$\lambda/(\lambda - 1)$. Hence

$$(1) \quad q \neq 0,\ 1,\ \lambda,\ 1-\lambda,\ \frac{\lambda}{\lambda-1},\ \frac{1}{\lambda},\ \frac{1}{1-\lambda},\ \frac{\lambda-1}{\lambda} \quad (\lambda \neq 0,\ 1).$$

Here the six functions of λ are the six elements of the cross-ratio group, and each differs from 0 and 1. Hence equalities arise only when $\lambda = -1,\ 2$, or $\frac{1}{2}$, or when $\lambda^2 - \lambda + 1 = 0$.

When x_1, x_2, x_3 are distinct, $qx_1 + x_2$ is six valued only in the following cases: (i) one of the x's is an arithmetical mean between the other two, with $q \neq 0,\ 1,\ -1,\ 2,\ \frac{1}{2}$; (ii) $\Sigma x_j^2 = \Sigma x_j x_k$, with $q \neq 0,\ 1,\ \lambda,\ 1/\lambda$ (where $\lambda^2 - \lambda + 1 = 0$); (iii) neither of the relations on the x's holding, with q not equal to one of the eight distinct values (1).

It may now be readily shown that there exist six valued linear functions of the roots x_i of a cubic in the $GF[p^n]$ when $p^n > 8$; when $p^n = 7$; and when $p^n = 5$ or 8, with the x_i not all in the $GF[p^n]$.

5. In conclusion it may be remarked that the Galois theory as presented in Weber's Algebra may readily be extended to apply to modular fields, provided his argument on page 500 (of volume 1 of the second edition) be replaced by that in § 2 above.

THE UNIVERSITY OF CHICAGO,
July, 1906.

NOTE ON THE VARIATION OF THE DEFINITE INTEGRAL.

BY MR. N. J. LENNES.

(Read before the Chicago Section of the American Mathematical Society. April 14, 1906.)

A function is said to be of limited variation on an interval ab if the set of sums

$$\left[\sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|\right]$$

is bounded for the set of all partitions of ab. The points $a = x_0, x_1, x_2 \cdots, x_{n-1}, x_n = b$ of each partition are ordered on the interval according to the subscripts. The least upper bound of this set of sums is denoted by $V_a^b f(x)$ and is called the variation of $f(x)$ on ab. We consider the variation of

$$\int_a^x f(x)dx$$

on an interval ab, where

$$\int_a^x f(x)dx$$

is a function of the upper limit of integration. Denote this variation by

$$V_a^b \int_a^x f(x)dx.$$

THEOREM I. *If $f(x)$ does not change sign on ab then*

$$V_a^b \int_a^x f(x)dx$$

exists on ab and

$$V_a^b \int_a^x f(x)dx = \int_a^b |f(x)|\, dx.$$

Proof: The theorem is obvious since for every partition of ab consisting of $n+1$ points

$$\sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)dx \right| = \int_a^b |f(x)|\, dx.$$

For convenience of reference a definition of the definite integral is inserted. Let ab be an interval upon which a function $f(x)$ is defined, single valued and bounded. Let π_δ stand for any partition of ab consisting of the points $x_0 = a, x_1, \cdots, x_{n-1}, x_n = b$ such that $\Delta_1 x = x_1 - a$, $\Delta_2 x = x_2 - x_1$, $\cdots$, $\Delta_n x = b - x_{n-1}$, each interval being numerically less than or equal to δ. Let $\xi_1, \xi_2, \cdots, \xi_n$ be any points of the intervals $x_0 x_1$, $x_1 x_2, \cdots, x_{n-1} x_n$ respectively and let

$$S_\delta = f(\xi_1)\Delta_1 x + f(\xi_2)\Delta_2 x + \cdots + f(\xi_n)\Delta_n x = \sum_{k=1}^{n} f(\xi_k)\Delta_k x.$$

If the many valued function S_δ of δ approaches a single limiting value as δ approaches zero, then

$$\mathop{\mathrm{L}}_{\delta \doteq 0} S_\delta = \int_a^b f(x)dx.$$

$S_a^b\delta$ represents the sum S_δ over the interval ab.

LEMMA I. *The function S_δ in the definition of the definite integral approaches its limit uniformly with respect to x, i. e., for every ϵ there exists a δ_ϵ such that*

$$\left| \int_{x_1}^{x_2} f'(x)dx - S_{x1}^{x_2}\delta_\epsilon \right| < \epsilon$$

for every $S_{x1}^{x_2}\delta_\epsilon$ and for every pair of points x_1 and x_2 on ab.

