THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE ninth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at Stanford University, on Saturday, February 24, 1906. The following sixteen members were present:

Professor R. E. Allardice, Mr. A. J. Champreux, Professor R. L. Green, Professor M. W. Haskell, Mr. Edwin Haviland, Professor L. M. Hoskins, Professor D. N. Lehmer, Dr. J. H. McDonald, Dr. W. A. Manning, Professor G. A. Miller, Professor H. C. Moreno, Dr. T. M. Putnam, Mr. Arthur Ranum, Professor Irving Stringham, Professor A. W. Whitney, Professor E. J. Wilczynski.

The attendance also included a number of teachers of mathematics and physics who are not members of the society. A morning and an afternoon session were held, Professor Allardice acting as chairman at both sessions. It was agreed to hold the next meeting at the University of California on September 29, 1906.

The following papers were read at this meeting:

(1) Dr. J. H. McDONALD: "The theory of the reduction of hyperelliptic integrals of the first kind and of genus 2 to elliptic integrals by a transformation of the nth order."

(2) Dr. W. A. MANNING: "On multiply transitive groups."

(3) Mr. ARTHUR RANUM: "A new kind of congruence group and its application to the group of isomorphisms of any abelian group" (preliminary report).

(4) Professor D. N. LEHMER: "On the orderly listing of substitutions."

(5) Professor D. N. LEHMER: "Note on the values of z of given modulus which give maximum or minimum values to the modulus of a given rational integral function of z."

(6) Professor R. E. ALLARDICE: "Note on Legendre's equation."

(7) Professor R. E. ALLARDICE: "On the multiple points of unicursal curves."

(8) Professor E. J. WILCZYNSKI: "Outline of a projective differential geometry of curved surfaces" (preliminary report).

(9) Mr. E. T. BELL: "Method of dealing with the problems connected with prime numbers" (preliminary report).

(10) Dr. T. M. PUTNAM: "Theorems on perfect numbers."

(11) Dr. J. H. McDONALD: "A method of simultaneous approximation to two consecutive roots of an algebraic equation of degree n all whose roots are real."

(12) Dr. J. H. McDONALD: "Remarks on the calculation of roots of Bessel functions."

(13) Professor M. W. HASKELL: "On collineations" (preliminary report).

(14) Professor G. A. MILLER: "Groups in which every subgroup of composite order is invariant."

(15) Professor G. A. MILLER: "The groups which contain exactly thirteen operators of order 2."

Mr. Bell was introduced by the Secretary of the Section. Professor Miller's second paper has appeared in the March number of the BULLETIN. Abstracts of the other papers are given below. The abstracts are numbered to correspond to the titles in the list above.

1. According to the Weierstrass-Picard theorem, when there corresponds to the algebraic relation $y^2 = R_6(x)$ (where $R_6(x)$ denotes a polynomial of the 6th degree) a single reducible integral, then a set of normal integrals may be found, both being reducible, for which the table of canonical periods is known. Dr. McDonald examined the problem in purely algebraic formulation and established a theorem which concerns the relations between the two reducible integrals and shows how to construct either when the other and its reducing substitution are known.

2. If a doubly transitive group of degree n has $r + 1$ transitive subgroups $H, H_1, \cdots, H_r$, of degrees $q, q + q_1, \cdots, q + q_1 + \cdots + q_r$, less than n, it is possible, as Dr. Manning proves, to group the letters of H q_r by q_i in systems of imprimitivity in at least

$$q_i\left(1 + \frac{1}{q_r} + \frac{1}{q_{r-1}} + \cdots + \frac{1}{q_i}\right)$$

distinct ways, for $i = 1, 2, \cdots, r$. This is a generalization on the theorems of Jordan and Marggraff on primitive groups with transitive subgroups of lower degree. A number of more special theorems on multiply transitive groups are given in this paper.

3. Mr. Ranum proved that the group of isomorphisms I of any abelian group G of order p^m having n invariants can be expressed as a linear congruence group in n variables, such that each row (or column) of a matrix has its own modulus, namely the corresponding invariant. A general formula is obtained for the order of any such group I. Its series of composition is found; and if $p > 3$, I is shown to be solvable when, and only when, no two of its invariants are equal, while if $p = 2$ or 3, I is solvable when no three of its invariants are equal.

