Proof: By theorem IV

$$\int_a^x f(x)dx$$

is of limited variation on ab. Hence there exists a number M such that for every set of intervals $[I]$

$$V_a^b \int_{x_i}^x f(x)dx < M.$$

But by Theorem II,

$$V_a^b \int_{x_i}^x f(x)dx = \int_{aI}^b |f(x)|\, dx < M,$$

which proves the theorem.*

CHICAGO,
July 17, 1906.

A NOTE ON TRANSITIVE GROUPS.

BY DR. W. A. MANNING.

(Read before the American Mathematical Society, September 3, 1906.)

THREE unconnected topics in the theory of transitive substitution groups are touched on in this note.

1.

THEOREM I. *The largest subgroup of a transitive group G of degree n, in which a subgroup H leaving fixed m $(0 < m < n)$ letters is invariant, has as many transitive constituents in these m letters as there are different conjugate sets in G_1 (a subgroup of G, that leaves one of the m letters fixed) which, under the substitutions of G, enter into the complete set of conjugates to which H belongs. Moreover, the degree of each of these constituents is proportional*

* For a general discussion of the improper definite integral see E. H. Moore, "Concerning Harnack's theory of improper definite integrals." *Transactions Amer. Math. Society*, volume 2, pp. 296–330, and pp 459–475 same volume. See also references given in this paper by Professor Moore. For a proof of Theorem V. of the present note see Jordan, Cours d'analyse, ed. 2, vol. 2 (1894), pp. 46 ff.

to the number of subgroups in the several conjugate sets of G_1 *in question.**

Let g be the order of G, and let those conjugates of H, which are found in G_1, lie in κ different sets, in so far as they are permuted by the substitutions of G_1 only, with c conjugates in the set including H, c_1 in a second set, and so on. In G_1, H is invariant in a subgroup of order g/nc, while H_1, a subgroup in the second set, is invariant in a group of order g/nc_1. The largest subgroup I of G in which H is invariant is of order $gm/n\sigma$ $(\sigma = c + c_1 + \cdots + c_{\kappa-1})$. Now I does not connect transitively the $n - m$ letters displaced by H and the m letters it leaves fixed. Since the largest subgroup of G_1 in which H is invariant is of order g/nc, I has one transitive constituent of degree $(gm/n\sigma) \div (g/nc) = mc/\sigma$ in letters left fixed by H. The group H_1 is transformed into H by some substitution S not in G_1. Then in the subgroup $G_2 = S^{-1}G_1S$, H belongs to a set of g/nc_1 conjugates (conjugate under the substitutions of G_2). Hence I has a transitive constituent of degree mc_1/σ in letters left fixed by H. Continuing thus, we see that I has transitive constituents of degrees mc/σ, $mc_1/\sigma, \cdots, mc_{\kappa-1}/\sigma$. These constituents involve all the letters left fixed by H. This completes the proof of the theorem.

COROLLARY. *If H is a Sylow subgroup of G_1, I has a transitive constituent of degree m.*

2.

THEOREM II. *If the class of a t-ply transitive group $(t > 3)$ is not less than $n - 2t + 5$, then the degree n is less than or equal to $\frac{1}{2}(t^2 - t + 2) + (t - 1)!$*

There are three cases to consider separately:

G contains a substitution of order 2 and (1) of degree $n - t + 2$, (2) of degree $n - t + 1$, (3) of degree $n - t - \epsilon$ $(0 \leqq \epsilon \leqq t - 5)$. Since the subgroup of G that leaves fixed $t - 2$ letters is of even order, it always contains a substitution of order 2 falling under one of the three following categories:

1. If there is one substitution of order 2 on just $n - t + 2$ letters in G, there are at least $[n(n-1)\cdots(n-t+3)(n-t+1)]/(t-2)!$ The total number of transpositions involved in these substitutions is $n(n-1)\cdots(n-t+1)/2(t-2)!$ Of these substitutions at least $n(n-1)\cdots(n-t+1)/n(n-1)$

*Cf. Burnside, Theory of Groups (1897), p. 202, where a special case is proved.

