geometry, but it is not yet possible to indicate the precise direction of such a development.§

Other theories belonging essentially to geometry im Grossen are the questions of analysis situs or topology to which reference has been made on several occasions, and the properties of the very general convex surfaces introduced by Minkowski in connection with his Geometrie der Zahlen.

Systems of Curves — Differential Equations.

Although projective geometry has for its domain the investigation of all properties unaltered by collineation, attention has

* *Göttinger Nachrichten*, 1899; *Math. Annalen*, vols. 53, 54.

† The entire projective plane, on the other hand, can be so depicted on a surface devised by W. Boy (Inaugural dissertation, Göttingen, 1901).

‡ According to Bernstein (*Math. Annalen*, vol. 59, 1904, p. 72), the proof given by Lütkemeyer (Inaugural dissertation, Göttingen, 1902) is not valid, though the conclusion is correct.

§ Lebesgue (*Comptes Rendus*, 1900) has examined the theory of surfaces applicable on a plane without assuming the existence of derivatives for the defining functions, and thereby obtains an example of a non-ruled developable. The validity of his conclusion depends on certain iterative constructions whose convergence has been questioned.

been confined almost exclusively to the algebraic configuration, so that projective is often confused with algebraic geometry. To the more general projective geometry belong, for example, the ideas of osculating conic of an arbitrary curve and the asymptotic lines of an arbitrary surface, and Mehmke's theorem which asserts that when two surfaces touch each other, the ratio of their gaussian curvatures at the point of contact is an (absolute) projective invariant. The field for investigation in this direction is of course very extensive, but we may mention as a problem of special importance the derivation of the conditions for the projective equivalence of surfaces in terms of their fundamental quadratic forms.

Coordinate with what has just been stated, that general configurations may be studied from the projective point of view, is the fact that algebraic configurations may be studied in relation to general transformation theory. One may object that, with respect to the group of all (analytic) point transformations, the algebraic configurations do not form a *body*,* that is, are not converted into algebraic configurations; but such a *body* is obtained by adjoining to the algebraic all those transcendental configurations which are equivalent to algebraic. As this appears to have been overlooked, it seems desirable to give a few concrete instances, of interest in showing the effect of looking at familiar objects from a new and more general point of view.

As a first example, consider the idea of a linear system of plane curves. In algebraic geometry, a linear system is understood to be one represented by an equation of the form

$$F_0 + \lambda_1 F_1 + \lambda_2 F_2 + \cdots + \lambda_\kappa F_\kappa = 0,$$

where the λ's are parameters and the F's are polynomials in x, y. On the other hand, in general (infinitesmal) geometry, a system is defined to be linear when it can be reduced (by the introduction of new parameters) to the same form where the F's are arbitrary functions. The first definition is invariant under the projective group; the second, under the group of all point transformations. If now we apply the second definition to algebraic curves, the result does not coincide with that given by the first definition. Thus, every one parameter system is

* The most extensive group for which the algebraic configurations form a body consists of all algebraic transformations. It is rather remarkable that even this theory has received no development.

linear in the general sense, while only pencils of curves are linear in the projective sense. The first case of real importance is, however, the two parameter system, since here each point of view gives restricted, though not identical, types. An example in point is furnished by the vertical parabolas tangent to a fixed line, the equation of the system being $y = (ax + b)^2$. From the algebraic or projective point of view, this is a quadratic system since the parameters are involved to the second degree; but, the system is linear from the general point of view since its equation may be written $ax + b - \sqrt{y} = 0$. This suggests the problem: Determine the systems of algebraic curves which are linear in the general sense.

As a second example, consider, from both points of view, the equivalence of pencils of straight lines in the plane. By means of collineations any two pencils may be converted into any other two; but if three pencils are given, it is necessary to distinguish the case where the three base points are in a straight line from the case where they are not so situated. We thus have two projectively distinct cases, which may be represented canonically by: (1) $x =$ const., $y =$ const., $x + y =$ const., and (2) $x =$ const., $y =$ const., $y/x =$ const. The first type may, however, be converted into the second by the transcendental transformation $x_1 = e^x$, $y_1 = e^y$, so that, in the general group of point-transformations, all sets of three pencils are equivalent. The discussion for four or more pencils yields the rather surprising result that the projective classification remains valid for the larger group.