Proof: Since by hypothesis

$$\mathop{\mathrm{L}}_{\delta \doteq 0} S_a^b\delta = \int_a^b f(x)dx,$$

here exists for a given ϵ a $\delta_{\epsilon/2}$ such that

$$(1) \qquad \left| \int_a^b f(x)dx - S_a^b\delta_{\epsilon/2} \right| < \frac{\epsilon}{2}$$

for every $S_a^b\delta_{\epsilon/2}$. Then for any pair of points x_1, x_2 on ab

$$(2) \qquad \left| \int_a^{x_1} f(x)dx - S_a^{x_1}\delta_{\epsilon/2} + \int_{x_1}^{x_2} f(x)dx - S_{x1}^{x_2}\delta_{\epsilon/2} \right.$$
$$\left. + \int_{x_2}^b f(x)dx - S_{x_2}^b\delta_{\epsilon/2} \right| < \frac{\epsilon}{2}.$$

Since $f(x)$ is integrable on ax_1 and x_2b, it follows that there exists a $\delta'_{\epsilon/2}$ for ax_1 and a $\delta''_{\epsilon/2}$ for x_2b such that

$$(3) \qquad \left| \int_a^{x_1} f(x)dx - S_a^{x_1}\delta'_{\epsilon/2} + \int_{x_2}^b f(x)dx - S_{x_2}^b\delta''_{\epsilon/2} \right| < \epsilon/2.$$

Then from (2) and (3)

$$\left| \int_{x_1}^{x_2} f(x)dx - S_{x1}^{x_2}\delta_{\epsilon/2} \right| < \epsilon.$$

Hence $\delta_{\epsilon/2}$ is the δ_ϵ required by the lemma.*

Definition: The difference between the least upper and the greatest lower bound of a function on an interval is the oscillation of the function on that interval.

* If $\delta_{\epsilon\ 2}$ is so chosen that $\delta_{\epsilon/2} \leqq \delta'_{\epsilon/2}$ and $\delta_{\epsilon/2} \leqq \delta''_{\epsilon/2}$, the argument is a little more obvious, though this restriction is not necessary.

LEMMA II. *If the oscillation of a function on an interval ab is less than $\epsilon/2$ then for any two partitions π and π' of ab*

$$\left|\sum_{i=1}^{n}|\Delta_i x|\cdot|f(\xi_i)|-\left|\sum_{i=1}^{n'}\Delta_i' x f(\xi_i')\right|\right|<\epsilon|b-a|$$

when n and n' are the numbers of intervals on the partitions π and π' respectively.

Proof: Let B be any value of $f(x)$ on ab. Then for any interval $x_i x_{i+1}$ of the partition π

$$\Big||f(\xi_i)|\cdot|\Delta_i x|-|\beta|\cdot|\Delta_i x|\Big|<\frac{\epsilon}{2}\cdot\Delta_i x.$$

Hence

$$(1) \qquad \left|\sum_{i=1}^{n}|\Delta_i x|\cdot|f(\xi_i)|-|\beta|\cdot|b-a|\right|<\frac{\epsilon}{2}|b-a|.$$

Similarly for any interval $x_i' x_{i+1}'$ of the partition π'

$$|f(\xi_i')\,\Delta_i' x-\beta\cdot\Delta_i' x|<\frac{\epsilon}{2}|\Delta_i' x|.$$

Hence

$$\left|\sum_{i=1}^{n'}f(\xi_i')\Delta_i' x-\beta\cdot(b-a)\right|<\frac{\epsilon}{2}|b-a|$$

and

$$(2) \qquad \left|\left|\sum_{i=1}^{n'}f(\xi_i')\Delta_i' x\right|-|\beta|\cdot|b-a|\right|<\frac{\epsilon}{2}|b-a|.$$

From (1) and (2) we have

$$\left|\sum_{i=1}^{n}|\Delta_i x|\,|f(\xi_i)|-\left|\sum_{i=1}^{n'}\Delta_i' x\cdot f(\xi_i')\right|\right|<\epsilon|b-a|.$$

