theorem, due to Sylow, is shown by Professor Miller to be included in the following: Every non-abelian group of order $p^m$ contains at least $p$ invariant commutator operators, and its commutator quotient group is always non-cyclic. The paper is devoted to a proof of this theorem and the following closely related theorems: It is possible to construct a non-abelian group having any arbitrary abelian group as a commutator quotient group. Every non-cyclic abelian group of order $p^a$ is the commutator quotient group of some non-abelian group of order $p^m$.

11. Mr. Quinn exhibited and explained a number of new linkages for describing the right line, in each of which the principle of inversion was applied.

16. A new form of thread model for ruled surfaces was exhibited by Professor Waldo, the frame of the model being conformed to the surface of a sphere; thus permitting the location of the points of attachment of the threads with much greater ease than in the ordinary forms, in which the limiting surface is discontinuous. The method of construction was also explained.

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## A CALCULUS FOR GEOMETERS.

*Cours D'Analyse.* By G. HUMBERT. Paris, Gauthier-Villars. 8vo, 2 vols.: Vol. I, 483 pp., 1903; Vol. II, 493 pp., 1904.

THE rapidity with which French treatises on the calculus follow one another is at times confusing to the American mind. Picard's monumental work is still unfinished; Jordan's is a recent production; Vallée-Poussin is as yet scarcely familiar; Goursat's second volume is only partially complete; and a number of others, more or less well-known, are certainly upon the horizon. Meanwhile, in the interim between the appearance of the separate volumes of some of the works just mentioned, another first class treatise — the subject of this review — has been published in its entirety.

In general, Humbert's work is characterized by a predominance of geometry, and in particular by applications to the theory of curves and surfaces. But this statement is to be taken relatively. Picard's Traité is described (by Picard himself) as a treatise on differential equations. The individual character of Jordan's Cours, at least in its second edition, arises to a considerable extent from his novel treatment of important questions in the theory of assemblages. Vallée-Poussin's work is strong upon the side of the theory of functions of a real variable. Goursat's Cours has been said * to be a well balanced presentation of the whole calculus, with no extraordinary stress on any one point. It is in this comparative sense that Humbert's Cours may be said to be characterized by the stress which is laid upon geometry. It is none the less a general treatise, however, and I shall attempt to bring out the fact that the methods of presentation employed in certain other connections are noteworthy and valuable for the American student and teacher of mathematics. In general it should be said, for the purpose of orientation, that Humbert's work is not especially adapted for use as a text in America, on account of the fact that he presupposes a knowledge of some subjects which are not throughly treated in our elementary courses on the calculus: notably the law of the mean, Taylor's series with and without a remainder, the general theory of series of constant terms, and the accurate introduction to incommensurable and imaginary quantities. Besides, there are certain portions with which the American student might have dispensed more readily, for instance the discussion of elementary geometric concepts in Chapter II of Part I and the discussion of elementary integrals in Chapter I of Part II (Volume I), in so far as those discussions are formal and non-rigorous. However, the fact that the book is not particularly suited to American conditions can scarcely be criticized.

Each volume is divided into three Parts. In the first volume the divisions,—Differential calculus, Integral calculus, Applications to geometry—are to some extent arbitrary; each part contains matter which might be placed under any one.

After a brief introduction on limits and extrema of sequences,† without stopping to discuss functions of a single (real) variable, the author passes immediately to functions of two (real) vari-

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*See a review by W. F. Osgood, BULLETIN, vol. 11, No. 10 (July, 1903), pp. 547–555.

† The word sequence (série, suite) is used in this review to denote an enumerable assemblage of numbers arranged in a definite order.

ables and establishes the general algorithm of infinitesimal rectangular subdivision of the plane, which certainly deserves all the prominence he gives it. In § 6, where it is shown that a continuous function (of two variables) actually attains its maximum, the fact that the theorems of §§ 1–2 were proved only for sequences seems to be overlooked, but it is easy to supply the argument. A similar oversight seems to manifest itself harmlessly in § 12 (definition of a derivative) and in § 13 (definition of an infinitesimal). To the reviewer the word "approach" does not seem to imply a notion of succession or order in which the variable should take on the values which it is permitted to assume, — so far have we departed from the original vulgar meaning of the technical word. The use of a sequence in such definitions as those mentioned does not, then, seem justifiable. At least the practical test for the existence of a derivative does *not* consist in the examination of all possible sequences of values of $h$ which approach zero as a limit,* and an attempt to base the definition upon approach through sequences cannot be useful without practically reverting to the rigorous $\epsilon$ definition.

