THE THIRTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE thirteenth summer meeting and the fifth colloquium of the Society were held at Yale University during the week September 3–8, 1906. The following forty-six members were in attendance:

Mr. G. D. Birkhoff, Professor G. A. Bliss, Professor Joseph Bowden, Professor E. W. Brown, Dr. A. B. Chace, Dr. J. E. Clarke, Dr. A. Cohen, Professor F. N. Cole, Professor L. L. Conant, Professor D. R. Curtiss, Mr. H. N. Davis, Professor L. P. Eisenhart, Professor W. B. Fite, Professor A. S. Gale, Professor C. N. Haskins, Dr. L. I. Hewes, Professor Edward Kasner, Professor O. D. Kellogg, Dr. W. R. Longley, Mr. E. B. Lytle, Professor T. E. McKinney, Professor James McMahon, Professor H. P. Manning, Professor Max Mason, Mr. E. A. Miller, Professor E. H. Moore, Professor W. F. Osgood, Professor James Pierpont, Dr. R. G. D. Richardson, Miss S. F. Richardson, Professor H. L. Rietz, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Dr. C. H. Sisam, Professor C. S. Slichter, Dr. Clara E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. R. P. Stephens, Professor W. E. Story, Professor E. B. Van Vleck, Professor A. G. Webster, Professor H. S. White, Professor T. W. D. Worthen, Professor J. W. Young.

The colloquium, of which a separate report appears in the BULLETIN, opened on Wednesday morning. The summer meeting proper extended through two sessions on Monday and a morning session on Tuesday. On Tuesday afternoon the visitors were conducted through the grounds and buildings of the University. The generous hospitality extended to the Society throughout the meeting by the University and its officers will long be remembered with grateful appreciation.

At the opening session on Monday morning Professor E. H. Moore presided. President W. F. Osgood occupied the chair during the remaining sessions. The Council announced the election of the following persons to membership in the Society: Professor William Beebe, Yale University; Mr. J. B. Clarke, Polytechnic High School, San Francisco; Dr. E. C. Colpitts, Cornell University; Brother Constantius, Christian Brothers College, St. Louis; Professor G. W. Droke, University of Arkansas; Mr. R. M. Ginnings, State Normal School, Kirksville, Mo.. Professor Harriet E. Glazier, Western College for Women; Professor C. O. Gunther, Stevens Institute of Technology; Mr. W. A. Hurwitz, University of Missouri; Dr. G. O. James, Washington University; Mr. B. F. Johnson, State Normal School, Cape Girardeau, Mo.; Mr. E. B. Morrow, Princeton University; Mr. G. B. Obear, Brown University; Dr. F. M. Pedersen, College of the City of New York; Professor G. A. Rose, Hardin College; Mr. R. L. Short, Chicago, Ill.; Miss Betty Trier, Mount Holyoke College; President J. W. Withers, Teachers College, St. Louis. Thirteen applications for membership in the Society were received.

A committee consisting of Professors Bôcher, E. B. Van Vleck, and Townsend was appointed by the President to prepare and report to the Council at the October meeting a list of nominations for officers and other members of the Council to be elected at the annual meeting in December. Steps were also taken toward amending the Constitution to include the Editorial Committee of the *Transactions* in the membership of the Council.

The following papers were read at the summer meeting:

(1) Mr. A. R. SCHWEITZER: "Systems of axioms for projective geometry."

(2) Mr. A. R. SCHWEITZER: "Concerning abstract geometric relations" (preliminary report).

(3) Professor O. D. KELLOGG: "The behavior on the boundary of harmonic functions of a region."

(4) Mr. F. R. SHARPE: "The motion of a viscous gas."

(5) Professor R. D. CARMICHAEL: "Multiply perfect numbers of three different primes."

(6) Professor LUDWIG STICKELBERGER: "Zur Theorie der vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören."

(7) Professor W. B. FITE: "Irreducible linear homogeneous groups whose orders are powers of a prime."

(8) Dr. ARTHUR RANUM: "The group of classes of congruent matrices and its application to the group of isomorphisms of any abelian group."

(9) Dr. R. G. D. RICHARDSON: "On the reduction of multiple integrals (second paper)."

(10) Mr. G. D. BIRKHOFF: "On a certain class of sets of normed orthogonal functions."

(11) Dr. W. B. CARVER: "Associated configurations of the Cayley-Veronese class."

