$\mu - 1)$ of powers of two properly chosen numbers $A = \sum_1^n {}_i a_i e_i$ and $B = \sum_1^n {}_i b_i e_i$ rational with respect to $\mathbf{R}(R, e_i)$. Any such system can be transformed into one consisting of ν mutually nilfactorial quadrates of order μ by a transformation rational with respect to the domain obtained by adjoining to R the roots of the equation $x^m + p_1(a)x^{m-1} + \cdots + p_{m-1}(a)x + p_m(a) = 0$ for a properly chosen number $A = \sum_1^n {}_i a_i e_i$ of the system rational with respect to $\mathbf{R}(Re_i)$.

11. In the January number of the BULLETIN, Dr. Burke Smith proves that the minimal surfaces and surfaces of translation whose generators are in perpendicular planes are the only surfaces of translation which can be deformed in a continuous manner in such a way that the generators continue to be generators. Dr. Eisenhart applies to these surfaces a theorem due to Adam and gets pairs of applicable surfaces of translation with the generators in correspondence; and the equations of these surfaces involve seven arbitrary parameters. The conditions to be satisfied in order that the generators be plane on these new surfaces are given, and a few examples are discussed.

F. N. COLE,
Secretary.

THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE sixteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held in the Northwestern University Building, Chicago, on December 30 and 31, 1904. The attendance was thirty-seven, including the following members of the Society:

Mr. R. P. Baker, Professor D. P. Bartlett, Professor G. A. Bliss, Dr. W. H. Bussey, Professor Florian Cajori, Professor D. F. Campbell, Professor L. E. Dickson, Dr. E. L. Dodd, Dr. Saul Epsteen, Professor A. G. Hall, Professor Thomas F. Holgate, Mr. N. J. Lennes, Mr. E. P. Lytle, Professor Heinrich Maschke, Professor G. W. Myers, Mr. Oscar Schmiedel, Miss I. M. Schottenfels, Professor H. E. Slaught, Dr. Burke Smith, Professor E. J. Townsend, Dr. Oswald Veblen, Professor A. L. P. Wernicke, Professor H. S. White, Mr. A. H. Wilson, Mr. N. R. Wilson, Dr. A. E. Young.

The first session opened at ten o'clock, Professor E. J. Townsend, of the University of Illinois, in the chair. The following officers were elected for the ensuing year: Secretary, Professor Thomas F. Holgate; additional members of the programme committee, Professor G. A. Bliss, Professor A. G. Hall.

The following papers were read:

(1) Professor L. E. DICKSON: "A definition of a group by independent postulates."

(2) Professor L. E. DICKSON: "Determination of the ternary modular groups."

(3) Mr. A. H. WILSON: "On the linear automorphic transformation of the binary quartic."

(4) Mr. N. J. LENNES: "Areas and volumes."

(5) Dr. SAUL EPSTEEN and Mr. J. MACLAGAN-WEDDERBURN: "On the composition of a hypercomplex number system."

(6) Professor L. E. DICKSON: "On the minimum degree of resolvents for the p-section of the periods of hyperelliptic functions of four periods."

(7) Professor L. E. DICKSON: "On semigroups and the general isomorphism between infinite groups."

(8) Dr. BURKE SMITH: "Deformation of surfaces with preservation of a conjugate system."

(9) Dr. OSWALD VEBLEN: "Non-metrical definition of the linear continuum."

(10) Dr. A. E. YOUNG: "Surfaces defined by the quadratic forms $\lambda(du^2 + dv^2)$ and $D(du^2 \pm dv^2) + 2D'dudv$.

(11) Professor H. S. WHITE: "Polar tetrads to a quartic surface upon a twisted cubic, with generalizations."

(12) Mr. N. J. LENNES: "On the necessary conditions for the convergence of an improper definite integral."

Some mathematical models were exhibited by Mr. R. P. Baker. Friday evening at six thirty o'clock dinner was served to those in attendance at the University Club of Chicago.

Abstracts of the papers presented follow below. The abstracts are numbered to correspond to the titles in the list above.

1. An abstract group may be defined by the four postulates: (1) product is in set; (2) composition is associative; (3) presence

of a right-hand identity i; (4) for a particular i (if existent) there occurs a right-hand inverse of each element a, namely a solution of $a \circ a' = i$. The postulates are proved to be sufficient, consistent and independent. In fact, if we add a postulate fixing the order of the set or the commutative postulate, the five postulates are independent. This definition was obtained by Professor Dickson as a simplification of Professor Moore's first definition (*Transactions*, 1902). In Professor Moore's recent proof that postulate $(3''_r)$ of his second definition was redundant, he obtained relations sufficient to establish the present definition, but did not set up the definition itself, but rather a very desirable simplification of his old second definition. This paper will appear, together with Professor Moore's paper, in the *Transactions* for April, 1905.

