ordered" classes and the transfinite numbers will serve as an introduction to the study of these most recent accessions to the list of mathematical concepts. The matter is of interest to the philosopher as well as to the mathematician; and the present exposition is intended especially for the general student of scientific method, who, without technical mathematical training, wishes to keep in touch with the modern development in the logic of mathematics. The mathematical prerequisites have been reduced to a minimum; the demonstrations are given in full; all new concepts are defined explicitly by sets of independent postulates; and in connection with each definition numerous examples are given, to illustrate, in a concrete way, not only the systems which have, but also those which have not, the property in question. The paper is being published in the *Annals of Mathematics* for July and October, 1905.*

The chapter headings are as follows: On classes in general; Ordered classes, or "series"; Discrete series, especially the type of order exhibited by the natural numbers; Dense series, especially the type of the rational numbers; Continuous series, especially the type of the real numbers; Continuous series in more than one dimension, with a note on multiply-ordered classes. An appendix treats of Cantor's "well-ordered" classes, and the transfinite numbers, and there is an index of technical terms. The paper contains also a bibliography of Cantor's writings on these subjects.

F. N. COLE,
Secretary.

A SET OF GENERATORS FOR TERNARY LINEAR GROUPS.

BY MISS IDA MAY SCHOTTENFELS.

(Read before the American Mathematical Society, September 17, 1904.)

The following is a proof that the substitutions

$$c_h: \quad x'_i = x_{i+1},\ x'_k = x_1 + hx_2 \qquad (i = 1, 2, \cdots k-1;\ h = 0, 1)$$

generate (1) the ternary linear substitution group with integral

* Reprints of this and other papers published in the *Annals* can be ordered from the Publication Office of Harvard University.

coefficients of determinant unity, and (2) the linear and linear-fractional Galois field groups $[2^1]$.*

In recent literature on linear and linear-fractional congruence groups, entrance to such groups has frequently been made by means of the above substitutions.†

In the proof of (1) the methods of Burnside ‡ are employed, and in (2) an analogue of this method is made use of for the Galois field $[2^1]$.

1. All substitutions of the form

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3,$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3,$$
$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3,$$

with coefficients rational integers of determinant unity can be formed by combinations of the two following substitutions:

$$T = (x_2 \cdot x_3 \cdot x_1), \qquad S = (x_2 \cdot x_3 \cdot x_2 + x_1).$$

The following combinations only are made use of in this proof:

$$P = TST = (x_1 \cdot x_2 + x_3 \cdot x_3), \quad Q = T^2S = (x_1 \cdot x_2 \cdot x_3 + x_1),$$
$$U = PV^{-1}P = (x_1 \cdot x_3 \cdot - x_2), \; W = ST^{-1} = (x_1 + x_2 \cdot x_2 \cdot x_3),$$
$$V = (T^2S^2T^{-1})^2(ST^{-1})^{-2}(T^2S)^{-2} = (x_1 \cdot x_2 \cdot x_3 + x_2).$$

From the form of P, Q, W, V and $T^n VT^{2n}$ $(n = 0, 1, 2)$ it is evident that T and S give rise to six substitutions in which two of the symbols are unchanged and the third is reinforced by either of the unchanged symbols.

From the form $T^n U^m T^{2n}$ $(n = 0, 1, 2; m = 1, 2, 3)$, derived from U, it is evident that T and S give rise to all substitutions in which one of the symbols remains unchanged.

The remainder of this proof consists in showing that corresponding to any substitution whatever, say Σ, a substitution

* Krazer, *Annali di Matematica*, (2) vol. 12, pp. 283–300. Kronecker, *Monatsberichte*, Berlin Akademie, Oct. 15, 1866, p. 597.

† Moore, "Concerning the general equations of the seventh and eighth degrees." *Mathematische Annalen*, vol. 51, p. 436. Schottenfels, *Annals of Mathematics*, 2d ser., vol. 1, No. 3; BULLETIN, 2d ser., vol. 6, pp. 440–443.

