Porter showed that there were points on the circle of convergence which are condensation points for the zeros of the polynomial convergents and that in the neighborhood of every point of this circle the set of these polynomials took on values less than any assigned number however small. He also showed that no set of these polynomials remained limited throughout any region lying outside the circle of convergence.

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SELECTED TOPICS IN THE THEORY OF BOUNDARY VALUE PROBLEMS OF DIFFERENTIAL EQUATIONS.

AN ABSTRACT OF FOUR LECTURES DELIVERED AT THE NEW HAVEN COLLOQUIUM, SEPTEMBER 5–8, 1906.

BY PROFESSOR MAX MASON.

§ 1.

FUNCTIONAL equations — particularly integral equations — have aroused much interest and activity in recent years. Important contributions to the subject have been made, notably by Volterra, Fredholm and Hilbert, and the results have found application in the field of mathematical physics and differential equations. The equations studied have been for the most part of a type difficult of solution, and the treatment correspondingly complicated.

There is however a type of functional equation whose solution may be obtained in a simple manner. Consider the equation

$$f = g + Sf, \tag{1}$$

where g is a known function, and S a *linear operator*, that is $S(u + v) = Su + Sv$. *The operator S will be called convergent if the infinite series*

$$\phi + S\phi + S^2\phi + S^3\phi + \cdots$$

converges for all functions ϕ which satisfy the conditions of con-

tinuity demanded of g and f, and if the convergence is such that the operation S may be performed on this series term by term.

Suppose S is convergent, and that a solution f of (1) exists. Then

$$\begin{aligned} f &= g + Sf = g + S(g + Sf) \\ &= g + Sg + S^2 f = g + Sg + S^2(g + Sf) \\ &= g + Sg + S\,g + S^3 f = \cdots, \end{aligned}$$

the equation

$$f = g + Sg + S^2 g + S^3 g + \cdots + S^{n-1} g + S^n f$$

being obtained after $n - 1$ substitutions of $g + Sf$ for f. If S is convergent, the limit for $n = \infty$ may be taken. It follows that if a solution of (1) exists it has the form

$$(2) \qquad f = g + Sg + S^2 g + S^3 g + \cdots.$$

Conversely, if S is a convergent operator the function defined by (2) is a solution of (1), for

$$\begin{aligned} Sf &= Sg + S^2 g + S^3 g + \cdots \\ &= f - g. \end{aligned}$$

To summarize: *If S is a convergent operator, the equation* (1) *admits a unique solution, given by equation* (2).

A simple example is furnished by the equation

$$(3) \qquad f(x) = g(x) + \int_a^x K(x, y) f(y) dy.$$

The operator S is convergent. For if Φ be the maximum of the absolute value of a function ϕ, and κ the maximum of K, then

$$| S^n \phi | \leqq \frac{(x - a)^n \kappa^n}{n!} \Phi.$$

These results will be applied to the study of certain boundary value problems of differential equations. This method has some advantages over the method of successive approximations, which has been used with such success, notably by Picard. The attention is at once directed to the essential difficulty, that of convergence, and the uniqueness of the solution is proved without special investigation. The method is applicable however only to linear equations.

§ 2.

The linear partial differential equation of the second order in two independent variables

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

is said to be of elliptic, hyperbolic, or parabolic type, according as $AC - B^2$ is positive, negative, or zero, in the region considered. The normal forms of these types are

$$u_{xx} + u_{yy} = au_x + bu_y + cu + f,$$

$$u_{xy} = au_x + bu_y + cu + f, \quad u_{xx} = au_x + bu_y + cu + f,$$

the general equation being reduced to the normal form of the type to which it belongs, by a transformation of the independent variables.

The most important representative of the elliptic type is the potential equation

$$u_{xx} + u_{yy} = 0, \tag{4}$$

and the boundary value problems for this equation are fundamental in the treatment of similar problems for the general equation of elliptic type. Dirichlet's problem — to determine a solution of (4) within a closed region Ω which assumes given values on the boundary of Ω — is the most famous of boundary value problems. *There exists a unique solution of this problem.* The proofs of this statement given by Schwarz, Neumann, and Poincaré are classic. In case the region Ω is convex, Neumann's method gives the simplest solution. The results of § 1 may be applied to interpret this method. The function

$$u(x, y) = \frac{1}{\pi}\int_0^l f(t)\frac{\partial}{\partial t}\Theta[x, y, \xi(t), \eta(t)]dt, \quad \Theta = \arctan\frac{\eta(t) - y}{\xi(t) - x},$$

containing the arbitrary function $f(t)$, is a solution of (4) at all points within Ω, the boundary of Ω being given by the curve

