THE NEW CALCULUS OF VARIATIONS.

THE title above will indicate the intention of the writer to allow himself more freedom of comment in the present paper than is usual in a review. In attempting to form a just estimate of Kneser's Lehrbuch* and in perusing Bolza's Lectures † as well as in the preparation of another paper ‡ dealing with the calculus of variations, the several books and memoirs which I shall mention have necessarily come to my attention, even though some of them have not been used extensively. It appears to me that a short paper which shall give an idea of the various books now before the public would serve two purposes: the valuable one of giving a comparative view of all, and the convenient one of condensing into one a number of separate reviews which might eventually weary the reader.

Of the old calculus of variations the mathematical public knows well. A book which will undoubtedly stand for all time as the last exposition of that theory in what was its best form is Pascal's Calcolo delle variazioni,§ which was published in German translation by Teubner in 1899.‖ That this translation — which is the edition to which we shall constantly refer — was thought worthy of publication and was actually the best extant treatise in 1899 is a curious commentary upon the suddenness with which the modern theory leaped into the public arena and upon the secrecy in which the previous developments of that theory had been veiled, especially when we note that the very next year is the date of Kneser's now famous Lehrbuch. To be sure Zermelo had in his thesis given the essence of the Weierstrass theory, and the papers by Zermelo and Kneser seem to have been familiar at least to the translator (see page 65 and footnote, page 65). But if evidence were needed that Weierstrass's theory was not generally known or that the papers mentioned had failed to make a noticeable impression upon the general mathematical public, one need not go beyond the present book in search of it, for the influence of the Weier-

* Lehrbuch der Variationsrechnung, by A. Kneser, Braunschweig, Vieweg, 1900, 8vo, 306 pp.

† Lectures on the Calculus of Variations, by O. Bolza. See BULLETIN, vol. 12, No. 2 (November, 1905) pp. 80–90.

‡ Article on the Calculus of Variations, Encyclopedia Americana, 1905.

§ Calcolo delle variazioni, by E. Pascal, Milan, Hoepli, 1897.

‖ Variationsrechnung, by E. Pascal, translated into German by A. Schepp, Leipzig, Teubner, 1899, 8vo, 146 pp.

strass theory upon this work is at most meagre. Pascal intended to make his book largely historical and he succeeded. This success now saves it from the oblivion which is the fate of many another book of scientifically the same class. The fourteen pages of historical matter at the beginning give only a partial idea of the amount of useful information of that nature in the book, for every chapter contains thorough references and some of the chapters are little else than abstracts of the theory and historical comment.

I shall not enter upon a detailed discussion of the contents, since the problems treated and the methods used are generally well known. Suffice it to say that nearly every problem of the old theory is mentioned in its general form: the simplest problem, the extensions to cases where higher derivatives enter the integrand, where multiple integrals occur, where auxiliary conditions are imposed, and so on. Thus most of the formal results are presented here in compact and convenient form. But the critical spirit is not so marked as it must be henceforth. As a single example may be mentioned the absence of exact statements concerning the very definition of a minimizing curve. Indeed the author points out (pages 16 and 17) that "certain categories of curves" are arbitrarily excluded as comparison curves on account of the fact that the nature of the "variations" allowed restricts the comparison curve so that "all the derivatives approach the derivatives of like order for the supposed solution." But this remark goes no further than did the almost identical remark made by Legendre (see § 30, page 109). Surely Zermelo's dissertation should have been mentioned in this connection, especially in the discussion of Newton's problem (§ 30),* and the failure to take account of Zermelo's exposition of Weierstrass's theory results, both here and elsewhere, in the well-known misstatements common to all the older works.