(12) Professor L. E. DICKSON: "On commutative linear algebras in which division is always uniquely possible."

(13) Professor L. E. DICKSON: "Uniform definitions of the abstract forms of the various known systems of linear groups."

(14) Professor L. E. DICKSON: "Criteria for the irreducibility of functions in a finite field."

(15) Professor L. E. DICKSON: "On the theory of equations in a modular field."

(16) Professor JAMES MCMAHON: "The differential geometry of the general vector field" (preliminary report).

(17) Dr. W. A. MANNING: "A note on transitive groups."

(18) Dr. C. H. SISAM: "On systems of conics lying on surfaces of the third, fourth, and fifth orders."

(19) Professor VIRGIL SNYDER: "Plane quintic curves which possess a group of linear transformations."

(20) Professor MAX MASON: "The expansion of an arbitrary function in terms of normal functions."

(21) Professor MAX MASON: "The boundary value problems of differential equations of hyperbolic type."

(22) Professor EDWARD KASNER: "An inverse problem of dynamics."

(23) Professor EDWARD KASNER: "The geometry of dynamical trajectories."

(24) Professor J. W. YOUNG: "General theory of approximation by functions with a given number of parameters."

(25) Professor J. I. HUTCHINSON: "On loci the coordinates of whose points are abelian functions of three parameters."

(26) Professor L. P. EISENHART: "Applicable surfaces with asymptotic lines of one surface corresponding to a conjugate system of another."

(27) Dr. H. B. LEONARD: "On the factoring of composite hypercomplex number systems."

(28) Professor FRANK MORLEY: "Reflexive geometry."

(29) Professor G. A. MILLER: Generalization of the groups of genus zero."

(30) Professor E. B. WILSON: "On divergence and curl."

(31) Professor E. B. WILSON: "Oblique reflections and unimodular strains."

(32) Professor E. B. WILSON: Double products and strains in n dimensions."

(33) Professor F. R. MOULTON: "A class of three dimensional periodic orbits in the problem of three bodies, with applications to the lunar theory."

(34) Professor OSKAR BOLZA: "Weierstrass's theorem and Kneser's theorem on transversals for the most general case of an extremum of a simple definite integral."

Professor Stickelberger's paper was communicated to the Society through Professor E. H. Moore. In the absence of the authors, Mr. Sharpe's paper was presented by Professor McMahon, Dr. Ranum's by Professor Moore, Dr. Carver's by Professor Snyder, Professor Hutchinson's by Professor Fite, and the papers of Professor Carmichael, Professor Stickelberger, Professor Dickson, Dr. Manning, Dr. Leonard, Professor Morley, Professor Miller, Professor Wilson, Professor Moulton and Professor Bolza were presented by title.

The third and fourth papers of Professor Dickson and the paper of Dr. Manning appeared in full in the October BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Schweitzer constructed systems of axioms for projective geometry of n dimensions ($n = 1, 2, 3, \cdots$), using two undefined symbols, viz., the element point and relation K_n between two ordered $(n + 2)$-ads of points. Under the respective systems the relation K_n is symmetric and transitive, *i. e.*, $(A_1A_2 \cdots A_{n+2})K_n(B_1B_2 \cdots B_{n+2})$ implies $(B_1B_2 \cdots B_{n+2}) K_n(A_1A_2 \cdots A_{n+2})$ and the statements $(A_1A_2 \cdots A_{n+2}) K_n(B_1B_2 \cdots B_{n+2})$, $(B_1B_2 \cdots B_{n+2})K_n(C_1C_2 \cdots C_{n+2})$ imply $(A_1A_2 \cdots A_{n+2})K_n(C_1C_2 \cdots C_{n+2})$. The existence of an ordered $(n + 2)$-ad implies no geometric axiom except the existence of points. Systems III, IV, $\cdots$ are sufficient for 3, 4, $\cdots$ dimensional projective geometry respectively. Systems, I, II, $\cdots$ are readily extended to higher dimensions. In each of the preceding systems, collinearity, separation, etc., are defined in terms of the corresponding relation K_n.