Corresponding to every characteristic subgroup H of G, it is shown that I has an invariant subgroup J whose isomorphisms transform all the operators of H into themselves and whose matrices are congruent to identity when the moduli are the invariants of H. Moreover, corresponding to the characteristic quotient group H' of the same type as H, I has an invariant subgroup J' of the same type as J, whose matrices are obtained from those of J by interchanging rows and columns.

Finally a still more general class of linear congruence groups is mentioned, in which each coefficient of a matrix has its own modulus, and it is shown that if all the moduli in the same rows are equal, the group is necessarily the group of isomorphisms of some abelian group.

4. Professor Lehmer discussed a notation by means of which a substitution is determined by a single number and the remarkable laws arising from the notation in the resulting multiplication tables of various groups were brought out. A method was explained for constructing the multiplication table of the symmetric group on any number of letters with no computation.

5. In Professor Lehmer's second paper certain theorems were brought out concerning the family of curves into which a system of concentric circles are transformed by means of a rational integral function $f(z)$. In particular the locus of the feet of normals from the origin on this family of curves was discussed and the theorem proved that not only the curve passes through the root points of $f(z) = 0$ but that the line joining a root point of $f(z) = 0$ to the origin is tangent to this locus at that point or else forms part of the locus. The paper will be offered to the *Annals of Mathematics* for publication.

6. The equation referred to by Professor Allardice is the linear differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

In Forsyth's Differential Equations (third edition, pages 155 and following), a method is given for finding a relation between the two solutions of this equation, by putting $y = uv - w$, where v stands for one of the solutions. The expression obtained for u is $\frac{1}{2}\log\{(x+1)/(x-1)\}$, and w is obtained in infinite series. This solution is stated to be valid in all cases, except the critical cases where $2n$ is a positive or negative odd integer; though there seems to be no good reason for excluding these cases, if the solution is valid in all other cases. Another form of w is obtained as a sum of functions P_r; but this solution is restricted to the case where n is a positive integer. Now the supposed solution vanishes identically in every case, except when n is a positive integer. In the latter case, certain terms become indeterminate; if these be excluded, a correct solution is obtained, but if they be included, the expression vanishes identically. In the expression in terms of functions P_r, only one term becomes indeterminate, and the limit of this term is Q_n; and thus the expression for Q_n in terms of functions P_r is obtained.

7. The second paper by Professor Allardice was concerned with the calculation of the multiple points of a curve when the coördinates of the curve are given explicitly as rational functions of a parameter. A simple proof was first given of the known theorem that the double point of a cubic is given by the hessian (or canonizant) of a certain binary cubic. By the same method an equation of the sixth degree was then obtained for the three double points of a quartic, in the form of a determinant of the third order. A quartic curve in space was next considered, and the condition for the existence of a double point was shown to be the vanishing of the invariant I_3 of a certain binary quartic. From this condition another form of the equation for the multiple points of a plane quartic was deduced, namely the equation obtained by equating to zero a certain canonizant of two binary quartics. The canonical forms of the quartics obtained by means of this canonizant are $(A, B, C, D, E)(x, y)^4$ and $(A', \lambda B, \lambda C, \lambda D, E')(x, y)^4$.

8. The system of differential equations

$$(1)\quad y_{uu} + 2ay_u + 2by_v + cy = 0, \quad y_{vv} + 2a'y_u + 2b'y_v + c'y = 0,$$

where $y_u = \partial y/\partial u$, $y_v = \partial y/\partial v$, $y_{uu} = \partial^2 y/\partial u^2$, etc., has four linearly independent solutions, which may be taken as homogeneous coördinates of a point on a surface. The curves $u =$ const. and $v =$ const. will be asymptotic curves upon this surface. Professor Wilczynski has begun the investigation of the invariants and covariants of (1), which theory, on its geometric side, is equivalent to a projective differential geometry of surfaces.

If the independent variables are not transformed, while y is put equal to $\lambda\bar{y}$, a new system is obtained from (1), satisfied by $\bar{y}$. The invariant formations (semi-invariants and semi-covariants) for this transformation are

$$(2)\quad a',\ b,\ e = a_v - b'_u,\ f = c - a_u - a^2 + 2bb',\ g = c' - b'_v - b'^2 + 2aa'$$

and

$$(3)\quad \begin{aligned} &y,\ z = y_u + ay, \quad \rho = y_v + b'y, \\ &\sigma = y_{uv} + b'y_u + ay_v + \tfrac{1}{2}(a_v + b'_u + 2ab')y. \end{aligned}$$

The invariants are functions of the quantities (2) left unchanged by arbitrary transformation of the independent variables. a', b, and e are themselves such invariants. All of them may be expressed in terms of these three and of

$$(4)\quad h = b^2(f - b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b_u^2,\ k = a'^2(g - a'_u) - \tfrac{1}{4}a'a'_{vv} + \tfrac{5}{16}a'^2_v$$

by the help of certain operations involving differentiation processes.