Dropping these special considerations on algebraic systems, let us pass to the theory of arbitrary systems of curves, or, what is equivalent, the geometry of differential equations. While belonging to the cycle of theories due primarily to Sophus Lie, it has received little development in the purely geometric direction. Most attention has been devoted to special classes of differential equations with respect to special groups of transformations. Thus there is an extensive theory of the homogeneous linear equations with respect to the group $x_1 = \xi(x)$, $y_1 = y\eta(x)$ which leaves the entire class invariant.* A special theory which deserves development is that of equations of the

* Halphen, Laguerre, Forsyth. This theory has been extended to simultaneous equations and applied geometrically by E. J. Wilczynski (*Trans. Amer. Math. Soc.*, 1901–1904).

first order with respect to the infinite group of conformal transformations.

As regards the general group of all point transformations, all equations of the first order are equivalent, so that the first case of interest is the theory of the two parameter systems.* The invariants of the differential equation of second order have been discussed most completely in the prize essay of A. Tresse (submitted to the Jablonowski Gesellschaft in 1896), with application to the equivalence problem. A specially important class, treated earlier by Lie and R. Liouville, consists of the equations of cubic type

$$y'' = Ay'^3 + By'^2 + Cy' + D,$$

where the coefficients are functions of x, y. It includes, in particular, the general linear system and all systems capable of representing the geodesics of any surface. While the analytical conditions which characterize these subclasses are known, little advance has been made in their geometric interpretation.

Perhaps the simplest configuration belonging to the field considered, that is, having properties invariant under all point transformations, is that composed of three simply infinite systems of curves, which may be represented analytically by an equation of third degree in y' with one-valued functions of x, y for coefficients. In the case of equations of the fourth and higher degree in y', certain invariants may be found immediately from the fact that when x and y undergo an arbitrary transformation, the derivative y' undergoes a fractional linear transformation (of special type). The invariants found from this algebraic principle are, however, in a sense, trivial, and the real problem remains almost untouched: to determine the essential invariants due to the differential relations connecting the coefficients in the linear transformation of the derivative.

General Theory of Transformations.

Closely connected with the geometry of differential equations that we have been considering, is the geometry of point transformations. In the former theory the transformations enter only as instruments, in the latter these instruments are made the subject matter of the investigation. The distinction is par-

* The elementary (metric) theory of curve systems has been too much neglected; it may be compared in interest and extent with the usual theory of surfaces.

allel to that which occurs in projective geometry between the theory of projective properties of curves and surfaces, and the properties of collineations. (It may be remarked, however, that although a transformation is generally regarded as dynamic and a configuration as static, the distinction is not at all essential. Thus a point transformation or correspondence between the points of a plane may be viewed as simply a donble infinity of point pairs; on the other hand, a curve in the plane may be regarded as the equivalent of a correspondence between the points of two straight lines.*)

We consider first two problems concerning the general (analytic) point transformation which are of interest and importance from the theoretic standpoint. The one relates to the discussion of the character of such a transformation in the neighborhood of a given point. Transon's theorem states that the effect of any analytic transformation upon an infinitesimal region is the same as that of a projective transformation. This is true, however, only in general; it ceases to hold when the derivatives of the defining functions vanish at the point considered. What is the character of the transformation in the neighborhood of such singular points?

