CRITERIA FOR THE IRREDUCIBILITY OF A RECIPROCAL EQUATION.

BY PROFESSOR L. E. DICKSON.

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1. A reciprocal equation $f(x) = x^m + \cdots = 0$ is one for which

$$x^m f(1/x) \equiv cf(x).$$

Replacing x by $1/x$, we see that $f \equiv c^2 f$, $c = \pm 1$. Now $f(x)$ has the factor $x \pm 1$ and hence is reducible, unless m is even and $c = +1$. Further discussion may therefore be limited to equations

$$(1)\quad F(x) \equiv x^{2n} + c_1 x^{2n-1} + c_2 x^{2n-2} + \cdots + c_2 x^2 + c_1 x + 1 = 0$$

of even degree and having

$$(2)\qquad x^{2n} F(1/x) \equiv F(x).$$

Let R be a domain of rationality containing the c's.

Under the substitution

$$(3)\qquad x + 1/x = y,$$

$x^{-n}F(x)$ becomes a polynomial in y,

$$(4)\qquad \phi(y) = y^n + k_1 y^{n-1} + \cdots + k_n,$$

with coefficients in R. By a suitable choice of the c's, the k's may be made equal to any assigned values.

We shall establish in §§ 2–7 the following:

THEOREM. *Necessary and sufficient conditions for the irreducibility of $F(x)$ in the domain R are*

(I) $\phi(y)$ *must be irreducible in R.*

(II) $F(x)$ *must not equal a product of two distinct irreducible functions of degree n.*

The second condition is discussed in §§ 8–10.

2. The irreducibility of $F(x)$ in R implies that of $\phi(y)$. For, if

$$\phi(y) = (y^l + \cdots)(y^m + \cdots), \quad (l + m = n, \quad l > 0, \quad m > 0),$$

then would

$$F(x) = x^l\{(x + 1/x)^l + \cdots\} \cdot x^m\{(x + 1/x)^m + \cdots\}.$$

3. If $F(x)$ has in R an irreducible factor

$$A(x) = x^{2r+1} + a_1x^{2r} + \cdots + a_{2r}x + a_{2r+1}$$

of odd degree, then $F(x)$ has the irreducible factor.

$$B(x) \equiv \frac{x^{2r+1}}{a_{2r+1}} A\left(\frac{1}{x}\right) = x^{2r+1} + \frac{a_{2r}}{a_{2r+1}} x^{2r} + \cdots + \frac{a_1}{a_{2r+1}} x + \frac{1}{a_{2r+1}},$$

not identical with $A(x)$. For, from $F \equiv AQ$ and (2) follows

$$F(x) \equiv B(x)Q', \quad Q' \equiv a_{2r+1}x^{2n-2r-1}Q(1/x).$$

Next, if $B \equiv A$, then

$$A = x^{2r+1} \pm 1 + a_1x(x^{2r-1} \pm 1) + a_2x^2(x^{2r-3} \pm 1) + \cdots + a_rx^r(x \pm 1),$$

so that A would have the factor $x \pm 1$ and be reducible.

4. If $\phi(y)$ is irreducible in R, $F(x)$ has in R no irreducible factor A of odd degree $< n$. For, if so, $P \equiv AB$, where B is given in § 3, would be a self-reciprocal factor of $F(x)$. In fact,

$$P(1/x) = A(1/x)A(x)/x^{2r+1}a_{2r+1}, \quad x^{2(2r+1)}P(1/x) = AB = P(x).$$

Hence, in view of (3), $x^{-(2r+1)}P(x)$ would equal a factor of degree $2r + 1$ of $\phi(y)$.

5. If $F(x)$ has in R an irreducible factor

$$A(x) = x^{2r} + a_1x^{2r-1} + \cdots + a_{2r-1}x + a_{2r}$$

of even degree, then $F(x)$ has the irreducible factor

$$B(x) = \frac{1}{a_{2r}} x^{2r} A\left(\frac{1}{x}\right).$$

If $B(x) \equiv A(x)$, A is self-reciprocal, viz.,

$$A(x) = x^{2r} + 1 + a_1(x^{2r-1} + x) + \cdots + a_{r-1}(x^{r+1} + x^{r-1}) + a_rx^r.$$

In fact, the conditions for $B \equiv A$ are

$$a_{2r} = \pm 1, \quad a_{2r-1} = \pm a_1, \quad a_{2r-2} = \pm a_2, \quad \cdots.$$

For the lower signs, A has the factor $x^2 - 1$, contrary to hypothesis.

6. If $\phi(y)$ is irreducible in R, $F(x)$ has in R no irreducible factor A of even degree $< n$. For, by § 5, either B is distinct from A so that AB is a self-reciprocal factor of $F(x)$, or else A itself is a self-reciprocal factor. In either case $\phi(y)$ would have in R a factor of degree $< n$.

7. It follows from §§ 4, 6 that, when $\phi(y)$ is irreducible in R, $F(x)$ has no irreducible factor of degree $< n$. Further, by §§ 3, 5, an irreducible factor $A(x)$ of degree n implies a second irreducible factor $x^n A(1/x)$, algebraically distinct from $A(x)$. The theorem of § 1 is therefore proved.