$(t-2)$! have some particular transposition as (ab) in common. Again by the same reasoning at least $n(n-1)\cdots(n-t+1)$ $(n-t)/n(n-1)(n-2)(n-3)(t-2)!$ that have (ab) in common have also a transposition (cd) in common. Finally the number of these substitutions which have $t-2$ common transpositions is at least

$$(1) \qquad \frac{n(n-1)\cdots(n-t)(n-t-2)\cdots(n-3t+8)}{(t-2)!\; n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq 1.$$

That this number cannot be greater than unity comes from the fact that G is by hypothesis of class $n-2t+5$, while the product of two substitutions with $t-2$ transpositions in common is of degree less than or equal to $n-2t+4$. This inequality is of the first degree in n, so that we obtain from it

$$(2) \qquad n \leqq \tfrac{1}{2}(t^2-3t+4)+(t-2)!.$$

2. In case G has a substitution of order 2 and degree $n-t+1$, we get by the same process

$$(3) \qquad \frac{n(n-1)\cdots(n-t+1)(n-t-1)\cdots(n-3t+7)}{(t-1)!\, n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq 1,$$

from which

$$(4) \qquad n \leqq \tfrac{1}{2}(t \;\; -t-2)+(t-1)!$$

When $t=4$, G is of class $n-3$, and the above inequality gives $n \leqq 11$.

3. The inequality in this case is

$$(5) \qquad \frac{n(n-1)\cdots(n-t+1)(n-t-\varepsilon)(n-t-\varepsilon-2)\cdots(n-3t-\varepsilon+6)}{(t+\varepsilon)(t+\varepsilon-1)\cdots(\varepsilon+1)(n)(n-1)\cdots(n-2t+3)} \leqq 1.$$

It is of the second degree in n, and gives

$$(6) \qquad n \leqq \tfrac{1}{4}(t^2-3t-8+2t\varepsilon-4\varepsilon)+\sqrt{(t+\varepsilon)!/\varepsilon}\,!$$

The limit $\frac{1}{2}(t^2-t+2)+(t-1)!$ is always the highest of the three.

In this connection it is easy to establish the following

THEOREM III. *If the subgroup leaving t letters fixed in a t-ply $(t>2)$ transitive group G is of even order, the degree of*

G is less than $\frac{1}{2}(t^2 - t + 6) + \varepsilon(t-1) + (t+\varepsilon)!/\varepsilon!$ *unless the class is less than* $n - 2t + 3$.

The inequality

$$\frac{n(n-1)\cdots(n-t+1)(n-t-\varepsilon)(n-t-\varepsilon-2)\cdots(n-3t-\varepsilon+4)}{n(n-1)\cdots(n-2t+6)(n-2t+5)} \leqq \frac{(t+\varepsilon)!}{\varepsilon!}$$

where $0 \leqq \varepsilon \leqq t-3$, is found as before, and from it the theorem follows.

3.

Another theorem of value in the applications is the following :

THEOREM IV. *A doubly transitive group cannot contain an invariant imprimitive subgroup unless its degree is a power of a prime. Then the group is a subgroup of the holomorph of the abelian group of order* p^a *and type* $(1, 1, \cdots)$.

On pages 193 and 194 of his Theory of Groups Burnside proves that the invariant imprimitive subgroup H is of degree n and class $n-1$ and that the $n-1$ substitutions of degree n in H form a single conjugate set under G. Then by Frobenius's theorem on groups of "class $n-1$," H contains a characteristic subgroup of degree and order n which is abelian with all its operators of the same order.

STANFORD UNIVERSITY,
June 19, 1906.

DIFFERENTIAL GEOMETRY OF n DIMENSIONAL SPACE.

Sur les Systèmes Triplement Indéterminés et sur les Systèmes Triple-Orthogonaux. Par C. GUICHARD. Scientia, no. 25. Gauthier-Villars, Paris, 1905. viii + 95 pp.

DURING the past ten years the field of differential geometry has been greatly enriched by the researches of M. Guichard. The eminent geometer has made a profound study of the properties of ordinary space by means of operations defined for space of n dimensions. He has introduced two elements depending upon two variables ; they are the *reseau* and the *congruence*. By definition, a point of space in n dimensions