tically. This may be seen by substituting the particular values $a_{11} = b_{22} = c_{33} = d_{44} = 1$, the remaining a_{ij}, etc., being zero. Hence these quantities F_{ijkl} are independent.

Finally, consider the quartic

$$\theta_2(A, B, C, D, E) = 0.$$

Let this equation be reduced to

$$A^2 + B^2 + C^2 + D^2 + E^2 = 0.$$

Equate the coefficients of the terms of this equation to the corresponding terms of

$$\Sigma k_{ijkl}\, x_i x_j x_k x_l = 0 \qquad (i \leqq j \leqq k \leqq l \leqq 4)$$

and determine the jacobian matrix as before. That the determinants of this matrix do not all vanish identically is seen by taking for A, B, etc., the particular expressions

$$A \equiv x_1^2,\quad B \equiv x_2^2,\quad C \equiv x_3^2,\quad D \equiv x_4^2,\quad E \equiv x_1 x_2.$$

Since these determinants do not vanish identically the k_{ijkl} are independent. Hence *the equation of an arbitrary quartic surface can be put into the form*

$$A^2 + B^2 + C^2 + D^2 + E^2 = 0.$$

URBANA, ILL.,
September, 1907.

SYMBOLIC LOGIC.

L'Algèbre de la Logique. Par LOUIS COUTURAT. Collection Scientia, No. 24. Gauthier-Villars, Paris, 1905. 100 pp.

Symbolic Logic and its Applications. By HUGH MACCOLL. Longmans, Green, and Co., London, 1906. xi + 141 pp.

The Development of Symbolic Logic; a Critical-Historical Study of the Logical Calculus. By A. T. SHEARMAN. Williams and Norgate, London, 1906. xi + 242 pp.

SYMBOLIC logic is in the interesting though somewhat precarious state of being little known, less used, and much scorned by the majority of mathematicians and philosophers, for whom it might supposedly offer a region of intimate contact and mutual admiration. Meanwhile it has its own ardent supporters whose proselytism is at times almost as fanatical and extravagant as it is unavailing. One is tempted to draw a parallel with the subject of quaternions, which was offered insistently but ineffectively at bargain prices to both mathematicians and physicists until the nearly simultaneous death of Tait and Joly caused a temporary and perhaps permanent lull in the propaganda.

How far the analogy may be carried cannot yet be determined; for neither quaternions nor symbolic logic has run its course. But even if we admit that quaternions can never realize the early hopes which were entertained of it, we cannot be blind to the fact that now, with its consort of grassmannian origin, it stands at the head of a large family of investigations which has allianced itself with the theories of groups and of matrices and which through vector analysis has actually found employment in the mathematical encyclopedia. It would perhaps be rash to forecast the future of symbolic logic by any too close translation of what has occurred in another field; but the attempt to look somewhat into the future has an irresistible lure. To get on safer ground and to reach a point from which the setting of the three books under review may be seen to better advantage, it will be well to run briefly through the history of symbolic logic.

The rudiments of a symbolic method in logic may be seen in Euler's diagrams. For the algebra of classes or of propositional functions they offer a ready expression of the relation of inclusion, and the inequality of mathematics is so vividly suggested that no surprise should be felt if the discovery were made that Euler had experimented with truly symbolic methods. Certain it is that Leibniz perceived the possibility and apparently believed in the desirability of the introduction of symbolism into logical doctrine. It was Boole, however, who first made public an extended treatment of logic in symbolic terms. It should not be overlooked that his point of view was to treat logic by mathematical methods; not mathematics by purely logical methods.

Following Boole, there was a considerable body of work done by those primarily interested in logic. We may mention Charles Peirce, Mrs. Ladd-Franklin, MacColl, and Schröder. Aside from affording a formulation of the simpler laws and processes of logical procedure, these investigations disclosed the possibility of constructing a boundless extension of formal or algebraic logic by mere manipulation of the formulas. In fact it was possible to go on and on in logical algebra just as in ordinary algebra. The huge structure thus raised in all its intricacies and complications must naturally have an even less vital bearing on the fundamental problems of logic than similar refinements of pure mathematics have in the discussion of the physical world. There would be the same fruitfulness and the same barrenness.