THEOREM II. If $\int_a^b f(x)dx$ *exists, then* $V_a^b\int_a^x f(x)dx$ *exists and*

$$V_a^b\int_a^x f(x)dx=\int_a^b|f(x)|\,dx.$$

Proof: Since for any two values of x, x_1 and x_2 on ab

$$(1) \qquad \left|\int_{x_1}^{x_2}f(x)dx\right|\leqq\int_{x_1}^{x_2}|f(x)|dx$$

and since by a well-known theorem

$$\int_{x_1}^{x_2} |f(x)|\, dx$$

exists if

$$\int_{x_1}^{x_2} f(x) dx$$

exists, it follows from (1) and Theorem I that

$$V_a^b \int_a^x f(x) dx$$

exists and that

$$V_a^b \int_a^x f(x) dx \leqq V_a^b \int_a^x |f(x)|\, dx. \tag{2}$$

It remains to show that the sign $<$ in (2) is impossible. Since $f(x)$ is integrable on ab it follows by a well known theorem that for any pair of positive numbers σ and λ there exists a partition π such that the sum of the intervals on which the oscillation of $f(x)$ is greater than σ is less than λ.

For a preassigned ϵ let $\lambda = \epsilon/M$, where M is the difference between the least upper and the greatest lower bound of $f(x)$ on ab. Let $\sigma = \epsilon/4\,|b - a|$. Denote by I the set of intervals on ab on which the oscillation is less than σ and by I' the complementary set of intervals. Let n be the number of intervals in I. By Lemma I there exists a δ such that

$$\left| \int_{x_1}^{x_2} f(x) dx - S_{x_1}^{x_2}\delta \right| < \frac{\epsilon}{2n} \tag{3}$$

for every pair of points x_1 and x_2 on ab and for every $S_{x_1}^{x_2}\delta$. Let x_1, x_2 be the extremities of a segment of the set I. By Lemma II

$$\Big| |S_{x_1}^{x_2}\delta| - |x_1 - x_2| \cdot |f(\xi_2)| \Big| < \frac{\epsilon}{2|b-a|} \cdot |x_1 - x_2|. \tag{4}$$

Denoting the integrals on the segments of the set I by

$$\int_i f(x) dx,$$

the lengths of the segments by Δ_i, and the sums S_δ on the segments of I by $S_{x_j}^{x_{j+1}}$, we have from (4)

$$\left| \sum_{j=1}^{j=2n-1} |S_{x_j}^{x_{j+1}}\delta| - \sum_{i=1}^{n} \Delta_i |f(\xi_i)| \right| < \frac{\epsilon}{2}$$

where $2n-1$ is the number of extremities of the segments of I and where j takes only odd values.

Then from (3)

$$\left| \sum_{j=1}^{2n-1} |S_{x_j}^{x_{j+1}}\delta| - \sum_{i=1}^{n} \left| \int_i f(x)dx \right| \right| < \frac{\epsilon}{2},$$

since if

$$\left| \int_{x_1}^{x_2} f(x)dx - S_{x_1}^{x_2}\delta \right| < \frac{\epsilon}{2n},$$

then

$$\left| \left| \int_{x_1}^{x_2} f(x)dx \right| - |S_{x_1}^{x_2}\delta| \right| < \frac{\epsilon}{2n}.$$

Hence

$$\left| \sum_{i=1}^{n} \Delta_i |f(\xi_i)| - \sum_{i=1}^{n} \left| \int_i f(x)dx \right| \right| < \epsilon.$$

Since

$$\mathop{\mathrm{L}}_{\delta \doteq 0} \sum_{j=1}^{n'} \Delta_j |f(\xi_j)| = \int_i |f(x)|dx$$

and since ϵ is arbitrary it is evident that the least upper bound of

$$\sum_{j=1}^{n'} \left| \int_j f(x)dx \right|$$

for the set of all partitions of I cannot be less than

$$\sum_{i=1}^{n} \int_i |f(x)|dx$$

(where

$$\int_j f(x)dx$$

denotes the integral on an interval formed by partitioning an interval of I.)

Since the sum of the lengths of the interval on which the oscillation is greater than σ is less than $\lambda = \epsilon / M$ and since by a known theorem

$$\sum_{i=1}^{n''} \int_{i'} |f(x)| dx < \lambda \cdot M = \epsilon$$

(where

$$\int_{i'} f|(x) dx$$

denotes the integral on a segment of I') it follows from the arbitrary character of ϵ that the least upper bound of

$$\left[\sum_{i=0}^{n} \left| \int_{x_i}^{x_{i+1}} f(x) dx \right| \right]$$

for the set of all partitions of ab cannot be less than

$$\int_a^b |f(x)| \, dx,$$

which proves the theorem.