$x = \xi(t), y = \eta(t)$. This function $f(t)$ must be so determined that u takes the value $g(s)$ at the point $x = \xi(s)$, $y = \eta(s)$. On account of the discontinuity of u in crossing the boundary this leads to the following integral equations for f:

$$f(s) = g(s) - \frac{1}{\pi}\int_0^l f(t)\frac{\partial\Theta(s, t)}{\partial t}dt, \tag{5}$$

where $\Theta(s, t) = \Theta(\xi(s), \eta(s), \xi(t), \eta(t))$. This equation is of the type considered in § 1, but the operator is not convergent. However, on adding $f(s)$ to both sides and dividing by two, the equation takes the form

$$f(s) = \tfrac{1}{2}g(s) + Sf,$$

where

$$Sf = -\frac{1}{2\pi}\int_0^l \{f(t) - f(s)\}\frac{\partial\Theta(s, t)}{\partial t}dt.$$

The convergence proof of Neumann's method shows that the operator S is convergent. Hence a unique solution of (5) exists and is given by

$$f = \tfrac{1}{2}\{g + Sg + S^2g + \cdots\}.$$

The function u produced by this function f is the desired solution of Dirichlet's problem.

On account of the above existence theorem, the existence of a Green's function is assured. This function has the form

$$G(x, y, \xi, \eta) = \log\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} + g(x, y, \xi, \eta),$$

where (ξ, η) is a parameter point within Ω, and G vanishes identically in ξ, η when the point (x, y) lies on the boundary of Ω. By the aid of this function a solution of the equation

$$u_{xx} + u_{yy} = f(x, y) \tag{6}$$

which assumes the values $g(s)$ on the boundary of Ω is given by the formula

$$u = u_0 - \frac{1}{2\pi}\int\int_\Omega G(x, y, \xi, \eta)f(\xi, \eta)d\xi d\eta, \tag{7}$$

where u_0 is the solution of Dirichlet's problem for the boundary values $g(s)$, provided that f satisfies a certain continuity condition.

A more general equation of elliptic type, and one of frequent occurrence in the applications, is

$$(8) \qquad v_{xx} + v_{yy} = cv + f.$$

Replacing f in equation (6) by $cv + f$ it is seen that the solution of (8) which assumes the boundary values $g(s)$ is given by the solution of the integral equation

$$(9) \qquad v = u - \frac{1}{2\pi}\int\int_{\Omega} G(x, y, \xi, \eta)c(\xi, \eta)v(\xi, \eta)d\xi d\eta,$$

where u is a known function, given by equation (7). It may be easily shown that the operator S in this equation is convergent, if the region Ω be sufficiently small, the proof depending on the fact that

$$\int\int_{\Omega} G d\xi d\eta$$

approaches zero with the area of Ω. *There exists one and only one solution of the boundary value problem of equation* (8) *if the region Ω is sufficiently small.*

The general equation of elliptic type,

$$(10) \qquad u_{xx} + v_{yy} = av_x + bv_y + cv + f$$

presents greater difficulties. To determine a solution of the boundary value problem, a functional equation of the form

$$(11) \qquad v = u - \frac{1}{2\pi}\int\int_{\Omega} G(av_x + bv_y + cv)d\xi d\eta$$

is to be solved. The operator S involves differentiation as well as integration. The proof that S is convergent is accordingly more difficult, and an investigation regarding continuity is necessary to show that a solution of (11) is also a solution of (10); the results are however the same as in the case of equation (8). This existence theorem was first obtained by Picard, who used the method of successive approximations. The restriction on the size of the region in the theorem is not one due to method alone. In point of fact, if the region be allowed to increase in size, a position is reached for which the theorem is no longer true. Some of the most interesting investigations of special boundary value problems deal with the equation

$$v_{xx} + v_{yy} = \lambda cv,$$

where λ is a parameter, the region Ω being fixed. The results are analogous to those obtained in § 4 for a similar problem in ordinary differential equations.

To Picard is due a remarkable theorem regarding the nature of solutions of the differential equation (10), viz.: *Every equation of elliptic type whose coefficients are analytic functions possesses only analytic solutions*; even though the solution assumes non-analytic boundary values, it will be analytic inside the region.

§ 3.

The equations of hyperbolic type are especially susceptible to treatment by the method of § 1. By the aid of this method a boundary problem may be solved which includes as special cases many problems whose solutions were originally obtained by quite distinct methods.

The equation

$$u_{xy} = au_x + bu_y + cu + f \tag{12}$$

is to be solved under the conditions

$$(u)_{y=\phi(x)} = U(x), \qquad (u_y)_{x=\psi(y)} = Y(y),$$

where ϕ, ψ, U, Y, are given functions.

Let $u_x = v$. The function

$$u = \int_{\phi(x)}^{y}\int_{\psi(\eta)}^{x} v(\xi, \eta)d\xi d\eta + \int_{\phi(x)}^{y} Y(\eta)dy + U(x)$$

satisfies the boundary conditions, whatever v be. On substituting this value u in (12), the equation becomes a functional equation for the determination of v, of the form

$$v = g + Sv,$$

where S is of somewhat complicated form, but involves integration only, the upper limits of each integral being x or y. On account of this fact the proof that S is a convergent operator may be made without difficulty, however large the region within which the point (x, y) may vary. The proof is analogous to that given in connection with equation (3). *It follows that there exists one and only one solution of the boundary problem.* If ϕ and ψ are inverse functions, so that $x = \psi(y)$ and $y = \phi(x)$ define the same curve, the problem is equivalent to finding a solution u of (12) when u and the normal derivative of u are given on a monotonic curve. If $\phi(x) = b$, $\psi(y) = a$, the boundary condition is equivalent to giving the values of u on the lines $x = a$, $y = b$. Other special cases may be easily obtained.