A more fundamental problem relates to the theory of equivalence. Consider a transformation T which puts in correspondence the points P and Q of a plane. Let the entire plane be subjected to a transformation S which converts P into P' and Q into Q'. We thus obtain a new transformation T' in which P' and Q' are corresponding points. This is termed the transform of T by means of S, the relation being expressed symbolically by $T' = S^{-1}TS$. The question then arises whether all transformations are equivalent, that is, can any one be converted into any other in the manner defined. The answer depends on certain functional equations which also arise in connection with the question whether an arbitrary transformation belongs to a continuous group. The problem deserves treatment not merely for the analytic transformations, but also for the algebraic and for the continuous transformations.†

* Geometry on a straight line, in its entirety, is as rich as geometry in a plane or in space of any number of dimensions.

† This problem is not to be confused with the similar (but simpler) question connected with Lie's division of (analytic) groups into *demokratisch* and *aristokratisch*. In those of the first kind all the infinitesimal transformations are equivalent, in those of the second there exist non-equivalent infinitesimal transformations. Lie shows that all finite groups are aristokratisch, while the groups of all (analytic) point and contact transformations are demokratisch. Cf. *Leipziger Berichte*, vol. 47 (1895), p. 271.

Aside from such fundamental questions, further development is desirable both in the study of the general properties (associated curve systems and contact relations) of an arbitrary transformation, and in the introduction of new special types of transformation, for instance, those which may be regarded as natural extensions of familiar types.

The main problems in the theory of point transformation are connected with certain fields of application which we now pass in review.

1. *Cartography.* A map may be regarded, abstractly, as the point by point representation of one surface upon another, the case of especial practical importance being, of course, the representation of a spherical or spheroidal surface upon the plane. As it is impossible to map any but the developable surfaces without distortion upon a plane, the chief types of available representation are characterized by the invariance of certain elements, as angles or areas, or the simple depiction of certain curves, as of geodesics by straight lines. Most attention has been devoted to the conformal type, but the question proposed by Gauss remains unsolved: what is the *best* conformal representation of a given surface on the plane, *i. e.*, the one accompanied by the minimum distortion? The answer, of course, depends on the criterion adopted for measuring the degree of distortion, and it is in this direction that progress is to be expected.

2. *Mathematical theory of elasticity.* As a geometric foundation for the mechanics of continua, it is necessary to study the most general deformation of space, defined say by putting x_1, y_1, z_1 equal to arbitrary functions of x, y, z. The most elegant analytical representation, as given for instance in the memoir of E. and F. Cosserat (*Annales de Toulouse*, volume 10), is obtained by introducing the elements of length ds and ds_1 before and after deformation, and the related quadratic differential form $ds_1^2 - ds^2 = 2\epsilon_1 dx^2 + 2\epsilon_2 dy^2 + 2\epsilon_3 dz^2 + 2\gamma_1 dydz + 2\gamma_2 dxdz + 2\gamma_3 dxdy$. The theory is thus seen to be analogous to, though of course more complicated than, the usual theory of surfaces. The six functions of x, y, z which appear as coefficients in this form are termed the components of the deformation. Their importance is due to the fact that they vanish only when the transformation is a rigid displacement, so that two deformations have the same components when, and only when, they differ by a displacement. The case where the components are constants leads to the homogeneous deformation (or affine transformation of

the geometers), the type considered almost exclusively in the usual discussions of elasticity. It would seem desirable to study in detail the next case which presents itself, namely, that in which the components are linear functions of x, y, z.

In the general deformation, the six components are not independent, but are connected by nine differential equations analogous to those of Codazzi. The fact that a transformation is defined by three independent functions indicates, however, that there should be only three distinct relations between the components. This means that the nine equations of condition which occur in the standard theory are themselves interdependent; but their relations (analogous to syzygies among syzygies in the algebra of forms) do not appear to have been worked out.

3. *Vector fields.* From its beginning in the Faraday-Maxwell theory of electricity until the present day, the course which the discussion of vector fields has followed has been guided almost entirely by external considerations, namely, the physical applications. While this is advantageous in many respects, it cannot be denied that it has led to lack of symmetry and generality. The time seems to be ripe for a more systematic mathematical development. The vector field deserves to be introduced as a standard form into geometry.