2. The determination of all m-ary linear homogeneous groups of determinant unity modulo p falls naturally into two problems, according as the order is a multiple of p or prime to p. The case $m = 3$ has been discussed at length by Professor Burnside, *Proceedings of the London Mathematical Society*, volume 27 (1894), pages 58–106, with limitations as to the number of factors of $p^2 + p + 1$. An error led him to overlook the groups with an invariant ternary quadratic form. The paper by Professor Dickson gives a brief but complete determination of the ternary groups of orders multiples of p, by methods quite different from those employed by Professor Burnside. The paper will appear in the *American Journal of Mathematics*.

3. The transformations of the tetrahedron group are given in the literature in the canonical form. Mr. Wilson derives expressions for these transformations without specialization of the coördinate system for the quartic, making use of its irrational covariants. Incidentally the group for the harmonic case of the quartic is also given.

4. The paper points out a certain minor change necessary in Hilbert's theory of areas: In regard to volumes, it is shown that while the present theory, which is built without the use of any continuity axioms, is a complete theory in terms of measure of volume, there is no such theory in terms of geometric congruence. Thus if of two polyhedrons P_1 and P_2, P_1 is con-

gruent to part of P_2 it is known that $M(P_1)$ (where $M(P_1)$ denotes the measure of volume of P_1) is less than $M(P_2)$. On the other hand, if $M(P_1) < M(P_2)$ we do not know that P_1 is congruent to part of P_2.

By means of the archimedean axiom the following theorem is proved: "If $M(P_1) < M(P_2)$, then P_1 is congruent to part of P_2." It is shown that this theorem cannot be proved without the archimedean axiom. The following definitions are then given: $P_1 < P_2$ if P_1 is congruent to part of P_2. P_1 is equal to P_2 if (1) P_1 is not congruent to part of P_2 and (2) P_2 is not congruent to part of P_1. It follows that for any two polyhedrons P_1 and P_2 one of the following relations must hold: $P_1 < P_2$, $P_1 = P_2$, $P_1 > P_2$ these relations being defined in terms of geometric congruences.

5. By the product EE_1 of an algebra $E \equiv e_1 \cdots e_m\ e_{m+1} \cdots e_n$ by one of its subcomplexes $E_1 = e_{m+1} \cdots e_n$ is meant the totality of independent units arising when every unit of E is multiplied in turn by every unit of E_1. If $EE_1 \leqq E_1$, $E_1E \leqq E_1$, the algebra E_1 is said to be invariant in E. According to Molien (*Mathematische Annalen*, volume 41, page 93), if E_1 is invariant in E, then there exists an algebra K_1 which accompanies E. K_1 is also said to be complementary to E_1 with respect to E.

The series $E, E_1, E_2, \cdots$ is said to be normal when E_s is a maximal invariant subalgebra of E_{s-1} ($E_0 = E$), and $K_1, K_2, \cdots$ is said to be a normal complementary series when K_s is complementary to E_s with respect to E_{s-1}. If n_s is the order (number of units) of E_s, then the integers l_s are defined as follows: $l_s = (n_s -$ order of maximal subalgebra of E_{s-1} containing $E_s)$, $s = 1, 2, \cdots$. In addition, the integers k_s are defined thus: $k_s = n_{s-1} - n_s$.

If C_s is the maximal subalgebra of C_{s-1} which is invariant in E, then $E, C_1, C_2, \cdots$ is said to be a *chief* series of E. The following theorems are demonstrated.

1) If E has two invariant subalgebras E_1, E_1' then (*a*) E is reducible in the sense that the product of any number of E by any number of E' is zero or belongs to their common subalgebra (Peirce, *American Journal of Mathematics*, volume 4, page 100); or else (*b*) $E_1' \leqq E_1$.

2) If E_1 and E_1' are two maximal invariant subalgebras of E and if their common part is E_1'', then E_1'' is a maximal invariant subalgebra of E_1 and E_1'.

3) The normal complementary series $K_1, K_2, \cdots$ is independent, apart from the sequence, of the normal series $E, E'_1, E'_2, \cdots$.

4) The sets of integers $l_1, l_2, \cdots$ and also the set of integers $k_1, k_2, \cdots$ are independent, apart from the sequence, of the normal series selected.

5) The chief complementary series $k_1^{(c)}, k_2^{(c)}, \cdots$ is independent, apart from the sequence, of the chief series $E, C_1, C_2, \cdots$.

6. The third paper by Professor Dickson is the outcome of an investigation conducted under the auspices of the Carnegie Institution of Washington. The chief result is the theorem that $\tau = (p^4 - 1)/(p - 1)$, when $p > 3$. For the case $p = 3$, the problem is equivalent to the determination of the 27 lines on a general cubic surface. Considerable headway has also been made towards the determination of all the subgroups of the quaternary linear abelian group modulo p, this group having the same relation to the hyperelliptic modular theory that the linear fractional congruence group bears to the elliptic modular theory. The paper was published in the *Transactions* for January, 1905.