The geometric interpretations are based upon the author's theory of ruled surfaces. In fact, there is associated with every asymptotic curve the ruled surface made up of the asymptotic tangents of the second set along this curve. For example, if $e = 0$, the osculating hyperboloids of the two ruled surfaces, which thus correspond to a given point of the given surface, coincide. If a' or b vanishes, the surface itself is ruled.

9. Mr. Bell's paper showed that the sifting of the primes from one to infinity can be made to depend on certain arith-

metical progressions, whose first terms are the squares of the odd numbers taken in order, and whose common differences are the odd numbers, also taken in order. This method gives all the primes uniformly, and can be used as a starting point from which to determine the number of primes less than a given integer. It is unnecessary to assume the knowledge of any prime number except 2 in dealing with the two problems of determining the primes from one to infinity, and the number of primes between given limits.

10. Dr. Putnam proved in his paper on perfect numbers that a number containing k distinct prime factors can not be perfect if all its primes are equal to or greater than $\frac{3}{2}k + 1$, the number being indeed deficient in this case. Odd perfect numbers being necessarily of the form $p^{\pi}A^2$, it was shown that if the primes in A all occur to the power 2, or all occur to the power four the number cannot be perfect.

11. The method of approximation developed in Dr. McDonald's second paper consists of a chain of quadratic equations depending on three arbitrary numbers, whose roots simultaneously converge to two consecutive roots of an equation of degree n all of whose roots are real.

12. Dr. McDonald's third paper was devoted to the following points:

(1) Genus of the function $J_n(x)$, Rayleigh's determination of least roots. (2) Calculation of early roots. (3) Determination of limit to value of possibly occurring common roots. (4) Relations among the polynomials deduced from recurring relations between Bessel functions.

13. Professor Haskell gave explicit formulas for reducing the collineations leaving a conicoid unchanged to Cremona substitutions in the plane, in such a way as to afford a basis for enumeration of the groups of such collineations, showing that the orders of such groups would be mn, or $2mn$, where m and n may take all values of the orders of groups of linear fractional substitutions of m variables. He developed the following theorem:

Any group leaving a conicoid unchanged is the product or twice the product of two groups of linear fractional substitutions on two independent variables; and conversely, any two such groups of linear fractional substitutions will generate a group leaving a conicoid unchanged.

14. Professor Miller's paper is devoted to a complete determination of the groups in which every subgroup of composite order is invariant but some subgroups of prime order are non-invariant. If the order of such a group is not a power of a single prime number it is of the form pq^2, where $p > 2$, $q + 1$ is divisible by p, and p, q are primes. The subgroup of order q^2 is of type (1,1) and is the only subgroup of composite order. Moreover, when a non-abelian group contains only one subgroup of composite order the order of the group is pq^2. The necessary and sufficient condition that every subgroup of composite order in a non-abelian and non-hamiltonian group of order p^m, $m > 5$, is invariant is that the group contains invariant operators of order p^{m-2}. If p is odd this condition is necessary and sufficient for every value of $m > 2$, and there are just two such groups for every value of m. When $p = 2$, there is one additional group when $m = 5$ and there is only one possible group when $m = 3$, viz. the octic group. Each of these groups of order p^m contains only one invariant subgroup of order p and has a commutator subgroup of order p. With the single exception of the given group of order 32, the group of cogredient isomorphisms is of order p^2. For the group of order 32 it is of order 16. The paper has been offered to the *Archiv der Mathematik und Physik* for publication.

G. A. MILLER,
Secretary of the Section.

AN APPLICATION OF THE THEORY OF DIFFERENTIAL INVARIANTS TO TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.

BY J. E. WRIGHT, M. A.

(Read before the American Mathematical Society, December 29, 1905.)

IT has been proved by Darboux * that a family of surfaces which makes part of a triply orthogonal system must satisfy a differential equation of the third order. This differential equa-

* "Sur les surfaces orthogonales" (*Bulletin de la Société philomath.*, 1866, p. 16), (*Annales de l'Ecole normale*, 1re séries, vol. 3 (1866), p 97); see Leçons sur les systèmes orthogonaux, pp. 13–14 for complete bibliography.