Abstractly, such a field is equivalent to a point transformation of space, since each is represented by three scalar relations in six variables. Instead of taking these variables as the coordinates of corresponding points, it is more convenient to consider three as the coordinates x, y, z of a particle and the other three as components u, v, w of its velocity; we thus picture the set of functional relations by means of the steady motion of a hypothetical space-filling fluid. This image should be of service even in abstract analysis; for its rôle is analogous to that of the curve in dealing with a single relation between two variables. The streaming of a material fluid is, of course, not sufficiently general for such a purpose, since, in virtue of the equation of continuity, it images only a particular class of vector fields.

In addition to the ordinary vector fields, physics makes use of so-called hypervector fields, which, geometrically, lead to configurations consisting of a triply infinite system of quadric surfaces, one for each point of space. In the special case of interest in hydrodynamics (irrotational motion), the configuration simplifies in that the quadrics are ellipsoids about the corresponding points as centers. This is equivalent to the *tensor* field which arises in studying the moments of inertia of an arbitrary distribution of mass. The more general case actually arises in Maxwell's theory of magnetism.

4. As a final domain of application we mention the class of questions which have received systematic treatment, under the title of *nomography*, only during the past few years. This subject deals with the methods of representing graphically, in a plane, functional relations containing any number of variables. Thus a function of two independent variables, $z = f(x, y)$, may be represented by the system of plane curves $f(x, y) = c$, each marked with the corresponding value of the parameter. This 'parametered' system is then a cartesian graphical table, which is the simplest type of abacus or nomogram.

By means of any point transformation, one nomogram is converted into another which may serve to represent the same functional relation. The importance of this process of conversion (the so-called *anamorphosis* of Lalanne and Massau) depends on the fact that it may replace a complicated table by a simpler. The problems which arise (for example, the determination of all relations between three variables which can be represented by a nomogram composed of three systems of straight lines *) are of both practical and theoretical interest. The literature is scattered through the French, Italian and German technological journals, but a systematic presentation of the main results is to be found in the Traité de Nomographie of d'Ocagne (Paris, 1899).

We return to the abstract theory of transformations. The type of transformation we have been considering, converting point into point, is only a special case of more general types. The most important extension hitherto made depends upon the introduction of differential elements. Thus the lineal element or directed point (x, y, y') leads to transformations which in general convert a point into a system of elements; when the latter form a curve, every curve is converted into a curve and the result is termed a contact transformation. Backlund has shown that no extension results from the elements of

* The case of three systems of circles has also been discussed. See d'Ocagne, *Journal de l'Ecole Polytechnique*, 1902.

second or higher order: osculation transformations are necessarily contact transformations. The discussion of elements of infinitely high order, defined by an infinite set of coordinates $(x, y, y', y'', \cdots)$, may perhaps lead to a real extension. The question may be put in this form: Are there transformations (in addition to ordinary contact transformations) which convert analytic curves into analytic curves in such a way that contact is an invariant relation? The idea of curve transformation in general will probably be worked out in the near future: what is the most general mode of setting up a correspondence which associates with every Jordan curve another Jordan curve? Such discussions are aspects of geometry with an infinite number of dimensions.

After a review of the kind given in this paper, one is tempted to ask: What is it which influences the mathematician in selecting *certain* (out of an infinite number of equally conceivable) problems for investigations? It is true, of course, that his subject is ideal, self-created, and that "Das Wesen der Mathematik liegt in ihrer Freiheit." Georg Cantor would indeed replace the term *pure* mathematics by *free* mathematics. This freedom, however, is not entirely caprice. The investigators of each age have always felt it their duty to deal with the unsolved questions and to generalize the results and conceptions inherited from the past, to correlate with other fields of contemporaneous thought, to keep in contact, as far as possible, with the whole body of truth. This is not all, however. The influence of esthetic considerations, though less subject to analysis, has been, and still is, of at least equal importance in guiding the course of mathematical development.

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