The above sets of axioms may be considered the projective analogues of the author's systems of descriptive axioms * for

* Cf. abstract No. 6, BULLETIN, June, 1906, p. 438; for n-ad read $(n + 1)$-ad. It should also there be stated that under the axioms the K_n is symmetric and transitive in the sense indicated above.

n dimensions $(n = 1, 2, 3, \cdots)$ involving the two undefined elements point and the symmetric, transitive relation K_n between two ordered $(n + 1)$-ads of points. A result of both investigations is that it is possible to construct axioms for n dimensions, $(n = 1, 2, 3, \cdots)$, which may be either descriptive or projective by means of two undefined symbols point and a transitive, symmetric relation; and in n-dimensional space this relation may be either linear, or planar, $\cdots$, or n-dimensional.

2. In his second paper Mr. Schweitzer investigates the kinds of undefined relations between points which may be used in the construction of geometric axioms, for example: 1) $\alpha R_n(\beta_1\beta_2 \cdots \beta_n)$, $(n = 1, 2, 3, \cdots)$, where the n-ad is ordered; 2) $[\alpha_1 \cdots \alpha_n] B_n [\beta_1\beta_2]$, $(n = 1, 2, 3, \cdots)$, where the dyad and n-ad are not ordered; 3) $(\alpha_1\alpha_2) S_{n+2}(\beta_1\beta_2 \cdots \beta_{n+1})(n = 1, 2, 3, \cdots)$, where the dyad and $(n + 1)$-ad are both ordered. In particular, the construction of descriptive axioms in terms of the relation R_n under 1) leads to essential simplifications of a treatment of geometric order given by Vahlen in his "Abstrakte Geometrie." This is shown in a note which will appear in the BULLETIN.

3. The object of Professor Kellogg's paper is the study of the behavior on the boundary of the derivatives of harmonic functions of a region, and allied questions. Some preliminary results for the circle are here given. The formulas of Hilbert,

$$f(s) = \int_0^1 g(t) \cot \pi(s - t)dt + \int_0^1 f(t)dt,$$

$$g(s) = -\int_0^1 f(t) \cot \pi(s - t)dt + \int_0^1 g(t)dt,$$

giving the boundary values of a harmonic function of the unit circle in terms of those of the conjugate function, still hold if we only postulate piecewise regularity of $f(s)$ and $g(s)$. By regularity we mean that a condition holds of the form $|f(s + h) - f(s)| < Ch^\alpha$, where C and α are positive constants, C as large and α as small as we please. A study of these formulas leads to the results: It is not sufficient for the continuity of the boundary values of a harmonic function that the boundary values of the conjugate function be continuous. It is sufficient that they be regular. In order that any first derivative of a harmonic function be continuous up to and upon the boundary, it is not sufficient that the tangential derivatives on the boundary be continuous. It is sufficient that they be regular. (This last result is proved for the circle in a slightly more general form by Dini, *Acta Mathematica*, volume 25, page 221 and following.) The necessary and sufficient conditions can be brought even nearer together, but it seems scarcely worth while. The extension of these results to more general regions will be the subject of a later report.

Another remark concerns the distinction between the conformality of the mapping of a function of a complex variable and the finiteness and non-vanishing of the derivative. That these are not the same is at once seen by an examination of the mapping given by $\zeta = z \log z$ in the neighborhood of the origin.

The paper will be offered to the *Transactions* for publication.

4. In discussing the motion of a gas it is usually assumed that every element neither gains nor loses heat, and consequently the pressure varies as some power of the density so long as we consider elements on the same stream line. In Mr. Sharpe's paper the adiabatic assumption is replaced by a new equation obtained by considering the total flow of energy, molar and molecular, which includes the effects of viscosity and conduction of heat. It is shown that this equation agrees with a similar one deduced from the kinetic theory of gases, and it is applied to the following problems: 1) heated air rising up a chimney, 2) gas flowing between two smooth parallel planes, 3) gas flowing between two rough parallel planes, 4) gas surrounding a rotating circular cylinder and inside a fixed concentric cylinder.

5. In this paper Professor Carmichael completes the investigation of the existence of multiply perfect numbers of (only) three different prime factors, showing that the only such numbers are $2^3 \cdot 3 \cdot 5$ and $2^5 \cdot 3 \cdot 7$, each of multiplicity 3. In a footnote he also points out that there exists no multiply perfect number which is the power of a prime, and that all perfect numbers of (only) two different primes are of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime. The paper will be published in the *Annals of Mathematics*.