‡ Burnside, *Messenger of Mathematics*, vol. 24, p. 109.

can be formed by a finite number of combinations and repetitions of T and S which, performed after Σ, leads to identity, and which is therefore Σ^{-1}. The inverse of Σ^{-1}, which in turn can be formed by a finite number of combinations and repetitions of T and S, is therefore equal to Σ.

Let

$$\Sigma = (\alpha x_1 + \beta x_2 + \gamma x_3,\ \alpha' x_1 + \beta' x_2 + \gamma' x_3,\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3)$$

where $\gamma \geqq \gamma'$.

$$\Sigma W^m = ([\alpha + m\alpha']x_1 + [\beta + m\beta']x_2 + [\gamma + m\gamma']x_3,$$
$$\alpha' x_1 + \beta' x_2 + \gamma' x_3,\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3),$$

or

$$\Sigma W^m = (\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3,\ \alpha' x_1 + \beta' x_2 + \gamma' x_3,$$
$$\alpha'' x_1 + \beta'' x_2 + \gamma'' x_3),$$

where the integer m, positive or negative, can always be chosen so that γ_1 is numerically less than γ'. Hence we have

$$\Sigma W^m (T^2 V T)^{m'} = (\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3,\ \alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3,$$
$$\alpha'' x_1 + \beta'' x_2 + \gamma'' x_3),$$

where m' can be chosen so that γ_2 is numerically less than γ_1.

After a finite number of such substitutions, a substitution must be arrived at in which one of the two coefficients corresponding to γ and γ' in Σ is zero; and, if it is the coefficient corresponding to γ', combining with T^2UT gives a substitution in which the coefficient corresponding to γ vanishes and Σ is reduced to

$$\Sigma' = (ax_1 + bx_2,\ a'x_1 + b'x_2 + c'x_3,\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3).$$

A precisely similar series of substitutions will reduce this to a form in which the coefficient corresponding to c' vanishes; and on the other hand since the determinant of the substitution remains unity, the coefficient corresponding to γ'' must become ± 1. By combining with U^2 if necessary, this last coefficient may be made $+1$, and since the coefficients a and b are unaffected by this set of substitutions, Σ' becomes

$$\Sigma'' = (ax_1 + bx_2,\ a_1 x_1 + b_1 x_2,\ a_2 x_1 + b_2 x_2 + x_3).$$

By repeating a set of substitutions similar to the first set, Σ'' may be reduced to a substitution in which b vanishes, and b_1 becomes $+1$; and since the determinant remains unity, a at the same time becomes $+1$, and Σ'' becomes

$$\Sigma''' = (x_1,\ a_1'x_1 + x_2,\ a_2x_1 + b_2x_2 + x_3).$$

Finally $(T^2UT)^{-a_1'}Q^{-a_2}.\ V^{-b_2}$ reduces Σ''' to

$$\Sigma^{\mathrm{IV}} = (x_1,\ x_2,\ x_3) \text{ or identity.}$$

2. All matrices of the form (α_{ij}) $(ij = 1, 2, 3)$

$$\begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{matrix}$$

where α_{ij} are marks of the Galois Field $[2^1]$ and where the matrix (α_{ij}) is such that its determinant is unity can be formed by combinations of the two following matrices

$$T = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad S = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

The following combinations only are made use of in this proof:

$$P = TST = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \qquad Q = T^2S = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \qquad W = ST^{-1} = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$V = (T^2S^2T^{-1})^2 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \qquad U = PV^{-1}P = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

As in case (1), P, Q, W, V and the form $T^nVT^{2n}(n = 0, 1, 2)$ give rise to six substitutions in which two of the symbols are unchanged and the third is reinforced by either of the unchanged symbols. Also the form $T^nUT^{2n}(n = 0, 1, 2)$ contains all the substitutions in which one of the symbols remains unchanged.