The treatment of uniform continuity (§ 9) should be mentioned: the consideration of $\eta$ as a function of $\epsilon$ and the clear statement that $\eta$ is a function of $\epsilon$ alone certainly deserve emulation.

On page 16, line 2, the words "et reste" should be inserted after "croît."

In the discussion of differentials (Section III) the law of the mean is assumed as previously familiar to the student. This would not be true of American students, at the time they would naturally study this book. In § 20 the usual statement that $dy[=hf'(x)]$ is the principal part of $\Delta y$ is, of course, untrue if $f'(x)=0$, and it may be untrue otherwise if $f'(x)$ is not continuous. The notation $d^2_{uv}z$, in § 43 speaks for itself, and illustrates the awkwardness of the notation of differentials. The difficulties which the author points out would seem to disappear in the simpler derivative notation $z_{uv}$.

*Such an investigation might exhaust all possible values of $h$, and still be inconclusive, as in the example $F(h)=h^{1/p}$ when $h=1/p^n$, $F(h)=0$ when $h \neq 1/p^n$, where $p$ is a prime integer: for any sequence of values of $h$ which includes no number $1/p^n$, and also for any sequence of the form $1/p$, $1/p^2$, $1/p^3$, $\cdots$, $F(h)$ approaches zero with $h$. But $F(h)$ does *not* approach zero with $h$, for $h$ can be chosen in any interval $(0, +\varepsilon)$ so that $F(h) > 1/10$, for example, by taking $h=1/P$ where $P$ is a prime greater than $1/\varepsilon$.

The next three chapters (II, III, IV) define rather clearly the standpoint of the author. The fact that Chapter II deals with infinitesimal geometry is mentioned expressly in the preface. Taylor's theorem with a remainder is stated without proof, and the elementary applications to geometry (the formal definition of curvature, etc.) occupy all the rest of the chapter. The problem of change of variable (Chapter III) is solved in two pages from the formal standpoint, by means of differentials. This is followed by twelve pages of particular (geometric) examples, and the chapter ends with a very clear presentation of contact transformations, including Legendre's transformation and Lie's line-sphere transformation. From the standpoint of geometry the author has already covered quite an amount of territory, closing with the essential properties of Dupin's cyclide. In Chapter IV (the formation of differential equations) the standpoint is again a purely formal one, the theory is restricted to a few pages, and the examples, which fill most of the pages, are the elementary differential equations of geometrical families. Both Chapters III and IV incline one to the belief that here also the author "a donné, autant que possible, à la géométrie le pas sur l'analyse," as he himself remarks in the preface regarding Chapter II. Except for the language, one might think oneself back reading Williamson: our English parentage has made these developments seem elementary, at least this formal view of them.

Not so with infinite series — the subject of the fifth chapter. Here the first pages — on series of constant terms — seem woefully meagre; but we gather that the French student already knows these things from that wonderful institution, le cours de Mathématiques spéciales. These first ten pages constitute at best a review text, and cannot be considered as a text for a beginner. Starting with page 133 (series of variable terms) we notice especially a very clear discussion of the definition of uniform convergence (§ 136), and the reasoning which follows is delightful to the end of the section (§ 141). The reviewer knows no book from which the beginner could gain such a grasp of this delicate subject in its elementary essentials, and these pages might well be made note of as a reference for students to whom the idea of uniform convergence is being presented for the first time. This portion is only too short, containing nothing but the definition, certain explicatory examples, the theorem of a dominant series of positive terms, and an extension of the definition to series of imaginary terms. Then follows the usual discussion of power series of imaginary terms.

The last three chapters of Part I present brief formal treatments of the theory of functions of an imaginary variable (which is more adequately treated in Vol. II), Taylor's series for functions of several variables, and maxima and minima, respectively. The theory is confined to a few pages; in the last chapter, for example, ten of the sixteen pages are devoted to geometrical examples, and the remaining six are hardly adequate for a thorough presentation of the subject.

A fundamental difference of opinion will doubtless exist for all time regarding the presentation of the integral calculus. There is of course something to be said in support of the theory that the indefinite integral should be presented to the young student before the definite integral is defined independently. Having done this once (in our elementary courses in calculus) it is unfortunate, for us, that the author treats the indefinite integral first, saying that the first problem of integral calculus is the discovery of a function whose derivative is given. Logically this is a problem in differential calculus, and this whole chapter (Chapter I of Part II) should fall in Part I.