It was at this point that Frege and Peano came forward with essentially new ideas — Frege in the early eighties and Peano in the late eighties of the last century. Unfortunately Frege's highly complicated symbolism prevented his keen philosophical reasoning from receiving much attention until very recently, and Peano's symbolism would very likely have proved a similar obstacle had his methods not appealed so strongly to his pupils as to build up a large school whose combined results naturally forced themselves on public attention. The new ideas were simply that mathematics was but a branch of logic, of symbolic logic. As early as 1889 Peano was very insistent on the fact that numbers and points were mere symbols. This was a great step to make.

And yet, when once made, it could hardly fail to appear inevitably suggested by what had already been accomplished. The well known geometries of Lobachevsky and Riemann were evidence enough that it was not our physical space but the system of axioms which afforded the logical foundation of geometry. And the geometric principles of correspondence, whether the simple dualities in plane and space or the more complicated transformations such as the line-sphere transformation of Lie, showed conclusively that it was not the undefined symbols and axioms but their interpretation which made the visual geometry. Thus on both sides the material for making the generalization lay right at hand. This new point of view brought to symbolic logic a new vitality, much as the discovery of a new relation between mathematics and physics has always brought fresh energy to mathematical investigations. The early point of view of Boole had now been changed to its opposite.

This change was far more vital than might at first be imagined. It emancipated logic from the typical form of reasoning found in the statements: All liberals are free-traders; X is a liberal; hence X is a free-trader. If logic is to handle all mathematics, it must deal successfully with the null class and the infinite class. The older logic was too vague, too full of tacit assumptions to stand such a strain. The researches of Cantor on transfinite numbers would alone show that the little word "all" had to be handled very gingerly. The wider field which was opened was not for symbolic logic merely, but for general logic as well. If the sufficiency of a definition or concept is to be tested, there is small value in selecting the simplest case as a test stone; the most complicated should be sought out, and Cantor's work has furnished the necessary complications for considerably clarifying, enlarging, and sharpening up our fundamental logical assumptions. The process is not yet finished.

Some of the advance which has been made towards a more perfect logic has been done without the use of what is strictly symbolic logic. The time may come when symbolism may be entirely discarded for such purposes and when symbolic logic as such may be a thing of the past. It should not then be forgotten, however, that those who first started upon the advance introduced an improved symbolic logic to help their progress, which without it would have been much slower and less effective. At present there remains much to accomplish which apparently cannot be done effectively without some acquaintance with symbolic methods. But it is not easy to admit that such a complete use of formalism as is advocated and illustrated by Peano in his Formulario can ever reach general acceptance. If it could, it may be true that a volume the size of the Jahrbuch might contain practically all the mathematical investigation of a year instead of merely short abstracts. But it is a recognized psychological fact that although condensation and abstraction up to a certain point aid the mind in grasping a result, yet after that point they render the task more difficult on account of the fatigue incident to excessive attention. The problem is essentially one in minimum fatigue, and notwithstanding the experiments on mental and physical dieting we shall probably as a majority stick to the feeling that we all need some hay with our grain.

Couturat's Algebra of logic aims to supply in brief and readily accessible form the elementary principles of the subject developed from the algebraic point of view rather than from the logical. It thus gives the mathematician an example of a non-quantitative algebra and the logician a chance to become acquainted with formal methods under auspicious circumstances. The terms and symbols are carefully explained and illustrated, and the algebraic work is for the most part maintained on a very simple and elementary level. The essentials only are given. For the learner this is a great advantage as compared with the detailed exposition of a large treatise like Schröder's. The logician will doubtless regret that the author did not give an article or two to the discussion of the different canonical types of syllogism, and indeed it might have been well to put in the text at an early point the elegant inconsistent triad of Mrs. Ladd-Franklin with a few remarks on its application. As it is, the author remains true to his intention of presenting the algebra of the subject without especial reference to the classical logic.