Definition: The length of a curve represented by the equation $y = f(x)$ on an interval ab is the least upper bound of the set of sums

$$\left[\sum_{i=1}^{n} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \right]$$

for the set of all partitions $a = x_1, x_2, \cdots, x_{n-1}, x_n = b$ of the interval ab, where y_i corresponds to x_i by the functional correspondence $y = f(x)$.

THEOREM III. *If on the interval ab and on every subinterval of ab the functions $f_1(x)$ and $f_2(x)$ are of equal variation then the curves represented by $y = f_1(x)$, and $y = f_2(x)$ are of equal length on the interval ab.*

Proof: Consider any partition π of ab consisting of the points $x_i (i = 0, \cdots, n)$. Then

$$\sum_{i=0}^{y} \sqrt{(x_i - x_{i+1})^2 + (f(x_i) - f(x_{i+1}))^2}$$

is one sum of the set of sums whose least upper bound is the length of the curve $y = f(x)$ on ab.

By hypothesis $f_1(x)$ and $f_2(x)$ are of the same variation on any interval $x_i x_{i+1}$ of π. Obviously the variation of $f_1(x)$ on $x_i\ x_{i+1}$ is equal to or greater than $|f_1(x_i) - f(x_{i+1})|$. Let ϵ be any preassigned positive number and let $\epsilon' = \epsilon/n$ where n is the number of interval in the partition π. Then there is a partition π'_i of $x_i\ x_{i+1}$ consisting of the points $x'_0 = x_i, x'_2, \cdots, x'_j, \cdots, x'_{n'} = x_{i+1}$ such that

$$(1) \qquad |f_1(x_i) - f_1(x_{i+1})| - \sum_{j-0}^{n'} |f_2(x_j) - f_2(x_{j+1})| < \epsilon'.$$

By means of a broken line whose segments are equal in length to the lengths of the segments connecting the points $(x_j, f_2(x_j))$ and $(x_{j+1}, f_2(x_{j+1}))$ connect the point $(x_i, f_2(x_i))$ with a point (x_{i+1}, y_1) on the line $x = x_{i+1}$ in such manner that the slopes of these segments are, say, all positive. Then by (1)

$$(2) \qquad |f_1(x_i) - f_1(x_{i+1})| - |f_2(x_1) - y_1| < \epsilon'.$$

Hence obviously

$$(3) \quad \sqrt{(x_i - x_{i+1})^2 + [f_1(x_i) - f_1(x_{i+1})]^2} - \sum_{j=\delta}^{n'} \sqrt{(x_j - x_{j+1})^2 + [f_2(x_j) - f_2(x_{j+1})]^2} < \epsilon' \cdots.$$

Let π'', consisting of the points $x_1 = a, x_2, \cdots, x_k, \cdots, x_{n''} = b$, be the partition of ab containing all the points of the partitions π'_i of the intervals $x_i x_{i+1}$, and let n'' be the number of intervals in this partition. Then from (3)

$$(4) \quad \sum_{i=0}^{n} \sqrt{(x_i - x_{i+1})^2 + [f_1(x_i) - f_1(x_{i+1})]^2} - \sum_{k=0}^{n''} \sqrt{(x_k - x_{k+1})^2 + [f_2(x_k) - f_2(x_{k+1})]^2} < \epsilon' n = \epsilon \cdots.$$

By a well known theorem the lengths on ab of the curves $y = f_1(x)$ and $y = f_2(x)$ both exist, since the functions are of limited variation. Moreover, it follows from (4) that the length of the curve $y = f_2(x)$ cannot be less than that of $y = f_1(x)$, ϵ being arbitrary. In the same manner it is shown that the length of the curve $y = f_1(x)$ cannot be less than the length of the curve $y = f_2(x)$. Hence the curves are of equal length and the theorem is proved.

Theorems II and III yield the following interesting

COROLLARY. *If $f(x)$ is such that*

$$\int_a^x f'(x)dx = f(x) - f(a)$$

on ab then the length on ab of the curve $y = f(x)$ can be found by finding the length of the curve

$$y = \int_a^x |f'(x)|\, dx$$

on this interval.

Proof: By Theorem II the curves

$$y = \int_a^x f'(x)dx = f(x) - f(a) \quad \text{and} \quad y = \int_a^x |f'(x)|\, dx$$

have the same variation on ab and on every subinterval of ab, whence by theorem III their lengths are equal.