Accepting the arrangement as it stands, Chapter I (Part II) is mainly very elementary matter. The exceptions are the proof (§ 198) of the principle of decomposition of a rational fraction into partial fractions, which is well given; and the discussion of unicursal curves, which is given in full. These two matters are worth noting for purposes of reference.

The matter in Chapter II on the reduction of hyperelliptic, elliptic, and abelian integrals for a curve of deficiency one is the usual presentation, with possibly a special stress upon the geometric side in the treatment of abelian integrals. The discussion of definite integrals in Chapter III suffers somewhat from the arrangement referred to above. Thus it seems to me that one of the principal reasons for defining the definite integral independently is that we do not know otherwise that any continuous function is the derivative of some other function. This is not included in the reasons given at the top of page 263, though the proper theorem is mentioned on page 271, with the remark that such a result was not one of the motives of the work. This chapter contains a satisfactory treatment of the definition and elementary properties of definite integrals, which is restricted however to the absolute essentials. For example, the definition of a definite integral (page 264) is given in terms of minima of the function only, a definition which does not permit of the distinction between integrable and non-integrable functions. On the other hand the work is accurate enough here and elsewhere. The theory in general is quite condensed, and geometric examples of areas and lengths consume half the pages. The definitions of area and of length are fairly satisfactory, and are reconsidered with more rigor in Volume II.

The last two chapters of this part deal with various properties of definite integrals. In the first, improper definite integrals are defined and discussed briefly, and the ordinary comparison theorems are proved. It should be noticed that the example treated in § 309 (pages 314–315) involves an essentially new definition, since the integrand becomes infinite at an infinite number of points; the definition tacitly assumed is the natural one. One misses in this chapter the comparison theorem of Cauchy relative to infinite series. In general the figures, the examples and the discussions are very clear, so far as the matter is carried. The last chapter treats term by term integration and differentiation of series, with a satisfactory proof of the first fundamental theorem in each case; and Leibniz's rule for differentiation under the integral sign. The discussion of the latter (§ 322) is very clearly put and deserves attention. The potentially beautiful theory of integration of total differentials, and Fourier's theorem, are also treated, but the discussion is formal in each case.

The third and final part of the first volume covers practically one-third of the book. Together with the comparatively disproportionate amount given to geometry in the first two parts, the remark that the book leans heavily toward geometry seems justified. It has often been said, apparently with justice, that geometry is less rigorous than arithmetic. This is, of course, untrue. The fact is that geometers are less rigorous than arithmeticians, — for what reason does not appear. In itself, can there be any doubt that the facts of geometry are absolutely rigorous? Is it possible that a theorem can have an exception, even though it relate to geometry, if that theorem has been proved? And yet it is wholly impossible to discuss this part in that sense. The theorems stated are those usually accepted and "proved" in most of the standard works, and none of these criticisms apply to Humbert's work in particular.

Thus Chapter I (Part III) is a clear presentation of the ordinary theory of envelopes of curves and surfaces and the allied concepts, including order of contact and so on. Questions of rigor might be raised at every point. In § 341, for example, where singular points are discussed with more care than in any other paragraph, and where the conditions imposed include one-to-one analytic correspondence between the points of the curve and the points of a segment of the $t$ axis, the statements made, particularly the statement at the top of page 345, can be justified only by allowing the parameter $t$ to assume imaginary values. For, if $t$ be restricted to real values, the theorem itself is evidently absolutely false, as the example $x = t^3, y = t^6$ shows; for the curve represented by these equations is precisely the parabola $y = x^2$, upon which the origin is in no sense peculiar. On the other hand, if $t$ be allowed to assume imaginary values, why should we say that the assemblage of pairs of values of $x$ and $y$ which satisfy the equations is adequately represented by a plane curve? The remark on page 348 shows that the author is not blind to the difficulties in question. Again, in the treatment of envelopes, the theorem that the curve (or curves) defined by the equations $f(x, y, c) = 0, f_c(x, y, c) = 0$ is tangent to every member of the family $f(x, y, c) = 0$, suggests a number of questions. Beside the one mentioned by the author (p. 360), consider the equation $f(x, y, c^2) = 0$, which represents the same curves as the given equation, or a portion of them; then $f(x, y, 0) = 0$ appears as a supposed envelope. These remarks leave untouched the nicer difficulties, the existence of the envelope, the conditions to be imposed upon $f(x, y, c)$, the discussion of one parameter families such that an infinite number of curves of the family pass through any given point in a two-dimensional region, the distinction between a one parameter family and a two parameter family, etc.