6. Professor Stickelberger gives a very simple proof of a lemma forming the basis of the article by Professor Loewy,

"Über die vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören" (*Transactions*, volume 6 (1905), pages 504–533). Moreover, there results an extension of Loewy's theorem. The paper will appear in the *Transactions*.

7. Certain irreducible linear homogeneous groups of different degrees are simply isomorphic; whereas, on the other hand, all the irreducible groups with which some groups are simply isomorphic are of the same degree. The question thus suggested as to the connection between the degree of an irreducible group and its abstract group properties forms the subject of Professor Fite's paper. The discussion is limited to groups whose orders are powers of a prime.

8. With respect to the n^2 moduli $p^{a_{ij}} (i, j = 1, \cdots n; p = \text{a prime})$ two matrices (l_{ij}) and (m_{ij}) are said to be congruent, if $l_{ij} \equiv m_{ij}$, mod $p^{a_{ij}}$. Dr. Ranum proves that the composition of classes of congruent matrices is possible if, and only if, the moduli satisfy the condition $a_{ij} + a_{jk} \geqq a_{ik}$ $(i, j, k = 1, \cdots, n)$ and the elements l_{ij} are divisible by $p^{\alpha_{ij}}$, where α_{ij} is the greatest of the $2n + 1$ integers $0, a_{ik} - a_{jk}, a_{kj} - a_{ki}$ $(k = 1, \cdots, n)$. If these conditions are satisfied, the totality of classes of matrices whose determinants are prime to p form a group, of which all other groups satisfying the same condition are subgroups. The order, invariant matrices, and composition series of this group, and its largest invariant subgroup of order a power of p, are found. The application of a special group of this kind to the group of isomorphisms of any abelian group is an amplification of a paper read before the San Francisco Section at its February meeting, entitled "A new kind of congruence group, etc." (abstract in BULLETIN, May, 1906, page 375).

9. In a former paper (April, 1906), Dr. Richardson showed that in all cases the existence of the double integral and the iterated integral is a sufficient condition for their equality. The present paper discusses the corresponding problem which arises when either the double integral or the iterated integral exists. In the *Journal de Mathématiques*, series 5, volume 5 (1899), de la Vallée-Poussin derived the relation

$$1) \quad \overline{\int_T} f dT \geqq \underline{\int_y} dy \overline{\int_x} m f dx \geqq \underline{\int_y} dy \underline{\int_x} m f dx \geqq \underline{\int_T} f dT \qquad (f \geqq 0),$$

where T is a portion of the plane bounded by a regular curve and the minimum function mf is defined at each point as the minimum of f in an infinitesimal area about that point. The introduction of the minimum function is shown to be unnecessary and relation 1) is true if f is substituted for mf. If the double integral exists, this becomes

$$\int_T f dT = \underline{\int}_y dy \overline{\int}_x f dx = \underline{\int}_y dy \underline{\int}_x f dx.$$

Other relations are derived under the hypothesis that the iterated integral alone exists. For the generalized integral introduced by Professor Pierpont, the theorems are valid; and if the fields of integration are $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, of $m + n$, m, and n dimensions respectively, the discussion applies. Finally, the theorems are extended to the case where the field of integration is unlimited.

10. Mr. Birkhoff's paper deals with certain infinite sets of normed orthogonal functions, $\phi_1(x)$, $\phi_2(x)$, $\cdots$, *i. e.*, sets of functions ϕ such that

$$\int_0^1 \phi_i(t)\phi_j(t)dt = \begin{matrix} 0, & (i \neq j) \\ 1, & (i = j) \end{matrix};$$

such a set is defined as the set of normed solutions of an equation

$$\phi(x) = \lambda \int_0^1 k(x, t)\phi(t)dt,$$

where λ is a parameter and where 1) $k(x, t)$ is defined for $0 \leqq x, t \leqq 1$ and is real, continuous, and symmetric in x and t; 2) $k(x, t)$ has continuous partial derivatives in x, t of first and second orders, except for $x = t$, where there is the discontinuity of a finite jump, at least in the first partial derivatives.

The principal theorems deal with the possibility of representing a prescribed function $f(x)$ on the interval (0, 1) as a series in the functions ϕ. The recent results of Fredholm, Hilbert and Schmidt in this direction are fundamental for the standpoint and theorems of the paper.