The first twenty articles give the reader a statement of the postulates and their immediate consequences. The matter is so presented as to admit of the dual interpretation conceptual and propositional, and these interpretations are constantly formulated in words. The postulates are chosen from the point of view which appears to the author as that most nearly in coincidence with the usual logical interpretation, and are not subjected to the critical examination which would be necessary if the chief interest were to set up a complete system of independent postulates. This is, of course, very much to the advantage of the ordinary reader who would merely be wearied by such refinements. On the other hand the author takes pains to make such comments as will inform the careful reader of many of the refinements which are really essential to a correct point of view. For instance on page 15, after showing how inclusions may be transformed into equalities, he states that if the relation of equality instead of the relation of inclusion had been taken as an undefined symbol, then the relation of inclusion could have been defined and the principle of the syllogism could have been proved. This statement contains the meat of what is perhaps the most fundamental idea in the whole subject, namely, that the definitions and theorems depend on the undefined symbols and postulates, which any author may assume with considerable arbitrariness, so that what is a theorem or definition for one author may be an undefined symbol or postulate for another, and vice versa. The presentation of the troublesome 0 and 1 (all or true) and of negation is such as to leave little room for doubt or misunderstanding. In explaining the law of duality, the distinction between primary and secondary propositions is insisted upon.

Articles 21 to 55 inclusive are of a different nature; they deal more especially with the logical algebra and contain numerous formulas which do not admit such an immediate logical interpretation as those of the preceding articles. The development of logical functions, the solution of equations in one or more unknowns, and the problem of elimination are among the subjects which might appeal most strongly to the mathematicians as affording differences and analogies to the corresponding problems of ordinary algebra. The laws of consequences and causes and the logical machine of Jevons would very likely be more of interest to the logician proper. Throughout these developments the author keeps the manipulation of the symbolism as simple as possible; but undoubtedly it would still offer obstacles to those quite unused to algebraic methods.

The concluding articles of the book are given over to the discussion of formulas which arise only in the calculus of propositions. As is known, it is only to a certain extent that the duality between propositions and classes may be carried. The principle of assertion introduces a fundamental difference. The postulate $(a = 1) = a$, which expresses this principle, enables a considerable reduction in the general formulas to be carried out — in short the calculus of propositions reduces to the calculus of classes where the universe of discourse has but one element.

In conclusion Couturat makes the point that this algebra of logic which he has set forth is of the nature of a formalization and extension of classical logic without going on to questions essentially different from those which have been treated since the time of Aristotle, namely, the relations of inclusion and implication, whereas a more general theory of relations seems necessary to a complete symbolic logic which shall suffice for the purposes of all mathematics.

From what has been said it will undoubtedly appear clear that Couturat has supplied a book which in no way duplicates anything in logical literature and which furnishes an easy and thorough introduction to the larger treatises. At the same time it is sufficiently advanced to cover more than the amount of symbolic logic which may be required by one not primarily interested in the subject. The author has had on hand for some time the preparation of a general treatise on logistics in which we may expect to find a completer discussion of symbolic logic as it now is.

For thirty-five years MacColl's name has been associated with the advance and with the criticism of symbolic logic. Aside from his numerous publications in various periodicals, he has given a somewhat lengthy account of his system in the third volume of the reports of the congress of philosophy at Paris in 1900. His present publication in book form appears to be an amplification of that account. One of his chief aims, as he states, has been to adopt notations which shall lead to a simple discussion of problems of probability and limits. It should be remarked that he uses limit in the sense in which it is used in the expression "limits of an integral" and not as a constant approached by a variable. For his purposes MacColl has seen fit to introduce a notation entirely different from any we have seen elsewhere. There is no suggestion, so far as the mere symbolism goes, of the relations of inclusion, whether subsumption or implication. A implies B is written $A:B$ and the statement that if A belongs to class B then C belongs to class D is formulated as $A^B:C^D$.