It seems that the old spirit of generality still pervades the traditions of geometry, so that a statement will escape criticism if put in geometric language, which would be utterly crushed under denouncements if stated in the forms usual in the theory of functions of real variables. To be sure, Jordan has awakened our sleeping consciences regarding the true inwardness of a plane curve, and Peano, Hilbert and others have taught us to speak with caution of closed curves and the like; but a family of curves is still spoken of with extreme freedom, as something considerably simpler, apparently, than a single one; the more abstruse the conception, the more glibly we refer to it. Let it be said clearly, however, that these remarks apply in general, and that they are not intended as a criticism of this particular book.

Chapters II and III deal briefly with plane curves and skew curves, respectively. Chapters IV and V present the usual matter concerning the element of length, the element of area, and the curvature of surfaces in the elementary essentials. The presentation is always clear and very effective. A brief consideration of Lie's theorem and an equally brief discussion of minimum surfaces constitute a somewhat unusual feature. The last chapter of the volume contains a sketch of the theory of transformation of one surface upon another.

It is a matter of considerable surprise that the theory of conformal transformation has been given so little space in a work otherwise geometric in its tendencies. Even in the discussion of the theory of functions of a complex variable in the second volume, no mention of this important question is made, and here, in the first volume, the discussion is limited to eight pages. It is obvious that only the first essentials are treated.

The divisions of the second volume — Supplement to integral calculus, Theory of functions, Differential equations — can be, and are more consistently observed than the divisions of the first volume.

The first chapter deals with multiple integrals. The author begins with a reconsideration of the definitions of area given in Volume I, and succeeds in placing the definition upon a comparatively trustworthy foundation, though there are still some omissions, — notably the definition and properties of a closed curve. In the definitions of double integrals, triple integrals, etc., which follow, the nature of the boundary of the field of integration is scarcely insisted upon, and might easily be lost sight of. Thus on page 13, footnote 2, the assumption would *seem* to be that the boundary is a rectifiable curve. Very great care will be necessary in reading the proof of the formula for change of variable, however, in regard to the one-to-one correspondance of the regions involved, and the signs of the corresponding jacobians. The matter is clearly stated on page 37, but before this, on page 32, there is danger of being carried outside the region of definition of the function to be integrated: the curve $\varphi(x, y) = 0$ may lie wholly outside $C$; and on page 35 the same difficulty is very prominent.

Chapter II, in its sixteen pages, comprises almost all there is in the whole work regarding line and surface integrals, Green's and Stokes's theorems, and (real) integrals independent of the path of integration.

The forms of Green's theorem which apply particularly to Laplace's equation (or other allied equations) are nowhere given. These omissions and curtailments* seem all the more striking in view of the statement (§ 63) that these theorems are of extreme importance both in analysis and in physics.

In the last two chapters of this part a great variety of topics are treated, of which the discussion of integration under the integral sign (page 83) and Hilbert's proof of the transcendental character of $e$ are perhaps the most noteworthy. The other pages contain discussions of more or less important problems, ranging from numerical examples to a general treatment of the $\Gamma$ function.

Part II (Functions of a complex variable and Elliptic functions) follows logically the twenty pages devoted to that subject in Volume I. The whole treatment of "analytic functions" (Chapter I) is a very lucid and comprehensive treatment, in fifty pages, of the essential facts of the Cauchy theory of single valued analytic functions. The fundamental definitions are forceful, as are many of the "remarks" throughout the chapter. Cauchy's integral theorem, Taylor's series, Laurent's series, Fourier's series, the elementary theorems on integral functions (including Weierstrass's and Mittag-Leffler's theorems), are all discussed in a manner quite adequate for a work of this kind. After some examples in the calculation of integrals and expansion in series of fractions, a dozen pages are devoted to the study of the integrals of algebraic functions. It seems unfortunate that so little space was given to the latter subject and that the methods of the Riemann school were not explained.