Finally, one feature which remains valuable should be mentioned. Practically the whole of the last thirty pages — nearly one-fourth of the book — is devoted to special problems and applications, including most of the famous problems and such general theories as those of minimum surfaces, geodetic lines, etc. While the treatments given do suffer from the lack of precision noted above, it should be kept in mind that the

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* Compare Kneser, Encyklopädie d. Math. Wiss., II A 8, p. 609.

solutions obtained must include the rigorous solutions at any rate, and that the curves which are declared to be minimizing curves would in general at least render the integral a "weak" minimum. In seeking a rigorous solution, therefore, these results are still by no means to be despised.

A very useful bibliography of the subject, which practically exhausts the literature up to 1890 at least, forms a fitting sequel to a book which Pascal has made curiously important from a historical standpoint.

In striking contrast to the spirit as well as to the contents of Pascal's book, there appeared in 1900 a new treatise by Kneser.* I have elsewhere † claimed that this book opened the doors of modern research in the subject to the general mathematical public, and its importance really merits a much more extended and probably a more favorable review than I shall be able to make at this time. The book would undoubtedly have been reviewed in these pages ere now, had the task seemed entirely easy and attractive to any one of several Americans who were otherwise interested in its appearance and its contents.

It is especially easy to draw broad comparisons between Kneser and Pascal. The latter's treatment is very clear, it gives a good general view of the subject as Pascal knew it; its theorems and other statements are well outlined and set in prominent types, it leans heavily toward historical comment; but it is lacking in rigor from the modern standpoint. Kneser's work, by contrast, is not lucid, its arrangement gives only a clouded view of the author's own conceptions and of the subject itself, the theorems and other similar statements are hidden amid a mass of discussion as if with conscious and consummate cunning, it shows an apparent tendency to conceal historical development, but it is a vast advance over any former work in its exactness.

Kneser published at the same time an article in the Encyklopädie,‡ which has been sent out only this year, but which was written and set up along with the Lehrbuch, and which should be considered at the same time. I shall refer to this second treatment as Kneser's *article*.

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* See first footnote, p. 172.

† See BULLETIN, loc. cit.

‡ Encyklopädie der Math. Wiss., II A 8., pp. 571–625. Article entitled Variationsrechnung.

In considering these two treatments by Kneser it should be noted that both of them formally appear in print under the date 1900, though the article has not been given to the public until very recently. It follows that we shall not expect to find in either of them any of that fundamental work which Hilbert has been doing since 1900. It will be very useful in this connection to refer to Bolza's Lectures on the calculus of variations, which has been reviewed in the BULLETIN,* for Bolza in his chapters numbered V and VI follows Kneser rather closely. On account of the greater clearness of style, and on account of the development of the theory in the interval between the two dates of publication, Bolza's presentation is an enlightening commentary upon Kneser's work — an almost indispensable aid in deciphering much that is obscure in Kneser's own point of view. Surely a review of Kneser's Lehrbuch is immensely simplified through the existence of Bolza's book.

The first striking characteristic of Kneser's Lehrbuch is his exclusive use of the parameter representation of Weierstrass. The standpoint assumed by Bolza in his Lectures † will appeal to most teachers on account of its pedagogical correctness. Thus Bolza apparently appreciates the parameter representation quite as fully as does Kneser, but he sees and utilizes the possibility of giving the main essentials of the theory in the simpler asymmetric form first, and then passing to the more involved parameter representation later, by means of generalizations which are often trivially simple when the main facts are already established. This advantage in style Kneser missed: in consequence his work suffers in clearness, though it might be possible to write a very readable book based on the parameter representation alone. Aside from this, however, I am inclined to believe with Bolza (Lectures, page 115) that it is unjustifiable to discard the asymmetrical forms entirely in favor of the parameter representation.

Let us pass in review briefly the various chapters. The first chapter contains a statement of the problem in parameter form and the formal transformation of the first variation. The theorems of Weierstrass concerning integrals which are independent of the choice of parameter, and the corresponding methods of transforming the first variation were previously known only through the meagre publications of Weierstrass's students, and

* Vol. 12, No. 2 (November, 1905), pp. 80–90.

