of Vailati, which we will write aR_1b. More generally, it is possible to construct a set of n-dimensional axioms ($n = 1, 2, 3, \cdots$) by means of the two undefined symbols point and the n-dimensional relation $aR_n(b_1b_2\cdots b_n)$, where the n-ad is ordered.

CHICAGO,
July, 1906.

ON THE ORDERLY LISTING OF SUBSTITUTIONS.

BY PROFESSOR D. N. LEHMER.

(Read before the San Francisco Section of the American Mathematical Society, February 24, 1906.)

1. In a substitution such as $\binom{12345}{34251}$ we shall speak of 34251 as the *permutation* belonging to the substitution. The following list of the permutations belonging to the substitutions on four letters exhibits what seems to be the most orderly arrangement:

$1 = 1234 = I$	$13 = 3124 = (132)$
$2 = 1243 = (34)$	$14 = 3142 = (1342)$
$3 = 1324 = (23)$	$15 = 3214 = (13)$
$4 = 1342 = (234)$	$16 = 3241 = (134)$
$5 = 1423 = (243)$	$17 = 3412 = (13)(24)$
$6 = 1432 = (24)$	$18 = 3421 = (1324)$
$7 = 2134 = (12)$	$19 = 4123 = (1432)$
$8 = 2143 = (12)(34)$	$20 = 4132 = (142)$
$9 = 2314 = (123)$	$21 = 4213 = (143)$
$10 = 2341 = (1234)$	$22 = 4231 = (14)$
$11 = 2413 = (1243)$	$23 = 4312 = (1423)$
$12 = 2431 = (124)$	$24 = 4321 = (14)(23)$

2. It is seen that the numbers of four digits expressed by the different permutations are in ascending order of magnitude. This order would be reversed by the interchange of 1 with 4 and 2 with 3.

3. It is easily possible to compute the rank of a given permutation or to write down a permutation of given rank. Thus the rank of the permutation 341562 is 240 plus the rank

of 41562, there being 120 beginning with 1 and 120 beginning with 2. The rank of 41562 is 48 plus the rank of 1562, since there are 24 beginning with 1 and 24 beginning with 2. The rank of 1562 is the same as the rank of 562 or, since our list gives the rank of any permutation on four elements, we can obtain the rank of 1562 from it as 4. The rank of 341562 is therefore 292. The finding of a permutation of given rank is not more difficult.

4. The permutation of rank 292 in seven letters is 1452673 obtained by prefixing unity and increasing each digit by unity. Similarly for eight letters the permutation of rank 292 is 12563784. The cyclic notations for the corresponding substitutions are

For six letters (13)(2456),
For seven letters (24)(3567),
For eight letters (35)(4678).

In accordance with this rule the multiplication table for the symmetric group on n letters will contain in the upper left hand corner the multiplication table for the symmetric group on $n-1$ letters.

5. The product of two substitutions S and S' in the order SS' is obtainable by operating on the permutation of S with the substitution S', as is readily seen by changing the order of the elements in the upper line and accordingly in the lower line of S'. This law enters in a remarkable way in the multiplication table of the symmetric group. One obvious result is seen in the last column under the substitution (14) (23). This, as we noted in § 2, reverses the order of the permutations so that we have in that column the substitutions in descending rank reading downward. In symbols $S_iS_{24} = S_{25-i}$, or since the permutation of rank $n!$ reverses the order of the arrangement we have generally

$$S_iS_{n!} = S_{n!-i+1}.$$

6. Suppose now that $S_iS_j = S_k$. Then

$$S_iS_jS_{n!} = S_kS_{n!},$$

whence by the above formula

$$S_iS_{n!-j+1} = S_{n!-k+1}.$$

From this it follows that two entries in the table which are on the same horizontal line and are equally distant from the ends add up $n! + 1$. Thus $12 \cdot 17 = 21$ and $12 \cdot 8 = 4$ and $21 + 4 = 25 = 4! + 1$.

7. Suppose again that

$$S_{n!}S_j = S_k.$$

Then

$$S_iS_{n!}S_j = S_iS_k,$$

or

$$S_{n!-i+1}S_i = S_jS_k.$$

From this formula it follows that the column in the table with k for its lowest element reads upward in the same way that the column with k for its topmost element reads downward. These two curious laws hold in their entirety for any subgroup of the symmetric group which contains the substitution of rank $n!$ This in particular will be the case with the alternating group when n is of the form $4m$ or $4m + 1$, since in these two cases the substitution $S_{n!}$ is an even one. The metacyclic group of order $p(p - 1)$ will also exhibit these laws since it contains the substitution $\binom{x_z}{x_{z+1}}$ which is of rank $n!$

8. The list of the permutations of the substitutions on n letters divides itself naturally into n sets. The first set comprises those beginning with the element 1, the second those beginning with the element 2 and so on. These sets we will call the primary sets. Each of these primary sets is divided into secondary sets $n - 1$ in number, each secondary set beginning with the same two numbers. These secondary sets divide up further into ternary sets and so on. We will examine the arrangement of these sets in the multiplication table of the symmetric group.