It may be readily seen from the expression for u in terms of v that the continuity properties of the functions ϕ, ψ, U, Y determine to a large extent the continuity properties of u throughout its region of definition. The equations of hyperbolic type thus differ fundamentally from the equations of elliptic type. In fact, S. Bernstein has recently proved that every equation of hyperbolic type possesses non-analytic solutions.

The general equation of parabolic type offers exceptional difficulties. But little is known of the nature of the solutions or of the boundary conditions which may be placed upon them. These equations form a limiting case between those of elliptic and hyperbolic types. Practically nothing is known regarding the boundary problems of equations which are of one type in a part of the region considered and of another in the remainder.

§ 4.

Many of the boundary value problems for partial differential equations may be reduced by the familiar substitution $u = X(x) \cdot Y(y)$, to boundary value problems for ordinary differential equations. The differential equation

$$y'' + \lambda A(x) y = 0, \tag{13}$$

where λ is a parameter, is typical of a large class, and has formed the subject of numerous investigations. The fundamental existence theorem for this equation may be easily proved by the method of § 1. The function

$$y = \int_a^x (x - \xi) u(\xi) d\xi + \alpha'(x - a) + \alpha$$

satisfies the conditions $y(a) = \alpha$, $y'(a) = \alpha'$, whatever be u. It is a solution of (13) if u is a solution of the integral equation

$$u = -\lambda A(x)[\alpha'(x-a) + \alpha] - \lambda A(x)\int_a^x (x-\xi)u(\xi)d\xi,$$

which is a special case of (3). There exists accordingly one and only one solution of (13) satisfying the initial conditions. From the form of the solution it is seen immediately that y is an integral transcendental function of the parameter λ.

A typical boundary problem for (13) is to determine a solution, not identically zero, which vanishes for two given values a and b of x. The general solution of (13) having the form

$$y = c_1\eta_1 + c_2\eta_2,$$

this problem may be solved when and only when λ is a zero of the transcendental function

$$\delta(\lambda) = \begin{vmatrix} \eta_1(a) & \eta_2(a) \\ \eta_1(b) & \eta_2(b) \end{vmatrix}$$

This problem was first studied by Sturm, and his results may be summarized in the following theorem — Sturm's theorem of oscillation: *If $A(x)$ does not change sign the parameter λ may be determined in one and only one way so that a solution of (13) exists which vanishes at a and at b, and n times between a and b.*

This result has since been proved by various methods, some of which are capable of greater generalization than the original methods of Sturm. The problem may be reduced to the solution of an integral equation. Any solution of the equation

$$y(x) = \lambda\int_a^b A(\xi)G(x,\xi)y(\xi)\,d\xi,$$

where

$$G(x,\xi) = \tfrac{1}{2}\left\{|x-\xi| - \frac{(b-x)(\xi-a)+(b-\xi)(x-a)}{b-a}\right\},$$

is a solution of (13) and vanishes at a and b. This integral equation is of a type studied by Fredholm and Hilbert. If

$A(x)$ does not change sign, it can be reduced to the form

$$\eta(x) = \lambda \int_a^b K(x, \xi)\eta(\xi)d\xi,$$

where K is symmetric, the only case to which Hilbert's results apply.* The application of these results proves immediately the existence of an infinite series of "normal parameter values" — zeros of $\delta(\lambda)$ — for each of which there exists a "normal function," a solution of the boundary value problem. Hilbert proves further that any function may be expanded in terms of these normal functions, if it is continuous together with its first and second derivatives. The most general result regarding the expansion of an arbitrary function have however been attained by Kneser, using a different method.

If the function $A(x)$ changes sign, the above method does not apply. The existence theorem may however be obtained by a method based on the consideration of certain minimum problems. Interesting properties of the normal functions are thus set in evidence, and by the aid of these properties a very simple proof for the expansion of a function in terms of normal functions is obtained, even in the case that $A(x)$ changes sign.

The field of boundary value problems is of enormous extent, and comparatively little is known territory. While the theorems stated above are typical, interesting variations result by generalization to more variables, higher derivatives, and more general boundary conditions. Some of these generalizations have been rigorously made, many merely guessed. In particular, the boundary value problems for systems of differential equations offer a wide field for research, and work in this field would be of especial value in the applications.

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* Since the above was written a fifth installment of Hilbert's Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen has appeared, in which the case where the function K is not symmetric is considered. (*Göttinger Nachrichten*, 1906, p. 439.)