† See second footnote, p. 172.

form a necessary and important part of the introduction. The assumption on page 10 that the second derivatives exist for the solution is now known to be unnecessary, through the DuBois-Reymond-Hilbert proof, but such an assumption does not vitally injure the work and might even be defended as justifiable by one who was conscious of its superfluity.* The non-parameter form is explained in § 5 as a special or dependent case. Kneser's article in the Encyklopädie shows a striking contrast to the treatment of the Lehrbuch in that the parameter representation is mentioned in the article only in the briefest possible manner. The difference may be due to the great limitation which the author imposes upon himself in the use of work due to Weierstrass in the article (see Encyklopädie, II, A, 8, page 608, footnote 121*a*). In the Lehrbuch, though Kneser limits himself to work actually published by others, he does at least permit himself enough freedom to use Weierstrass's essential methods as presented by Zermelo and others. Again, the article is historical to a large extent, so that Weierstrass's results consume relatively little space, and the impossibility of giving an account of earlier investigations in parameter form may well account for the extreme difference between the Lehrbuch and the article.

The second chapter of the Lehrbuch (pages 10–43) is devoted to the derivation of necessary conditions. The proof of Euler's (or Lagrange's) condition follows easily from the transformed first variation by means of DuBois-Reymond's lemma (§8). In this connection, Kneser dwells upon the now familiar idea, which is due to Kneser himself, of the extremals, i. e., the solutions of Euler's equation. After giving a list of examples, Kneser proceeds in § 10 to what is perhaps the key note of his whole theory :† the consideration of integrals in which the end points are variable. As a matter of fact, these problems lead naturally to a generalization of the theorems concerning geodetic lines on a surface, especially after the introduction of the important idea of transversal curves. The whole problem of variable end points may also be thought of as an analogon of the boundary value problems of the theory of differential equations ; it is perhaps even more interesting from this point of view than from the analogy to geodetics, but both points of view should be kept in mind and both might

* Compare, for example, the results in § 17, p. 58.

† See Bolza, Lectures, Chapter V.

have been emphasized even more than Kneser has seen fit to do, without overstepping the proper bounds of the calculus of variations. A brief consideration of the isoperimetric problem closes the chapter.

One expects to find next a treatment of the so-called "second variation," according to the methods of Jacobi and his school. This expectation is based wholly on tradition and on the stress which Kneser lays in his article on the same methods. Here, however, Kneser leaves the beaten paths absolutely and begins at once a consideration of the sufficient conditions. The ideas developed are new and important in almost every instance. The idea of a field of extremals (§14) is the first and most fundamental. The following article on the extension of Gauss's theorems concerning geodetics is scarcely less important. Then follow the essential methods according to Kneser's theory for the proof of the sufficient conditions (§§ 16, 17), including the introduction of Kneser's curvilinear coordinates based upon a set of extremals and a corresponding set of transversals, and the essential distinction between strong and weak minima. These articles lead up to sufficient conditions, which Kneser sums up on page 60 under the titles "Jacobi's condition" and "Legendre's condition." The former is essentially the same condition as that elsewhere given as Jacobi's condition. The condition called "Legendre's" is that known to Legendre only in the case of weak minima. In the case of strong minima the condition is radically different, since it practically involves "Weierstrass's" condition, and might well be called "Weierstrass's" condition rather than "Legendre's." I shall not discuss in detail the remainder of this chapter, though it is important throughout. Among the topics considered are the methods of Weierstrass, including the E function; the envelope of a field of extremals; the Jacobi-Hamilton theorems. An amount of this matter will be found either reproduced or independently collected in Bolza's Lectures. A few errors or oversights in this chapter have been noted by Bolza in his Lectures and by Zermelo and Hahn in the Encyklopädie. Thus Kneser's proof of the existence of a field (§ 14) must be supplemented by a theorem due to Osgood (see Bolza, Lectures, page 176); and the statements of Jacobi's condition require certain revision (see Encyklopädie II, pages 630, 632), which have been given by Osgood, Bolza, and Bliss in memoirs quoted by Zermelo and

Hahn. The latter revisions have been conducted by means of considerations of the so-called "second variation," which would seem to indicate a present necessity for retaining the older considerations of Jacobi, at least until the proofs in question have been made upon the basis of the newer theories.