7. The fourth paper by Professor Dickson treats of the general mutual correspondence between the elements of two groups A and B, such that to each element of either correspond one or more elements of the other, and such that products correspond. Let $A'[B']$ be the set of elements of $A[B]$ which correspond to the identity of $B[A]$. The sets A' and B' are semigroups, not necessarily groups if A and B are infinite. This was shown by explicit examples, in one of which neither A' nor B' contains the identity. But if one of the semigroups A', B' is a group, the other is also. If we omit his postulate that the set is finite, we obtain from Weber's definition of a group a set of independent postulates for a semigroup. A semigroup contains at most one identity and at most one inverse of any element. The paper will appear in the April number of the *Transactions.*

8. In Dr. Smith's paper the surfaces whose total curvature, in terms of the parameters of the asymptotic lines, is of the form $k = -1/[\phi(u) + \psi(v)]^2$ are classified, and their associate surfaces studied. The discussion leads to the determination of the

cartesian coördinates of a new class of surfaces depending on three arbitrary functions, which may be deformed with preservation of a conjugate system, one family of the invariant conjugate system being lines of zero length. They are the associates of a class of surfaces first considered by Stäckel, on which the two families of lines of curvature fall together into a single family.

9. Dr. Veblen's paper contains a set of independent postulates for the linear continuum. They are stated in terms of order relations alone, with no notion of laws of operation or equality of segments. The main element of novelty is a "postulate of uniformity" which replaces Cantor's assumption of a numerable everywhere dense subset and leads to the following proposition:

About every element x of the continuum there is a sequence of segments $\{\sigma_k^{(x)}\}$, $(k = 1, 2, 3, \cdots)$ such that: (1) $\sigma_k^{(x)}$ includes $\sigma_{k+1}^{(x)}$; (2) x is the only point that lies on every $\sigma_k^{(x)}$ $(k = 1, 2, 3, \cdots)$; (3) if σ is any segment whatever, there exists an n (independent of x) such that no segment $\sigma_n^{(x)}$ contains every point of σ.

10. Dr. A. E. Young concludes the discussion of surfaces defined by the quadratic forms $\lambda(du^2+dv^2)$ and $D(du^2 \pm dv^2) + 2D'dudv$. In a paper given last spring before the Society he showed the existence of surfaces having the linear element $(u+v)^2\left(\frac{du^2}{f(u)}+\frac{dv^2}{\phi(v)}\right)$ when referred to lines of curvature. Proceeding with the discussion of the surfaces having this linear element, he finds that the cartesian coördinates may be obtained without the solution of Riccati equations as is customary. Making use of this fact, he finds the cartesian coördinates of the surfaces in question. In the most general case, at least one set of the lines of curvature are plane curves, either transcendental curves resembling somewhat the cycloids, or else the well-known nodal cubic curves. However, for certain forms of the functions involved, both sets of the lines of curvature are plane curves. In particular, there are surfaces on which both sets of the lines of curvature are nodal cubics. A model of these latter surfaces was shown.

11. A polar tetrad to a quadric surface is constituted by 4 points, every pair of which are conjugate with respect to that

surface. Can such a tetrad vary while its points remain upon a twisted cubic curve? Professor White showed that while this is not possible for every cubic and quadric, it can occur for special quadrics invariantively conditioned by the cubic curves. Given the curve and two polar tetrads upon it, the equation of the surface was explicitly determined, a surface triply tangent to the curve. The problem was extended to polar pentads of a cubic surface, to forms of all orders above the first in 4-space, and to cubics, quartics, etc., in the plane, the vertices moving always upon the rational norm-curve of the space under consideration.

EVANSTON,
February 4, 1905.

THOMAS F. HOLGATE,
Secretary of the Section.

MATHEMATICS AT THE ST. LOUIS CONGRESS, SEPTEMBER 20, 22, AND 24, 1904.

IN the scheme of the Congress of Arts and Science connected with the Louisiana Purchase exposition, mathematics was classified as a department under the division of normative science, philosophy being the other department of that division. At 10 o'clock on Tuesday, September 20, both departments met and listened to the divisional address entitled "The science of the ideal," by Professor Josiah Royce, of Harvard University. An abstract of a paper itself so condensed as this is hardly a possibility; the paper has been published in *Science* (October 7, 1904).

Immediately following this joint session was the opening session of the department of mathematics, later to be subdivided into three sections. The officers of this session were Professor H. S. White, chairman, and Professor G. A. Bliss, secretary. Two addresses had been provided for this session, both of a highly general character, as distinguished from the more special discussions of sectional meetings. The first, by Professor Maxime Bôcher of Harvard University, was upon "The fundamental conceptions and methods of mathematics"; the second, by Professor James Pierpont of Yale University, on "The history of mathematics in the nineteenth century." Both have appeared already in this BULLETIN.* The auditors at this, as

* This volume, pages 115–135 and pages 136–159 respectively.