The remainder of this proof proceeds as in case (1). Let

$$\Sigma = \begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix};$$

if $\gamma < \gamma'$, then $\gamma = 0$ and Σ becomes

$$\Sigma' = \begin{matrix} \alpha & \beta & 0 \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix};$$

otherwise $\gamma \geqq \gamma'$.

$$\Sigma W = \begin{matrix} \alpha + \alpha' & \beta + \beta' & \gamma + \gamma' \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix} \quad \text{or} \quad \Sigma W = \begin{matrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix}.$$

(1) If $\gamma = \gamma'$, then $\gamma_1 = 0$. (2) If $\gamma > \gamma'$, then $\gamma' = 0$.

In (1) ΣW reduces to Σ'. In (2) we combine ΣW with $T^2 UT$ and again get

$$\Sigma' = \Sigma W T^2 UT = \begin{matrix} \alpha' & \beta' & \gamma' \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha'' & \beta'' & \gamma'' \end{matrix},$$

where $\gamma' = 0$,

$$\Sigma' P = \begin{matrix} \alpha' & \beta' & 0 \\ \alpha_1 + \alpha'' & \beta_1 + \beta'' & \gamma_1 + \gamma'' \\ \alpha'' & \beta'' & \gamma'' \end{matrix} \quad \text{or} \quad \Sigma' P = \begin{matrix} \alpha' & \beta' & 0 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha'' & \beta'' & \gamma'' \end{matrix}.$$

If $\gamma_1 < \gamma''$, then $\gamma_1 = 0$, and Σ' becomes

$$\Sigma'' = \begin{matrix} \alpha' & \beta' & 0 \\ \alpha_1 & \beta_1 & 0 \\ \alpha'' & \beta'' & \gamma'' \end{matrix}.$$

In the remaining case $\gamma_1 \geqq \gamma''$.
(1) If $\gamma_1 = \gamma''$ then $\gamma_2 = 0$. (2) If $\gamma_1 > \gamma''$ then $\gamma'' = 0$.

In (1) $\Sigma' P$ reduces to Σ''. In (2) $\Sigma' P$ combined with U gives

$$\Sigma'' = \begin{matrix} \alpha' & \beta' & 0 \\ \alpha'' & \beta'' & 0. \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

$$\Sigma'' W = \begin{matrix} \alpha' + \alpha'' & \beta' + \beta'' & 0 \\ \alpha'' & \beta'' & 0, \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix} \quad \text{or} \quad \Sigma'' W = \begin{matrix} \alpha_3 & \beta_3 & 0 \\ \alpha'' & \beta'' & 0. \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

If $\beta' < \beta''$ then $\beta' = 0$ and Σ'' becomes

$$\Sigma''' = \begin{matrix} \alpha' & 0 & 0 \\ \alpha'' & \beta'' & 0. \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

In the remaining case $\beta' \geqq \beta''$.

(1) If $\beta' = \beta''$ then $\beta_3 = 0$. (2) If $\beta' > \beta''$ then $\beta'' = 0$. In (1) $\Sigma'' W$ reduces to Σ'''. In (2) $\Sigma'' W$ combined with $T^2 UT$ gives

$$\Sigma''' = \begin{matrix} \alpha'' & 0 & 0 \\ \alpha_3 & \beta_3 & 0, \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

Finally $\Sigma''' T^2 UTQV$ gives Σ^{IV}, which is identity,

$$\Sigma^{\mathrm{IV}} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0. \\ 0 & 0 & 1 \end{matrix}$$

In conclusion, the generators T and S give rise to the simple group of order 168 with generational definition as follows:

$$G_{168} = \{T^3 = S^7 = (ST^2)^2 = (TS^2)^4 = I\}.*$$

NEW YORK, *June*, 1905.

* Schottenfels: "Upon the non-isomorphism of two simple groups of order 8!/2 or 20160," BULLETIN, 2d ser. vol. 8, no. 1, p. 26, § 2.