It may be interesting to go somewhat into detail as to the author's notation. He has five general exponents, τ, ι, ϵ, η, θ, which are used respectively to denote that the proposition to which they may be affixed is true, false, certain, impossible, variable. The difference between true and certain is this: A^τ means that A is true in some particular instance, whereas A^ϵ means that it is always true, that is, of probability 1. Again A^ι means that A is false in some particular instance, whereas A^η means that A contradicts some datum or definition, that its probability is 0. To quote further we find that "A^θ asserts that A is neither impossible nor certain, that is, that A is possible but uncertain. In other words, A^θ asserts that the probability of A is neither 0 nor 1, but some proper fraction between the two." As the author apparently does not define probability our interpretation of what he means by it must belong to class θ.

According to the generally accepted definition, we may agree that the probability of a certainty is 1 and of an impossibility is 0; but unless the class of objects is finite we cannot agree that the statements "possible but uncertain" and "having a probability which is a proper fraction" are identical. For suppose we mark off an interval of length unity on a line and let us further mark the middle point of the interval with the letter M and let P represent any point of the interval. Then the proposition P is M appears to be neither certain nor impossible but variable and so with the proposition P is not M. Yet in the former case the probability would appear to be 0 and in the latter case to be 1. Whether the author has been insufficiently careful in his statement concerning A^θ or not, the fact remains that probability is a dangerous thing and might better be left out. As a mathematician, one would hesitate a long while before he was willing to commit himself to a definition of probability which could be applied to a proposition derived from a propositional function which had an infinite number of applications in the universe of discourse.

The first six chapters of MacColl's book are used to set up his logical system and develop his analysis. It would take an expert a long time to decide which of the many statements the author intends to have regarded as postulates or axioms. It is probable that his method of procedure is not all in sympathy with the recent postulational tendency and that he reserves the right to introduce any statement at any time without making much ado over it, provided only that it appears to him to be correct. In this he is quite in touch with the everyday logician. Yet on the whole he is explicit in his statements and not readily misunderstood; it is sometimes difficult to understand him at all. For instance, there is the long standing dispute between him and some other logicians as to whether a proposition may be sometimes true and sometimes false. This seems to be largely a matter of definition. As MacColl's definition of proposition does not differ much from the usual definition of propositional function, we may be justified in concluding with some of his milder opponents that according to his own definitions he is right.

Chapters VII to XIII inclusive treat the usual subjects of classical logic. The treatment is careful and critical and affords a large amount of material for any one interested in the logical side of symbolic logic. On page 47 is found the startling statement that not one syllogism of the traditional logic is valid in the form in which it is usually presented in our textbooks. The author merely wishes to emphasize the fact that "Every A is B; every B is C; therefore every A is C" is incorrect whereas "If every A is B and if every B is C, then every A is C" is correct. And he is perfectly right in emphasizing it. The logician might attempt to wriggle out of the difficulty in one of two ways: 1° by saying that he meant the if's and the then instead of the assertions and the therefore, or 2° by claiming that it was always assumed that the asserted premises were true. In the former case we could certainly reply that it was his business to say what he meant, and in the latter case we should point out that the useful principle of reductio ad absurdum depends on drawing valid inferences from hypothetical or else erroneous premises.

The author next takes up the nineteen classical syllogisms as revised by the insertion of if's and then's. Four of these types, namely, Darapti, Felapton, Bramantip, Fesapo, have generally been held by symbolic logicians to be incorrect forms of inference or impure syllogisms. It is certain that they do not satisfy the inconsistent triad of Mrs. Ladd-Franklin and that they are the only ones that do not. Darapti reads as follows: If every Y is Z and every Y is X, then some X is Z. The difficulty with the triad in this case is that if no Y exists there is no way of connecting X and Z, and most symbolic logicians have been willing to add $Y \neq 0$ to make the inference sure.* Of course if it be admitted that every Y implies the existence of at least one Y or that some X does not imply the existence of any X, there is no difficulty. There remains therefore only the case where Y is null and where some is taken to imply existence. MacColl claims that the inference is sure in this case; and a large number of logicians would probably agree with him. Perhaps they are right; but part of MacColl's reasoning looks suspiciously as if it could be applied equally well to cases which everybody would admit were false.