Kneser's fourth chapter treats the isoperimetric problem both for fixed and for variable end points. The principal methods and results of the chapter are presented in somewhat simpler form by Bolza (Lectures, Chapter VI). While the theorems are frequently essentially the same as those previously stated, some of the results and some of the methods employed are new. Any future work on this subject will necessarily use many of the ideas which Kneser introduces in this chapter.

The last four chapters are devoted to discontinuous solutions, the problem in which higher derivatives enter in the integrand, the most general problem of the calculus of variations, and double integrals, respectively. These chapters are certainly important, and they make the work complete in a sense in which it could not be without them. The previous chapters give the character to the book, however, and they are the especially interesting and important portions. Such an explanation is necessary in passing them over with no detailed mention, in order to avoid a misconception on the part of the reader. The high plane upon which Kneser has placed his work in the earlier chapters is fully maintained to the end.

In general, Kneser's Lehrbuch must surely be assigned a very high scientific value, and its many contributions will surely remain essential for the further development of the subject for many years. It is only from the pedagogic side that essential criticism can justly be made. The work is of monumental importance; it might have been of truly immense influence upon all classes of students of mathematics, had it not been for the unfortunate style of the author and for the resulting unnecessary complication and intricacy of statement and proof which characterize it from the standpoint of presentation. A previous remark of mine * to the same effect has been misunderstood in a curious manner by Professor Haussner (*Fortschritte der Mathematik*, volume 33, page 379), who himself repeats my own criticism and my own praise in slightly greater detail. For this reason I might now insist that a presentation can justly be criticized for its complication when and only when it is unneces-

* BULLETIN, vol. 9, No. 1, October, 1902, p. 24.

sarily intricate, i. e., in so far as the complication arises from the style of the writer rather than from the nature of the problem. And can there be any doubt that any intrinsic difficulty or intricacy in the subject matter itself calls for especial care upon the part of a writer who wishes adequately to present the subject or any portion of it?

Kneser's article in the Encyklopädie, which has been quoted above, is scarcely to be classed with his Lehrbuch. Its whole tone is rather historical than scientific. Its value is further limited by the attitude of the author toward Weierstrass's work, which, though well known through practically public lecture notes, is refused recognition here except when accidentally published by another author. The scruple against presenting the unpublished work of another man is honorable in general, but in this instance no possible harm could have been done, and the fame of Weierstrass would be better guarded by a frank reference to his unpublished work than by silence at a time when a large part of the modern developments in the calculus of variations can be traced directly or indirectly to his influence. The references given in footnotes supplement to a very considerable degree the lack of such references in the Lehrbuch; otherwise the Lehrbuch is a more systematic presentation of the subject and is, of course, much more detailed in its treatment than the article. It would therefore seem that the only present value of the article is its historical value, including its copious notes.

The article by Zermelo and Hahn,* which immediately follows Kneser's article in the Encyklopädie, is much more important at the present moment, since the work reviewed in it is not as yet fully reproduced in any treatise. To be sure a number of the results are given by Bolza in his Lectures, but there are very many points of interest upon which Bolza does not touch. This article covers the period from 1900 (when Kneser's article was finished) to 1904 (when both articles actually appeared). It is probable that no equal period of time has seen such an advancement in the subject as this, and it is certain that at no time have so many minds been devoted to its investigation. The article covers only fifteen pages and, therefore, it is necessarily only a summary of facts. The footnotes, however, con-