8. It remains to consider the case $F = AB$,

$$A = x^n + a_1 x^{n-1} + \cdots + a_n,$$

$$B = x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_1}{a_n} x + \frac{1}{a_n},$$

where A and B are distinct irreducible functions in R. To determine the a_i, we have n distinct relations

$$(5) \quad a_1 + a_{n-1}/a_n = c_1, \quad a_2 + (a_1 a_{n-1} + a_{n-2})/a_n = c_2, \cdots$$

We may eliminate $a_1, \cdots, a_{n-1}$ and obtain an equation for a_n. As shown in § 9, this equation is of degree 2^n. Except for certain sets of values of the c_i, we may express $a_1, \cdots, a_{n-1}$ rationally in terms of a_n; the problem is then reduced to the consideration of the rationality of a root of the equation of degree 2^n. This equation for a_n is a reciprocal equation. In fact, if we set

$$A_1 = a_{n-1}/a_n, \quad \cdots, \quad A_{n-1} = a_1/a_n, \quad A_n = 1/a_n,$$

equations (5) become

$$(5') \; A_1 + A_{n-1}/A_n = c_1, \; A_2 + (A_1 A_{n-1} + A_{n-2})/A_n = c_2, \cdots.$$

That the equations (5′) are throughout of the same form as equations (5) is evident from the fact that we have merely interchanged the rôles of the factors A and B of F. Hence the equation in a_n, obtained by eliminating $a_1, \cdots, a_{n-1}$ from

(5), is identical with the equation in $A_n = 1/a_n$, obtained from (5′).

9. Denote the roots of $F = 0$ by

$$(6) \qquad \alpha_1, \quad \alpha_1^{-1}, \quad \alpha_2, \quad \alpha_2^{-1}, \quad \cdots, \quad \alpha_n, \quad \alpha_n^{-1}.$$

A factorization $F = AB$, of the kind considered in § 8, corresponds uniquely to a separation of the roots (6) into two sets each of n roots, such that reciprocal roots belong to different sets. Hence the roots of the first set may be selected in

$$\frac{2n(2n-2)(2n-4)\cdots 2}{n\,!} = 2^n$$

ways. The number of factors A is thus 2^n.

10. For $n = 2$, we set $\alpha_1 = \alpha$, $\alpha_2 = \beta$, and have

$$(7) \quad \alpha+\beta+\alpha^{-1}+\beta^{-1}=c_1, \; 2+\alpha\beta^{-1}+\beta\alpha^{-1}+\alpha\beta+\alpha^{-1}\beta^{-1}=c_2.$$

From these we derive

$$\alpha^2 + \beta^2 + \alpha^{-2} + \beta^{-2} = c_1^2 - 2c_2.$$

Hence $\alpha\beta + \alpha^{-1}\beta^{-1}$ and $\alpha\beta^{-1} + \alpha^{-1}\beta$ are the roots of

$$(8) \qquad z^2 - (c_2 - 2)z + c_1^2 - 2c_2 = 0.$$

The quartic for a_2(§ 8) is obtained by setting

$$(9) \qquad z = a_2 + a_2^{-1}.$$

By (7_1), $\alpha + \beta$ is a rational function of $\alpha\beta$ and c_1 when $c_1 \neq 0$. Hence, for $c_1 \neq 0$, the necessary and sufficient conditions for the factorization $F = AB$ in R are that the roots

$$(10) \qquad z_\pm = \tfrac{1}{2}(c_2 - 2) \pm [(1 + \tfrac{1}{2}c_2)^2 - c_1^2]^{1/2}$$

of (8) be rational and that one of the values $(z_\pm^2 - 4)^{1/2}$ be rational, so that (9) shall lead to a rational value of a_2. Incorporating the condition that (4) shall be irreducible in R, we obtain the

THEOREM.* *The necessary and sufficient conditions that*

$$x^4 + c_1x^3 + c_2x^2 + c_1x + 1 \quad (c_1 \neq 0)$$

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* For other proofs by the writer, see *Amer. Math. Monthly*, vol. 10 (1903), p. 221; vol. 15 (1908), p. 75. The first paper cited also treats reciprocal sextic equations.

shall be irreducible in a domain R are that $(c_1^2 - 4c_2 + 8)^{1/2}$ *be irrational, and that either* $l = [(1 + \frac{1}{2}c_2)^2 - c_1^2]^{1/2}$ *be irrational or else l rational and* $[\frac{1}{2}c_2^2 - c_1^2 - 2 \pm (c_2 - 2)l]^{1/2}$ *both irrational.*

11. The only linear fractional transformations which replace a reciprocal equation by a reciprocal equation are

$$x' = \pm \frac{\alpha x + \beta}{\beta x + \alpha} \qquad (\alpha^2 \neq \beta^2). \tag{11}$$

Then y, given by (3), undergoes the transformation

$$y' = \pm \frac{(\alpha^2 + \beta^2)y + 4\alpha\beta}{\alpha\beta y + \alpha^2 + \beta^2}. \tag{12}$$

The transformation on $\frac{1}{2}y$ is the square of (11).

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A NEW GRAPHICAL METHOD FOR QUATERNIONS.

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1. ANY quaternion q may be written in the form $q = (w + xi) + (y + zi)j$. For convenience let us represent numbers of the form $w + xi$ (practically equivalent to ordinary complex numbers save in their products by j) by Greek characters, so that q may be written

$$q = \alpha + \beta j,$$

where for any number β we have $\beta j = j\bar{\beta}$, $\bar{\beta}$ being the conjugate of β.

The tensor of q is then the square root of the sum of the squares of the moduli of α, β. Also the scalar of q is $\frac{1}{2}(\alpha + \bar{\alpha})$, that is, the real part of α.

2. The product of $q = \alpha + \beta j$ and $r = \gamma + \delta j$ is

$$qr = (\alpha\gamma - \beta\bar{\delta}) + (\alpha\delta + \beta\bar{\gamma})j,$$

and also we have

$$rq = (\alpha\gamma - \bar{\beta}\delta) + (\bar{\alpha}\delta + \beta\gamma)j.$$