After offering so much in disapprobation of the book under review, it is a great pleasure to be able heartily to approve of the manner in which the author treats the so-called canons of syllogistic validity. These are: 1) Every syllogism has three and only three terms. 2) Every syllogism consists of only three propositions. 3) The middle term must be distributed

* It may be of interest to quote Schroeder's example. We omit the if's because the premises are so stated as to be incontestable. Every rectangular equilateral triangle is rectangular; every rectangular equilateral triangle is equilateral; hence some rectangular triangle is equilateral. The conclusion is true in spherical geometry where there are equilateral rectangular triangles but untrue in the plane where there are none.

at least once in the premises; and it must not be ambiguous.* 4) No term must be distributed in the conclusion unless it is also distributed in one of the premises. 5) We can infer nothing from two negative premises. 6) If one premise be negative, the conclusion must be so also; and conversely. The author proceeds to demolish this structure. Let us take a simple example in connection with the fifth canon. Suppose the premises: No X is Y, no Y is Z (the if's as usual are omitted). Can nothing be inferred? It is unnecessary to wake up the old-school logician for such a matter. Canon 5) is obviously sufficient. Suppose by an extremely artificial rearrangement of language we write: All Y is not-X, all Y is not-Z. Can nothing be inferred? A foolish question: this is obviously the valid Darapti and some not-X is not-Y!

It is by such chicanery as this that the logicians of the old school persist in saving the face of their canons and at the same time render the study of their logic distasteful to many and disgusting to those who are used to drawing conclusions rapidly and accurately. The author is quite justified in maintaining that his formulas are more reasonable. If one is willing to discard the four disputed cases, the inconsistent triad is even more convenient. The statements some X is Y and some Y is X are obviously equivalent and may be written $XY \neq 0$. The statements no X is Y and no Y is X are written $XY = 0$. The statement all X is Y is equivalent to no X is not-Y and is written $XY' = 0$. The triad may then be written

$$(XY = 0)(Y'Z = 0)(XZ \neq 0) \text{ is impossible,}$$

or $(XY = 0)(Y'Z = 0)$ implies $(XZ = 0)$, universal syllogism,

or $(XY = 0)(XZ \neq 0)$ implies $(Y'Z \neq 0)$, particular syllogism.

It takes less time to acquire a ready working knowledge of this rule than to learn the poem about Barbara, to say nothing of learning how to apply it.

The author next goes on to the question of supplying the missing premise in an enthymeme and to similar questions. He then takes up some disputed points such as the alleged duality between classes and inclusion on the one hand and propositions and implication on the other. MacColl was one

* It is extraordinary that anybody should think it worth while to canonize this latter statement!

of the first to argue that the duality is by no means complete, and by this time his contention is pretty well established despite the fact that he himself is not careful in drawing a distinction between propositions and propositional functions. Another matter in which the author has always been very active is the discussion of the null class and the paradoxes which arise from its introduction into the logical system. That there are difficulties in the theory of the null class and that these difficulties, and those connected with the class of all, perhaps more than anything else stand in the way of the ordinary logicians when they attempt to follow the symbolists no one will deny. The solution which the author gives by introducing a theory of unrealities does not, however, appear any more satisfactory to most symbolists than their theory appears to him.