9. In the product SS' let S' remain fixed while S runs through the list of $n!$ substitutions in order. This by § 5 amounts to operating on all the permutations in order with the substitution S'. It is clear that none of the sets are separated by this operation, the substitution S' merely interchanging the sets as wholes and permuting the subsets among themselves. Moreover the permutation of the primary sets will be of the same rank as the permutation of S'. Thus in the column headed 10 in the group on four elements it is seen that the primary sets are listed reading down in the order 2341 which is the permutation of rank 10.

10. Consider now any primary set beginning with the element k. The secondary sets in it are clearly permuted among themselves in accordance with the effect of S' on all the elements except k. The permutation of the secondary sets may then be obtained by striking out k from the permutation of S'. Thus in column 10 as before referred to the order of the secondary sets in the first primary set is 341; in the second primary set is 241; in the third primary set is 231, and in the fourth 234. The order for the ternary sets is similarly obtained so that for instance the ternary sets in the first primary set have the orders 41, 31, 34, obtained by striking out 3, 4 and 1 in succession from the order of the secondary set 341.

The multiplication table for the symmetric group on four letters is appended; also the table for the alternating group.

Symmetric Group.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23
3	5	1	6	2	4	9	11	7	12	8	10	15	17	13	18	14	16	21	23	19	24	20	22
4	6	2	5	1	3	10	12	8	11	7	9	16	18	14	17	13	15	22	24	20	23	19	21
5	3	6	1	4	2	11	9	12	7	10	8	17	15	18	13	16	14	23	21	24	19	22	20
6	4	5	2	3	1	12	10	11	8	9	7	18	16	17	14	15	13	24	22	23	20	21	19
7	8	13	14	19	20	1	2	15	16	21	22	3	4	9	10	23	24	5	6	11	12	17	18
8	7	14	13	20	19	2	1	16	15	22	21	4	3	10	9	24	23	6	5	12	11	18	17
9	11	15	17	21	23	3	5	13	18	19	24	1	6	7	12	20	22	2	4	8	10	14	16
10	12	16	18	22	24	4	6	14	17	20	23	2	5	8	11	19	21	1	3	7	9	13	15
11	9	17	15	23	21	5	3	18	13	24	19	6	1	12	7	22	20	4	2	10	8	16	14
12	10	18	16	24	22	6	4	17	14	23	20	5	2	11	8	21	19	3	1	9	7	15	13
13	19	7	20	8	14	15	21	1	22	2	16	9	23	3	24	4	10	11	17	5	18	6	12
14	20	8	19	7	13	16	22	2	21	1	15	10	24	4	23	3	9	12	18	6	17	5	11
15	21	9	23	11	17	13	19	3	24	5	18	7	20	1	22	6	12	8	14	2	16	4	10
16	22	10	24	12	18	14	20	4	23	6	17	8	19	2	21	5	11	7	13	1	15	3	9
17	23	11	21	9	15	18	24	5	19	3	13	12	22	6	20	1	7	10	16	4	14	2	8
18	24	12	22	10	16	17	23	6	20	4	14	11	21	5	19	2	8	9	15	3	13	1	7
19	13	20	7	14	8	21	15	22	1	16	2	23	9	24	3	10	4	17	11	18	5	12	6
20	14	19	8	13	7	22	16	21	2	15	1	24	10	23	4	9	3	18	12	17	6	11	5
21	15	23	9	17	11	19	13	24	3	18	5	20	7	22	1	12	6	14	8	16	2	10	4
22	16	24	10	18	12	20	14	23	4	17	6	19	8	21	2	11	5	13	7	15	1	9	3
23	17	21	11	15	9	24	18	19	5	13	3	22	12	20	6	7	1	16	10	14	4	8	2
24	18	22	12	16	10	23	17	20	6	14	4	21	11	19	5	8	2	15	9	13	3	7	1

Alternating Group.

1	4	5	8	9	12	13	16	17	20	21	24
4	5	1	12	8	9	16	17	13	24	20	21
5	1	4	9	12	8	17	13	16	21	24	20
8	13	20	1	16	21	4	9	24	5	12	17
9	17	21	5	13	24	1	12	20	4	8	16
12	16	24	4	17	20	5	8	21	1	9	13
13	20	8	21	1	16	9	24	4	17	5	12
16	24	12	20	4	17	8	21	5	13	1	9
17	21	9	24	5	13	12	20	1	16	4	8
20	8	13	16	21	1	24	4	9	12	17	5
21	9	17	13	24	5	20	1	12	8	16	4
24	12	16	17	20	4	21	5	8	9	13	1