This corollary reduces the problem of finding the length of a curve of the class specified in the corollary to the finding of the length of a non-oscillating curve.

We now use theorem II to give a fresh proof of a theorem on improper definite integrals.

Definition: A function is integrable at a point x_0 if there exists an interval containing x_0 as an interior point on which $f(x)$ is properly integrable. (A function is properly integrable on an interval only in case it is bounded on that interval.)

Consider a function $f(x)$ which is integrable at every point of the interval ab except at a set of points $[P]$ which is of content zero.*

Let $[I]$ be any finite set of non-overlapping intervals on ab such that no point of $[P]$ lies on an interval of $[I]$. Then the integral of $f(x)$ exists properly on every interval of $[I]$. Denote by

$$\int_{aI}^b f(x)dx$$

the sum of the integrals of $f(x)$ on the intervals of $[I]$.

* A set of points $[P]$ is of content zero if for every positive number ε there exists a finite set of intervals $[I']$ such that every point of $[P]$ lies on at least one interval of $[I']$ and further such that the sum of the lengths of the intervals of $[I']$ is less than ε.

Denote by $m[I]$ the sum of the lengths of these intervals and let $D = |a - b|$.

If the limit

$$\underset{m[I] \doteq D}{\mathrm{L}} \int_{aI}^{b} f(x)dx$$

exists and is finite, this limit is said to be the improper definite integral of $f(x)$ on ab.

THEOREM IV. *The improper definite integral regarded as a function of the upper limit of integration is of limited variation on any interval ab.*

Proof: Since by hypothesis the improper definite integral of $f(x)$ exists on ab it follows that there exists an M such that for every set of interval $[I]$

$$(1) \qquad \left| \int_{aI}^{b} f(x)dx \right| < M.$$

If the theorem fails to hold, *i. e.*, if for every M' there is a partition π of ab such that

$$(2) \qquad \sum_{i=0}^{n} \left| \int_{x_i}^{x_{i+1}} f(x)dx \right| > M',$$

then for a certain subset $x_j\, x_{j+1}$ of the set of interval $x_i\, x_{i+1}$

$$(3) \qquad \sum_{j=0}^{n'} \left| \int_{x_j}^{x_{j+1}} f(x)dx \right| > \frac{M'}{2}.$$

(The integrals in (3) are improper definite integrals.) Since the improper definite integral exists on each of the intervals $x_j\, x_{j+1}$ there exists a set of segments $[I']$ such that

$$(4) \qquad \int_{aI'}^{b} f(x)dx > \frac{M'}{2}.$$

If $M' = 2M$ then (4) contradicts (1). Hence the theorem is proved.

THEOREM V. *If the improper definite integral of $f(x)$ exists on the interval ab then the improper definite integral of $|f(x)|$ exists on $a\,b$.*

Proof: By theorem IV

$$\int_a^x f(x)dx$$

is of limited variation on ab. Hence there exists a number M such that for every set of intervals $[I]$

$$V_a^b \int_{x_i}^x f(x)dx < M.$$

But by Theorem II,

$$V_a^b \int_{x_i}^x f(x)dx = \int_{aI}^b |f(x)|\, dx < M,$$

which proves the theorem.*

CHICAGO,
July 17, 1906.

A NOTE ON TRANSITIVE GROUPS.

BY DR. W. A. MANNING.

(Read before the American Mathematical Society, September 3, 1906.)

THREE unconnected topics in the theory of transitive substitution groups are touched on in this note.

1.

THEOREM I. *The largest subgroup of a transitive group G of degree n, in which a subgroup H leaving fixed m $(0 < m < n)$ letters is invariant, has as many transitive constituents in these m letters as there are different conjugate sets in G_1 (a subgroup of G, that leaves one of the m letters fixed) which, under the substitutions of G, enter into the complete set of conjugates to which H belongs. Moreover, the degree of each of these constituents is proportional*

* For a general discussion of the improper definite integral see E. H. Moore, "Concerning Harnack's theory of improper definite integrals." *Transactions Amer. Math. Society*, volume 2, pp. 296–330, and pp 459–475 same volume. See also references given in this paper by Professor Moore. For a proof of Theorem V. of the present note see Jordan, Cours d'analyse, ed. 2, vol. 2 (1894), pp. 46 ff.