Two very useful chapters are those on the solutions of some questions taken from recent examination papers and on the definitions of technical terms. One of the difficulties which a reader who is not especially trained in the nomenclature of logic experiences is the sudden appearance of some term such as modus tollendo ponens or complex constructive dilemma, and so on. The author's systematic tabulation of definitions of technical terms is therefore a great convenience. It is impossible, however, to pass over one of the author's definitions without a word of protest. He says: "Much confusion of ideas is caused by the fact that each of these words (infinite and infinitesimal) is used in different senses by mathematicians. Hence arise most of the strange and inadmissible paradoxes of the various non-euclidean geometries. To avoid all ambiguities I will define. . . . The symbol α denotes any positive quantity or ratio too large to be expressible in any recognized notation and any such ratio is called a positive infinity." The first sentence is correct although the confusion as far as mathematicians is concerned is not insuperably great. The second, about non-euclidean geometries, appears to be a bit of news. The third is apparently both ambiguous and unintelligible. Is the α beyond the alephs and omegas? As a matter of fact the α does not appear to be anywhere in particular. And this was in 1906.

The last five chapters, XIV to XVIII, deal with the calculus of limits. It is this portion of the work which the author might expect to find most fully reviewed here. But the mention will only be short. The problem is to find those values of the variable for which certain expressions remain positive or become zero or remain negative, and more especially to determining the limits of integration for multiple integrals. The question of the limits of integrals was treated some long time since by the author in the *Proceedings of the London Mathematical Society*. He has also contributed other mathematical articles to that periodical and to the Mathematical questions and solutions from the *Educational Times*. The great advantage of his method is, as he states, that it is independent of diagrams and is equally applicable when the number of variables is great, whereas diagrams become complicated in three dimensions and impossible in a higher number. But like the old-school logicians, most mathematicians have an obstinate fondness for their own methods and will probably struggle on with them the best they can.

As regards the numerous points still in dispute between the author and other logicians one further word should be said. The disputations have been largely controversial. In some cases MacColl has won and in some he has lost; and a similar division of the spoils is not unlikely in the future. The unfortunate thing is that there has been so much controversy and so little proof whether on one side or on the other. In fact the mass of verbiage and illustration is often quite bewildering; so that the reader of the communications pro and con is unable to decide on anything. If in such cases as these the disputants would only refrain as much as possible from talking and devote their energies to setting up sets of postulates as definitely as they could and then developing their consequences with reference to alternative systems, some distinct and intelligible results might be obtained in short order. These results might be largely negative in that they showed that different logical systems led to similar conclusions with regard to all points not in dispute and to different conclusions with respects to points in dispute; but even that much conclusively shown would amount to a great deal in clearing the atmosphere. In other words why, pray, do not logicians use to a greater extent the refinements of deductive logic when they are discussing delicate points; mathematicians have adopted that point of view in dealing with geometries. Evidently the doctors do not take their own medicine.

Shearman, if his text be taken in evidence, is precisely one of these controversial and dialectic logicians. His book, being in itself a history and a criticism of symbolic logic, is particularly hard to criticize in detail. He has much more of interest to offer to logicians, and especially to such as stand aloof from symbolic logic, than to mathematicians. For instance, he starts his introduction with the statement that it is his object to show that during the last fifty years there has been a definite advance made in symbolic logic. As there was very little symbolic logic in 1856 no one will dispute his contention, and as far as mathematicians are concerned one may say that the chief advance in symbolic logic has been made in the last twenty years. And this work is left almost untouched by the author. As his point of view is partly historical and partly critical, and as there is no very sharp distinction made between the two in the text, it is at times practically impossible to decide whether any particular series of statements is to be interpreted as a quotation from past reasoning which led to certain conclusions or as an argument of the author's.

The first chapter deals with the question whether letters should be taken to represent classes or propositions. After thirteen or fourteen pages of dialectic we reach this conclusion : "I think, then, that it is correct to hold that our letters may represent either classes or propositions, but that we must be careful to notice that the rules to be adopted in working out problems are not the same in the two cases. This view has been adopted by Boole, Venn, and Schröder, who, so far as the point in question is concerned, are thus seen to have been proceeding in the right direction." This is typical of the author's critical-historical conclusions, and the dialectic which goes before is typical of his analysis. To one at all accustomed to the use of symbols it might seem obvious that letters may be used to represent either classes or propositions or anything else, provided that the proper rules were in each case observed ; but the author has seen fit to hedge a bit in a footnote, where he states that his argument has reference to the earlier problems of symbolic logic and that "when we come to deal with problems that are not included within the scope of the Boolian treatment, I admit that it is better to let symbols stand primarily for propositions."