* Encyklopädie der Math. Wiss. II A 8*a*, article entitled "Weiterentwickelung der Variationsrechnung in den letzten Jahren," pp. 626–641.

tain complete references to the original sources. The recent work of the Hilbert school seems to predominate in the exposition in the constant recurrence of the important ideas of the Hilbert invariant integral and the Hilbert a priori existence proof. Of course many authors in no way allied with Hilbert are repeatedly mentioned. In particular the work of Weierstrass was an essential precursor of that of Hilbert, and Kneser's work is of great importance, independently of any of Hilbert's work. The next most prominent names seem to be those of Osgood, Bolza, and Bliss; these three Americans and to a lesser extent a few other Americans have not only made the modern calculus of variations familiar to the American public, they have advanced to a marked degree the general knowledge of the subject in the important papers referred to in this article. The topics are the Weierstrass theory, the simplest problem, the isoperimetric problem, the general problem, applications, and the existence proofs. In many particulars this article is at present the most authoritative and exhaustive presentation of the results of recent investigations in the subject.

The last book I shall mention is also of American origin.* It represents substantially the lectures delivered in regular courses on the calculus of variations by the author, Professor Harris Hancock, at the University of Cincinnati. Professor Hancock bases his work almost exclusively upon the lectures of Weierstrass and Schwarz which he heard at Berlin, and consciously disregards the developments of later years. Since the work of Weierstrass is already fairly well known in even more authoritative form through the well-known lecture notes at Berlin, the present book is shorn of much of the value it might otherwise have from a historical standpoint. An official publication of Weierstrass's work by those who now hold his original papers would possess extreme interest in showing precisely Weierstrass's point of view, whereas it is not always evident in the present book whether the statements and the proofs, in the precise form in which they are given, are due to Weierstrass or to Hancock. For this reason it is difficult and rather unnecessary to review the book in detail. It may be noted that a considerable portion of the contents has already been published by Professor Hancock in the *Annals of Mathematics* (first series, volumes 9–12).

* Lectures on the Calculus of Variations, by H. Hancock, Cincinnati, University Press, 1904, 8vo, 292 pp.

Beside the books and articles mentioned in this review, several of the recent treatises on the calculus contain a chapter on the subject. Among these should be mentioned at least the new edition of Serret-Bohlmann, Differential and Integralrechung, volume 3, and Goursat, Cours d'Analyse, volume 2. These books will doubtless be reviewed in their entirety in the BULLETIN, and I therefore satisfy myself here with a mere mention.

The works mentioned attest in the strongest possible manner the extraordinary vogue into which the calculus of variations has suddenly sprung. The cause is not far to seek: it is the revelation through the work of Weierstrass, Kneser, Hilbert and others that the calculus of variations is susceptible of the same exquisite rigor which had previously existed only in the theory of functions of a real variable, and that a wide field of research and rich discovery was opened by such methods.

Although the end of these investigations has by no means been reached in this single subject, it is not premature to suggest the analogous development of other mathematical theories along equally rigorous lines, and also the construction of a supplementary theory in each of them which shall be as rigorously applicable to general geometric problems as is the Weierstrass theory in the calculus of variations. E. R. HEDRICK.

COLUMBIA, MO.,
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GRANVILLE'S DIFFERENTIAL AND INTEGRAL CALCULUS.

Elements of the Differential and Integral Calculus. By W. A. GRANVILLE, Ph.D., with the editorial cooperation of PERCY F. SMITH, Ph.D. Ginn & Co., 1904. 463 pp.

So many text-books have been written upon the elementary branches of college mathematics that a raison d'etre can properly be asked upon the appearance of each new work. The great number of American text-books upon such subjects as college algebra, trigonometry and calculus, duplicating one another in aim and character, is in striking contrast with the paucity of our text-books upon more advanced mathematical subjects. What, then, we naturally ask, is the purpose of this new treatise, and what does it seek to accomplish which has not been already accomplished?