11. Beginning with the idea of $n + 2$ points in S_n associated with $n + 2$ S_{n-1}'s, one may define a $C^{n+1}_{n+2,\,r}$ associated with a

given $C^n_{n+2,\,r}$ (the notation being that used by Dr. Carver in an earlier paper, "On the Cayley-Veronese class of configurations," *Transactions*, October, 1905). A method is given for constructing the polar point of a $C^{n-1}_{n+1,\,r}$ with respect to a $C^n_{n+1,\,r}$; and also for constructing a polar S_{k-1} of a $C^{n-k}_{n+1,\,r}$ with respect to a $C^n_{n+1,\,r}$.

12. In his first paper, Professor Dickson considers commutative linear algebras in $2n$ units, with coördinates in a general field F, such that n of the units define a subalgebra forming a field $F(J)$. The elements of the algebra may be exhibited in the form $A + BI$, although the algebra is not binary. Algebras VIII and X, on four and six units, respectively, given in the July number of the *Transactions*, may now be given a more luminous form (valid for n arbitrary), in which the product of any two elements is

$$(A + BI)(X + YI) = AX + B'Y'J + (AY + BX)I,$$

where $B' = B(J')$, J' being a second root of the normal equation satisfied by J. If the constant term of the latter equation is a not-square in F, division is uniquely possible in the algebra.

By an investigation of the general commutative linear algebra in $2n$ units in which division is unique, we are led to algebras with the law of multiplication $(A + BI)(X + YI) = R + SI$, where

$$R = AX + L(BY - B'Y') + JB'Y',$$

$$S = AY + BX + U(BY - B'Y').$$

For $n = 2$ and $J^2 - C_1J + C_2 = 0$ the conditions for unique division are

$$L = J^2U'^2/4C_2 - \tfrac{1}{4}U^2, \quad C_2 = \text{not-square in } F,$$

$$(J'U^2 + JU'^2 - 4C_2)^2 - 4C_2U^2U'^2 = \text{square in } F.$$

Aside from the case $U = 0$ (giving the first algebra above), these algebras are equivalent, under linear transformation, in sets of four, while the identity is the only transformation into itself of such an algebra. The article will appear in the *Transactions*.

13. In his second paper, Professor Dickson defines abstractly various systems of linear groups with coëfficients in an arbitrary field F, viz., the general linear group, the abelian and hyperabelian groups, and the systems of linear groups defined by a quadratic or a hermitian form. The simplicity of the sets of generational relations is due to the choice of the generators (all of period p when F has modulus p; all non-periodic when F is non-modular). The great majority of the relations express the commutators of each pair of generators. The defining relations have intimate, but not entirely obvious, connections with Lie's commutator relations for the corresponding continuous groups. Two papers incorporating the results for the general linear and the abelian groups have been offered for publication in the *Quarterly Journal of Mathematics*.

16. A systematic study of the geometry of the general vector field without immediate reference to physical applications would be of considerable mathematical interest, and, if carried far enough, must ultimately throw new light on various physical problems. In Professor McMahon's preliminary paper, the vector lines of the field are viewed as a family of twisted curves, which may or may not have a family of orthogonal surfaces. The jacobian determinant of u, v, w, the three rectangular components of the field at (x, y, z), is studied as a fundamental invariant. The linear transformation of any vector (l, m, n), by using a matrix composed of the elements of the jacobian of (u, v, w), produces another vector that bears important relations to the field. The conjugate and reciprocal matrices produce other transformed vectors whose invariant relations are studied, and interpreted geometrically. The differential equations of various other invariant lines are derived, such as the locus of inflexions, lines along which the field vector preserves its direction or magnitude, points of maximum curvature, direction of greatest and least derivative of vector or of tensor. The properties of the auxiliary quadrics and of the general vector tube are considered.

18. Dr. Sisam discusses in this paper the properties of algebraic surfaces generated by systems of conics. A number of general properties of such surfaces are determined and a classification is made of the surfaces so generated which are of order not higher than five.

19. The purpose of Professor Snyder's paper was to find those quintic curves which are left invariant by linear transformations. The only collineations that can transform a plane quintic into itself must all have the same invariant triangle. Apart from a few obvious types and admitting one cyclic collineation, thirteen forms were found. In particular the triangular-symmetric curve has a group of order 150. The largest number of harmonic homologies in any group is 15; in two different types 5 can exist, and one has 3. The remaining types have not more than one.