The last three chapters (III, IV, V) (pp. 184–261) deal with elliptic functions and their applications. While this amount of space is not excessive, it seems to me that the point of view might profitably have been broader, to include some of the theory according to Riemann.—As it is, these pages constitute the usual treatment according to the Cauchy-Weierstrass methods.

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* For example, the distinction between one-sided and two-sided surfaces is not pointed out, and the final statement of Stokes's theorem (p. 82) should be accepted with caution.

The treatment of differential equations in the last Part (pp. 263–493) is more characteristic, more individual, than either of the other parts of this volume. The author seems to return to a very large extent to the geometric point of view; much of the matter is geometric in character, as in Volume I, and the discussion is to a large extent formal. In the first chapter (the integration of the simpler classes of ordinary equations of the first order), I might mention the proof (page 282) that the cross ratio of any four solutions of a Riccati equation is constant, and the geometric portions (pages 296–312), which form a supplement to Part III of Volume I, particularly on the topics evolutes, lines of curvature, asymptotic lines, and conjugate systems.

It is not my purpose to criticize this chapter on the side of rigor, since there is no attempt made to emphasize that side. It seems to me, however, that the argument in § 271 might as well be omitted, since the assumptions made are as questionable as the theorem itself.

Chapter I shows a tendency to return to geometry; Chapter II develops that tendency to an extreme. Of the thirty pages, twenty-six are devoted to geometric applications. In fact the whole chapter (called differential equations of any order) is mainly a discussion of the geodetic lines on a surface. As such it is an excellent presentation of the elementary theory and of the application of that theory to important examples, but the discussion is rendered quite formal by the assumption (page 330) that a geodetic line is a shortest line.*

The proofs of the Cauchy existence theorems are given in the next chapter according to Cauchy. The question whether or not other solutions than those given by the Cauchy process exist is mentioned by the author in footnotes to pages 367 and 370, and reference is made to an exercise (XVIII, page 402) in which the proof of the non-existence of other solutions is given for a single equation of the first order.† Since no other case is treated, certain theorems stated in the chapter cannot be said to have been proved. Passing over the usual treatment of linear equations in the fourth chapter, mention should be made of the rather unusual treatment (pp. 407–434) in the fifth chapter of the integrals of ordinary linear equations of the second order. While this matter is to be found in several books which are

* Compare E. R. Hedrick, "On the sufficient conditions in the calculus of variations," BULLETIN, vol. 9, No. 1 (October, 1902), p. 18.

† Compare my thesis "Ueber den analytischen Character der Lösungen von Differentialgleichungen," Göttingen, 1901, p. 14.

entirely devoted to differential equations, its occurrence here increases the value of the book for purposes of reference.

The last chapter, on partial differential equations, is almost entirely devoted to the formal Lagrange theory of equations of the first order. Practically only one differential equation of the second order is discussed, one which enters in the solution of a geometric problem.

In general, the treatise covers a vast territory. There are many topics which are treated in a detail not usual in a work of this class: notably infinitesimal geometry in general, but also elliptic functions and their applications, and the integrals of linear differential equations of the second order. There are certain other problems which are presented with peculiar clearness: uniform convergence, the deficiency of curves, the Cauchy theory of functions, and the geodetic lines on a surface. There are also some strange omissions: notably the theory of conformal tranformations, partial differential equations of the second order — in particular Laplace's equation, and the calculus of variations; but also the Riemann point of view in the theory of functions and an adequate treatment of multiply valued functions, the properties of a Jordan curve, implicit functions, and an adequate treatment of Green's and allied theorems.

I think I have shown that the work as a whole has an extraordinary tendency toward geometry. But many topics which are not geometric are treated at great length, sometimes with unusual accuracy, and a great majority of subjects which are of general importance in mathematics are at least mentioned. Hence the book is to be classed, not as a treatise on geometry, but as a general calculus. But it would seem peculiarly valuable to a student who intended to make geometry a specialty, or as a book of reference on subjects connected with geometry.

We have recently had "The Calculus for Engineers" in many forms; we have had at least one "Calculus for Science Students" (Nernst-Schoenflies); "The Calculus for Analysts" appears to have been a ruling favorite in German and Italian circles; we have had also "The Calculus in General," and such a one seems useful to more persons than any one of the more special types. The ruling motive of Humbert's work is not altogether novel, for the same theme appears in many of the treatises of the past century, but this is probably the best modern example of a "Calculus for Geometers." E. R. HEDRICK.

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*December*, 1904.