This sort of elaborate treatment of questions of notation, this hedging about every statement with a qualification, this mixture of criticism and history to a point where the reader cannot tell what is meant, this formulating of conclusions in the first person with the author's own position as a final test for correctness and rightness abound throughout the book. They are not peculiar to Shearman alone, but permeate the work of a large number of logicians who apply the adjective symbolic to their work, although they rarely if ever do the smallest piece of research with truly symbolic methods. What real justification they have in applying the term to themselves is merely a matter of conjecture. Perhaps they do not know what other term to use. They certainly are not classical logicians, because they have been so bold as to see that reading Attic Greek is not a sufficient condition for becoming a complete logician, and they are certainly not up-to-date symbolic logicians, because they have not perceived that reading modern Italian is a necessary condition. Notwithstanding, they attack each other with ardor, assault a really conscientious symbolist like MacColl or Mrs. Ladd-Franklin with virulence, and freely distribute their willing or unwilling applause to such authorities as Boole, Schroeder, Peano, Frege, Couturat, and Russell. Although they are thus proficient in controversy, it may be doubted if they have any even embryonic notion of what really constitutes a proof, a definition, or a postulate.

Shearman's second chapter is on the symbols of operation. He states on page 34 that most writers on the subject of symbolic logic have undoubtedly introduced symbols of operation. It is interesting to note the implication that one may perfectly feasibly construct a symbolic logic without any symbols of operation. This is but corroborative of our statement that many logicians who term themselves symbolic eschew symbolism to a very great extent. It would appear that they might better use the adjective formal in place of symbolic. For any logic which is other than an investigation in the psychology of the mental processes of reaching conclusions (as Wundt's Logik, for instance) is bound to be symbolic to a certain extent. The mere employment of language at all is symbolism. To go a little further the use of X's and Y's and Z's in stating the syllogisms is symbolic. To this extent all formal logic is symbolic. But the real essence of symbolic logic as contrasted with merely formal logic is its recognition of the value of a preponderating use of symbols as against language of the ordinary sort on account of the greater precision, the greater abstraction, and the readier manipulation which are thereby obtained.

Although a large amount of the author's second chapter will appear either trite or misguided to those acquainted with symbolism, it may be of value to others; and some of the discussions, such as that connected with the difference between exclusive and non-exclusive alternatives, are illuminating to a larger circle of readers. The next chapter on the process of solution gives an idea of solution whether analytic or diagrammatic. The presentation would gain considerably in clearness if the author had inserted at least one diagram and had outlined the analytic method with more analytic detail. The next chapter discusses the question of the extensive versus the intensive basis for the calculus of logic. The treatment is mainly carried on as a critique of Castillon's system, which was based on intension. This is particularly interesting inasmuch as Castillon's logic is comparatively little known. The author inclines very strongly to the opinion that the basis should be extensive — in fact he is by no means completely in sympathy with Russell's idea that it is in a domain intermediate between pure intension and pure extension that symbolic logic has its lair. But just what the author would do in handling the infinite classes does not appear. He avoids the very subject which has been mainly responsible for the recent advances in symbolic logic.

Shearman spends chapter V in trying to show that Jevons and MacColl have not contributed much to symbolic logic. Jevons's deficiency was due to his lack "in power of originating important logical generalizations and his failure to appreciate the full significance of the work done by other logicians." This is perhaps true; but the statement is so extreme that it would apply with equal or greater force to so many others as to nullify its value as a criticism of Jevons in particular. MacColl's difficulty is his lack of cooperation with other symbolists. The words about Jevons are few; but more than twenty pages are devoted to reducing MacColl to an infinitesimal. The length of the argument shows one thing very conclusively, namely, MacColl's prominent position in symbolic logic during the last thirty years. As has been stated above, the points in dispute with MacColl are partly matters of definition in which any author is more or less free to use his own discretion, partly matters of notation in which an author is bound by little more than his regard for the ease of his readers, partly matters connected with much mooted questions which are not yet settled entirely to the satisfaction of any one class of logicians, to say nothing of all classes together.