20. Professor Mason gave a proof for the expansion of an arbitrary function which has a sectionally continuous first derivative in terms of normal functions of the differential equation

$$\frac{d^2y}{dx^2} + \lambda A(x)y = 0.$$

Application was made of the fact that the normal functions are solutions of certain minimum problems.

21. In Professor Mason's second paper a boundary value problem for the differential equation

$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu + f$$

was treated which includes several of the boundary problems previously considered. The resolution of the problem was made to depend on an indefinite integral equation, the solution being obtained by successive substitutions.

22. The determination of the trajectories of a given positional force requires the integration of the equations of motion. The inverse problem which forms the subject of Professor Kasner's paper consists in determining the field of force when the trajectories are given. Analytically the problem is solved by a quadrature. The object of the paper is to obtain a direct geometric construction of the field. This is attained by employing the auxiliary system of curves, termed velocity curves, which the author has studied in a previous paper. This related system is decomposable in a simple way into ∞^1 subsystems, from any

one of which the field can be constructed. The problem is solved for both plane and space fields.

23. In previous papers Professor Kasner has studied the motion of a particle acted upon by an arbitrary (positional) field of force in the plane. A complete set of geometric properties of the trajectories is presented in the *Transactions* for July, 1906. The present paper discusses the general three-dimensional problem. The results are not only more complicated, as is to be expected, but also quite different in character. The osculating spheres of the twisted curve take the place of the osculating parabolas employed in the plane theory to interpret differential elements of the third order. It is shown that the osculating spheres of the ∞^1 trajectories passing through a given point in a given direction have their centers on a straight line; and that, as the initial direction varies, the straight line generates a congruence consisting of the secants of a twisted cubic. Converse questions are considered and ultimately a characteristic set of properties is obtained.

24. Professor Young's paper considers a general problem of approximation, which he states as follows: Given a class $\mathfrak{C}_n$ of functions $f = f(x) = f(x;\ a_0, a_1, \cdots, a_n)$ of the real variable x and a functional operation U, such that every $U(f)$ is a fully defined function of x at every point of a finite interval (a, b) as soon as the parameters a_i are given; to determine these a_i so that the maximum of $|U(f)|$, as x varies over (a, b), shall be as small as possible. The existence of a solution, which is called a "function of approximation in $\mathfrak{C}_n$ with reference to U," is proved under very general conditions on $U(f)$. The author then assigns to the functions f a more definite form; *i. e.*, he considers a class $\mathfrak{S}_n$ of functions

$$S_n = a_0 s_0 + a_1 s_1 + \cdots + a_n s_n,$$

where the s_i are given functions of x, and seeks to determine the a_i so that the minimum of $|V(S_n) - \phi|$, as x varies over (a, b), shall be as small as possible. Here V is a distributive functional operation and ϕ a given function of x. Under certain general conditions on V, s_i, and ϕ the problem is shown to have a unique solution, and certain necessary and sufficient conditions are derived that a function in $\mathfrak{S}_n$ shall be a "function of approximation in $\mathfrak{S}_n$ with reference to ϕ and V." The theorems thus proved form a two-fold generalization of the theorems of Tchebychev considered by the author in a paper presented to the Society at its last April meeting, and show that the properties expressed in the older theorems are characteristic of a much larger class of problems than those to which they originally applied.

25. The paper by Professor Hutchinson considers the loci in space of four dimensions determined by taking the five homogeneous coördinates of a point proportional to linearly independent theta functions of three variables of the same order and characteristic, and all odd, or all even. It is found that only a finite number of such loci exist having a given degree. By equating to zero a linear combination of the given theta functions, algebraic surfaces in space of three dimensions are determined. Various results are deduced by means of the properties of the theta functions.

26. Professor Eisenhart considers pairs of surfaces applicable to one another with the asymptotic lines of one surface corresponding to a conjugate system on the other. He finds that each of the two surfaces is an associate of a spherical surface— a surface whose gaussian curvature is positive and constant— and the two spherical surfaces are the Hazzidakis transforms of one another. Moreover, every surface associate to a spherical surface is applicable to a surface associate to the Hazzidakis transform of the spherical surface, in such a way that the asymptotic lines of the one correspond to a conjugate system of the other. To asymptotic lines on the associate surface corresponds a conjugate system with equal tangential invariants on the given spherical surface, and on its Hazzidakis transform a conjugate system with equal point invariants. When a conjugate system upon a spherical surface has equal point and tangential invariants, two pairs of applicable surfaces of the kind sought can be found by quadratures; a particular study is made of this case. When a surface is referred to its asymptotic lines, the equations of condition to be satisfied in order that the surface admit an applicable surface upon which the parametric lines form a conjugate system are of such a form that it is seen that if there are more than two such applicable surfaces there are an infinity. In this case the fundamental coefficients of the applicable surfaces can be found directly and the infinity of associate spherical surfaces are given by quadratures.

27. In the *Mathematische Annalen* (volume 39 (1891), pages 324–326) Scheffers calls the algebra $EF \equiv \epsilon_{ij} = e_i f_j = f_j e_i$ $(i = 1, \cdots, n; j = 1, \cdots, r)$ the product (compound) of the algebras $E \equiv e_1 \cdots e_n$, $F \equiv f_1 \cdots f_r$. He suggests inversely the desirability of theorems concerning the roots of the characteristic equation of the composite algebra, of a criterion for determining whether a given algebra is composite, and also of theorems concerning the division of zero in the composite algebra.

These problems are studied in a thesis prepared by Mr. Leonard in candidacy for the degree of doctor of philosophy from the University of Colorado. In § 1 it is shown that if G_E and G_F are the groups of the algebras E and F, then the transverse of the compound of these groups is a subgroup of the group G_{EF} of the compound algebra EF. In § 2 the first of the above problems is treated and the following theorem proved: If $\mu_1, \cdots, \mu_n$ are the roots of the characteristic equation of the number $X = \sum_i x_i e_i$ of the algebra E and if $\nu_1, \cdots, \nu_r$ are the roots of the characteristic equation of the number $\bar{X} = \sum_j \bar{x}_j f_j$ of the algebra F, then the roots of the corresponding number of the compound algebra are $\mu_i \nu_j (i = 1, \cdots, n; j = 1, \cdots, r)$.

The second of the above problems is treated from two distinct points of view. In § 3 the characteristic equation of the composite algebra EF being given, the characteristic equations of the factor algebras are sought. In § 4 the multiplication table of the composite algebra being given, the multiplication tables of the factor algebras are determined by means of Peirce's matrix representation of linear associative algebras. For example, by both methods the algebra

	ϵ_1	ϵ_2	ϵ_3	ϵ_4
ϵ_1	ϵ_1	ϵ_2	ϵ_3	ϵ_4
ϵ_2	ϵ_2	0	ϵ_4	0
ϵ_3	ϵ_3	ϵ_4	$-\epsilon_1$	$-\epsilon_2$
ϵ_4	ϵ_4	0	$-\epsilon_2$	0

is factored into the algebras

	e_1	e_2
e_1	e_1	e_2
e_2	e_2	0

	f_1	f_2
f_1	f_1	f_2
f_2	f_2	$-f_1$

Thus one comes to the interesting classification of algebras into prime and composite, exactly like ordinary integers, and this leads to a theory of linear associative algebras analogous to the theory of integers.

If one of the factors is a divisor of zero, it is shown in § 5 that the composite number is a divisor of zero, and conversely.

28. Professor Morley's paper on reflexive geometry is in continuation of two which appeared in the *Transactions*, volumes 1 and 4. Given n lines of a plane, a certain curve C^{n-1} is thereby given, and the discussion of this curve in relation to the lines is the main theme. The curve C^2 is a circle, the curve C^3 a cardioid. The C^{m-1} of any m of the n lines is an osculant of the C^{n-1} of all the lines. If we build up all Jonquières curves of order $n-2$, with multiple point of order $n-3$ at a point x, and passing through the reflexions of x in the n lines, then such a curve has (projectively speaking) coincident points at infinity when x is on C^{n-1}, and it has equiangular asymptotes when x is on the centric circle. Incidentally it appears that n lines belong to and define a set of $2n-3$ lines. The case when for n odd the n lines form such a set is exceptional.

29. The object of Professor Miller's paper is to study all the groups which result if the defining equations of the groups of genus zero are generalized along certain lines. It is a continuation of the paper in volume 9 of the *Archiv der Mathematik und Physik* under the heading "The groups generated by two operators which have a common square." Some of the results are as follows: There are exactly four groups whose two generators s_1, s_2 satisfy the relations $s_1^3 = s_2^3$, $(s_1s_2)^2 = 1$. They are the tetrahedral group, the direct product of this group and the cyclic group of orders 4 and 2, and a certain group of order 96. There are also just four groups whose two generators satisfy the conditions $s_1^2 = s_2^3$, $(s_1s_2)^3 = 1$, while there is an infinite system of groups which satisfy the conditions $s_1^3 = s_2^3$, $(s_1s_2^{-1})^2 = 1$. The similar generalizations of the defining equations of the octahedral and the icosahedral groups are considered and all the possible groups are completely determined. The paper has been offered to the *Transactions* for publication.