Until this triplicate division of the dispute is more carefully observed and until more refinements have been employed than are available in a mere argument, we cannot agree with the author in stating that the controversies have been settled, even if a good deal of light has been thrown on them.

Chapter VI is on the later logical doctrines, namely, the doctrine of multiple quantification, the logic of relatives, and the new symbolic logic. The first will be passed without comment. In treating the logic of relatives, the author appears to be unacquainted with Russell's or Royce's investigations on the subject, and without this any present discussion of relatives is to a large extent lacking in completeness. This defect is partly remedied in the treatment of the new symbolic logic. Here there is some account of the work of Frege, Peano, Russell, and Whitehead. This will probably afford interesting reading for logicians rather than for mathematicians, who will prefer to go to the original sources instead of relying on what is necessarily an inadequate account. In the concluding chapter on the utility of symbolic logic, the author lapses into a metaphysics or something similar that apparently does not advance him very far toward showing that symbolic logic has any particular utility.

In concluding with this critical-historical study of the logical calculus, it should be stated that the mathematician must not turn to it in expectation of finding what he is most likely to want. There is nothing in it which will throw light on the nature of a defintion, a postulate, or a proof, or of what constitutes a deductive system; little or nothing which will suggest the existence of reasoning other than syllogistic; and not much more which would indicate the analytic elegance of the threefold calculus of propositions, classes, and relations. The mathematician is almost certain to find more of value to him in the excellent though brief account of symbolic logic by E. V. Huntington and Mrs. Ladd-Franklin in the ninth volume of the Encyclopedia Americana. It would be difficult to cite a better reference.* Logicians may find in Shearman's book much that is interesting and valuable to them; but as a mathematician writing in a mathematical journal, it is impossible to

* We should not omit to cite also the admirable and quite recent brochure entitled Uno sguardo al nuovo indirizzo logico-matematico delle scienze deduttive; discorso letto dal professore M. Pieri inaugurandosi l'anno accademico 1906–'07 nella Reale Università di Catania. Francesco Galati, Catania, 1907. 62 pp.

refrain from cautioning them against imagining that they are becoming acquainted with symbolic logic, or the deductive system in general, as mathematicians know it and use it. It may be hoped that logicians, too, will see fit to consult the article in the encyclopedia just mentioned or some other source of similar character, such as Pieri's inaugural address, before they permit themselves to form an opinion on the accomplishments, value, and recent advances of symbolic logic.

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SHORTER NOTICES.

H. Durège, Elemente der Theorie der Funktionen einer komplexen veränderlichen Grösse, in fünfter Auflage, neu bearbeitet, von LUDWIG MAURER, mit 41 Figuren im Text. Leipzig, B. G. Teubner, 1906. 397 + x pp.

THE present work, although styled the fifth edition of Durège's well-known "Elements," is in reality a new treatise. The title and the short historical introduction have been retained ; aside from these, we have not remarked a trace of the original work. Nor could it be otherwise. The theory of functions has grown enormously since the days of Riemann. On the one hand, new fields have been opened up and explored, on the other the old tools of research have been given a greater refinement and many new ones have been added. The present author, in preparing a new edition, quite rightly decided not to patch up the old edifice, but to tear it down completely and erect a new one in harmony with the needs and tendencies of the present day. The result is an up-to-date treatise of moderate proportions, clearly and attractively written, which will surely have a widespread and well-deserved popularity.

The book starts out with an introductory chapter on real variables. Dedekind's theory of irrational numbers is sketched; such notions as simple and multiple limits, upper and lower limits, uniform convergence, also a few notions from the theory of point aggregates are briefly treated. The subject of integration is developed more fully and terminates with Gauss's relation between line and double integrals, which is later used to prove Cauchy's fundamental theorem.