30. In the paper on divergence and curl Professor Wilson starts with the integral definitions of these differentiators. This

leads to a closer contact with physics and hydrodynamics, and brings out almost intuitively the theorems of Gauss and Stokes. The introduction of the linear vector function $\nabla \mathbf{V}$ (where $\mathbf{V}$ is a vector function of position in space) makes it possible to obtain div $\mathbf{V}$ and curl $\mathbf{V}$ as $\nabla \cdot V$ and $\nabla \times V$ respectively without the usual recourse to integral calculations relative to an infinitesimal cube or rectangle. The paper appears in the *American Journal of Science*, 1906.

31. After a short introduction giving the history of the efforts to study various subgroups of the projective group by means of involutory transformations, Professor Wilson proceeds to a systematic study of involutory projective transformations and to their special cases in various subgroups. An algebraic interest is added to the discussion by pointing out the fact that these transformations may be regarded as square roots of the idemfactor. In the main body of the paper the discussion of the composition of oblique reflection into strains and the resolution of unimodular strains into reflections is treated in detail. The analytic method employed is Gibbs's vector analysis—in particular his theory of dyadics. Among other things it is proved that: Any root of the minor idemfactor may be resolved into the product of two square roots in ∞^1 ways. The necessary and sufficient condition that a strain be resolved into two reflections is that the scalar cubic corresponding to the identical equation in the matrix representing the strain shall be a reciprocal equation. Any unimodular strain may be resolved into three reflections and the line of the first reflection is arbitrary except possibly for at most three directions.

32. In the paper on double products Professor Wilson takes up somewhat at length the generalizations of some of the questions treated in chapters V and VI of his edition of Gibbs's Vector Analysis. The present paper is perhaps the first published account of the late Professor Gibbs's ideas on multiple algebra, except for the general papers "On multiple algebra" by J. Willard Gibbs, *Proceedings of the American Association for the Advancement of Science*, volume 35 (1886), and "On products in additive fields" by the present author, Verhandlungen des III internationalen Mathematiker-Kongresses in Heidelberg, 1904. The paper will be offered for publication in the *Transactions*.

33. The paper of Professor Moulton is devoted to the consideration of a class of periodic orbits in which the mean motion of the line of nodes of the orbit of one the bodies, referred to the mean plane of the motion of the other two, is arbitrary except for the condition that it shall be commensurable with the synodic mean motion of the three bodies. The discussion involves a new treatment of Hill's linear differential equation with periodic coefficients.

As applied to the lunar theory, the paper shows how to construct expressions for the coordinates of a body having the same synodical period and mean rate of revolution of the line of nodes as observations show the moon has. These are the terms which are said to depend upon the mean motions of the sun and moon, the parallax of the moon, and the latitude of the moon.

34. Professor Bolza's paper gives an extension of Weierstrass's theorem on the expression of the total variation by means of the E-function and of Kneser's theorem on transversals to the so-called most general case of an extremum of a simple definite integral, in which it is required to minimize an integral of the form

$$I = \int_{x_0}^{x_1} f(x, y_1, \cdots, y_n; y_1', \cdots, y_n')dx$$

involving n unknown functions $y_1, \cdots, y_n$ of x and their first partial derivatives $y_1', \cdots, y_n'$, connected by $r < n$ differential equations $f_\rho(x, y_1, \cdots, y_n; y_1', \cdots, y_n') = 0$ $(\rho = 1, 2, \cdots, r)$.

F. N. COLE,
Secretary.

THE NEW HAVEN COLLOQUIUM.

THE Fifth Colloquium of the AMERICAN MATHEMATICAL SOCIETY was held, at the close of the thirteenth summer meeting, at Yale University, New Haven, Conn., opening on Wednesday morning, September 5, 1906 and extending until noon of the following Saturday. Since the colloquium has become a highly important element in the Society's activities, an outline of its historic development may here be of interest.